TR with coefficients

joint with Achim Krause and Thomas Nikolaus.

The construction

$$R \longmapsto W(R)$$

has a refinement to spectra: $\pi_0 TR(R) \simeq W(R)$.

More flexible: $(R, M) \mapsto W(R, M)$

Dotto - Krause - Nikolaus - Patchkoria.

Question: variant of TR with coefficients?

TR is built from THH:

$$\underbrace{THH(R)}_{\text{carries a cyclotomic structure:}} = \left| \cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} R \otimes R \otimes R \substack{\rightarrow\\ \rightarrow\\ \rightarrow} R \otimes R \rightrightarrows R \right| \in Sp$$

carries a cyclotomic structure:

(1) $\mathrm{THH}(R) \in \mathrm{Sp}^{BS^1}$.

(2) $\varphi_p : \mathrm{THH}(R) \longrightarrow \mathrm{THH}(R)^{tC_p}$ S^1-equivariant.

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_1} \arrow[r, "\mathrm{THH}"] & \mathrm{CycSp} \arrow[r, bend left, "\mathrm{TC}"] \arrow[r, bend right, "\mathrm{TR}"'] & \mathrm{Sp} \end{tikzcd}$$

$\mathrm{TC}(X) \simeq \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}, X)$ Nikolaus - Scholze.

$\mathrm{TR}(X) \simeq \mathrm{map}_{\mathrm{CycSp}}(\widetilde{\mathrm{THH}}(\mathbb{S}[t]), X)$ Blumberg Mandell.

M.

With coefficients: R $\mathbb{E}_1$-ring, M bimodule.

$$\mathrm{THH}(R, M) = \left| \cdots \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}} M \otimes R \otimes R \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} M \otimes R \rightrightarrows M \right| \in \mathrm{Sp}.$$

Warning does not admit an S^1-action.

Insight (Lindenstrauss–McCarthy)

There is a suitable struct. on $THH(R,M)$ which allows for a constr. of $TR(R,M)$ but not of $TC(R,M)$.

Definition A polygonic spectrum consists of:

(1) A collection $\{X_n\}_{n\geq 1}$, $X_n \in Sp^{BC_n}$.

(2) For every prime p and $n \in \mathbb{N}$, map

$$X_n \longrightarrow (X_{pn})^{tC_p}$$

which is C_n-equivariant.

$$Sp^{\hexagon}$$

Examples (1) $\mathbb{S} \in Sp^{\hexagon}$ w/ $\mathbb{S}_n = \mathbb{S}$.

$$\mathrm{CycSp} \xