Reconstruction of sheaves on stratified ∞-topoi.

Consider étale topos X_{et} of étale sheaves on qcqs scheme X.

Ex: $X = \mathrm{Spec}\, k$, k field, $G = \mathrm{Gal}(k^{sep}/k)$, then $X_{et} \simeq \mathrm{Spc}^{BG}$ (pro-sense).

Problem: Describe sheaves $F \in X_{et}$ in terms of simpler data.

(e.g. stalks at closed points, gluing data, ...)

Analogy: X top. space. $X \xrightarrow{f} P$ cts map to a finite poset.

Then (Lurie) shows $\mathrm{Shv}^{P\text{-cstr}}(X) \simeq \mathrm{Fun}(\mathrm{Exit}(X), \mathrm{Spc})$

here $\mathrm{Exit}(X) \xrightarrow{\pi} P$ conservative functor to P } call $\mathrm{Exit}(X)$

s.t. fibers are $\mathrm{Sing}(X_p)$ $\forall p \in P$ } P-stratified space.

In alg. geom, étale topos X_{et} is canonically stratified / underlying space X_{zar}

(X qcqs $\Rightarrow$ X_{zar} spectral space $\overset{\text{Hochster}}{\iff}$ profinite poset $X_{zar} \simeq \varprojlim_{P \in FC(X)} P$)

get $X_{et} \to \mathrm{Shv}(X_{zar}) \to \mathrm{Shv}(P)$. P-stratification of X_{et}.

Thm (BGH): $\exists$ profinite stratified space $\mathrm{Gal}(X) / X_{zar}$, X_{zar} profinite poset

s.t. $X_{et} \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}\, X, \mathrm{Spc})$ of ∞-topoi.

Specializes: $X_{et}^{cstr} \simeq \mathrm{Fun}^{cts}(\mathrm{Gal}(X), \mathrm{Spc}_\pi)$ π-finite spaces.

Here: $\mathrm{Gal}(X)$ pro-object in π-finite strat. spaces.

$\mathrm{Gal}(X) \simeq \{\mathrm{Gal}(X/P)\}_{P \in FC(X)}$, $\mathrm{Gal}(X/P) \in \mathrm{Pro}(\mathrm{Strat}^{\pi\text{-fin}}_{/P})$
$\{\mathrm{Gal}(X/P)_\alpha\}$

$$\mathrm{Fun}^{cts}(\mathrm{Gal}\, X, \mathrm{Spc}) := \varinjlim_{P} \varinjlim_{\alpha} \mathrm{Fun}(\mathrm{Gal}(X/P)_\alpha, \mathrm{Spc}).$$

BGH: construct $\mathrm{Gal}(X)$ as profinite stratified shape of $\begin{matrix} X_{et} \\ \downarrow \\ \mathrm{Shv}(X_{zar}) \end{matrix}$.

Goal today: study abstract thy of stratified ∞-topoi.

See how, for a finite poset P, P-stratifications on ∞-topos $\mathcal{X}$ is generalization of recollement.

Def: P finite poset, a P-stratified ∞-topos $\mathcal{X} \xrightarrow{f_*} \tilde{P} := \mathrm{Shv}(P)$
$\simeq \mathrm{Fun}(P, \mathrm{Spc})$.

Ex: For any ∞-topos $\mathcal{X}$, have 0-localic reflection.

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[rr, bend left] & \mathrm{Shv}(\mathrm{Open}(\mathcal{X})) \arrow[r] & \mathrm{Shv}(P) \end{tikzcd}.$$

Given P-stratification $\mathcal{X} \xrightarrow{f_*} P$, let

$$\begin{tikzcd} \mathcal{X}_p \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \arrow[d, "f_*"] \\ \{p\} \arrow[r, hook] & \tilde{P} \end{tikzcd}$$

in RTop_∞.

$$G(\mathcal{X}) = \{ (x,p) \in \mathcal{X} \times P^{op} \mid x \in \mathcal{X}_p \} \longrightarrow P^{op}.$$

Claim: locally cocart fibration.

and $\mathrm{Fun}^{\mathrm{cocart}}_{/P^{op}}(\mathrm{sd}(P^{op}), G(\mathcal{X})) \underset{\simeq}{\xrightarrow{\lim}} \mathcal{X}$ Thm (S.) Reconstruction of $\mathcal{X}$ from its strata.

pres. locally cocart edges

$$\left(\begin{tikzcd} \mathrm{sd}(P^{op}) \arrow[r, "\phi"] \arrow[dr, "\mathrm{max.}"'] & G(\mathcal{X}) \arrow[d] \\ & P^{op} \end{tikzcd} \right) \longmapsto \lim_{\mathrm{sd}(P^{op})}(\mathrm{pr}_{\mathcal{X}} \circ \phi)$$

$\mathrm{sd}(P^{op})$ = totally ordered subsets $\Sigma \subseteq P^{op}$

Recall thy of recollements of ∞-topoi: U (-1)-truncated obj in $\mathcal{X}$
$U \in \mathrm{Open}(\mathcal{X})$

$$\mathcal{X}_{/U} \underset{j_*}{\overset{j_!}{\rightleftarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{X} \setminus U = \{ Z \in \mathcal{X} \mid Z \times U \simeq U \}.$$

(j^*: $\mathcal{X} \to \mathcal{X}_{/U}$)

j_* open immersion, i_* closed immersion.

$\mathcal{F} \in \mathcal{X}$:

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & i_* i^* \mathcal{F} \arrow[d] \\ j_* j^* \mathcal{F} \arrow[r] & i_* i^* j_* j^* \mathcal{F} \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathcal{X} \setminus U) \arrow[d, "ev_1"] \\ \mathcal{X}_{/U} \arrow[r, "i^* j_*"'] & \mathcal{X} \setminus U \end{tikzcd}$$

expresses $\mathcal{F} \in \mathcal{X}$ as $(j^* \mathcal{F},\ i^* \mathcal{F} \to i^* j_* j^* \mathcal{F})$ tuple.

let $P = [1]$ in above theory: $sd([1]^{op})$:
$$\begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] & 0<1 \end{tikzcd}$$

and $\begin{tikzcd} G(\mathcal{X}) \arrow[d] \\ [1]^{op} \end{tikzcd}$ classified by gluing functor

$$\begin{tikzcd} {[1]^{op}} \arrow[r] & Cat_\infty \\ 1 \arrow[d] & \mathcal{X}^{cu} \arrow[d, "i^* j_*"] \\ 0 & \mathcal{X}\setminus U \end{tikzcd}$$

Then $\mathrm{Fun}^{cocart}_{[1]^{op}}(sd([1]^{op}), G(\mathcal{X})) \simeq \mathcal{X}$

retrieving recollement description of $\mathcal{X}$.

Go back general finite poset P, $\begin{tikzcd} \mathcal{X}_p \arrow[r, hook, "e_p"'] & \mathcal{X} \arrow[l, bend right, "\Phi^p"'] \end{tikzcd}$

factors as composition of immersions, $\mathcal{X}_p \underset{closed}{\hookrightarrow} \mathcal{X}_{P_{\geq p}} \underset{open}{\hookrightarrow} \mathcal{X}$
$\qquad \mathcal{X}_{P_{\geq p}} = \Phi^{\geq p}_{\hat{p}}\mathcal{X}$

and then $\begin{tikzcd} G(\mathcal{X}) \arrow[d] \\ P^{op} \end{tikzcd}$, functoriality $\begin{tikzcd} & \mathcal{X} \arrow[dr, "\Phi^q"] & \\ \mathcal{X}_p \arrow[ur, "e_p"] \arrow[rr, "p_q^p"] & & \mathcal{X}_q \end{tikzcd} = \Phi^q \circ e_p$.

$p \geq q$

So: what is: $F \in \mathcal{X}$ is data: of $(\Phi^p F \in \mathcal{X}_p$, + $\forall p \geq q$,
$\Phi^q F \to i^q_p \Phi^p F$, +
higher coherences)
...

Contrast w/ description of G-spectrum in terms of geometric fixed points
(Glasman, Ayala-Mazel Gee-Rozenblyum.)

Indeed: Open($\mathcal{X}$) in toposic case
$\iff$ $\underbrace{\text{Balmer}(\mathcal{C}^\omega)}_{\text{coherent locale}}$ for $\mathcal{C}$ pres. stable SMC s.t. $\otimes | \mathcal{C}^\omega$.

BGH: they think of P-stratified ∞-topoi in terms of "decollage" strata + links.
dual perspective to my description of P-stratified ∞-topoi.

$$\mathrm{RTop}/_P \simeq \mathrm{LocCoCart}_{\mathrm{Pop}}^{\mathrm{topnice}}$$

BGH $\quad \cup\!\!\!\!\!\mathrm{S}$ +adjectives

$$\mathrm{Dec}_P \subseteq \mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathrm{RTop})$$

$P = [1] \qquad (0 \to 1) \mapsto \mathcal{U} \overset{\star}{\underset{\mathcal{X}}{\leftarrow}} \mathcal{Z} \qquad$ lmk.

$$\mathcal{U} \rightleftarrows \mathcal{X} \rightleftarrows \mathcal{Z}$$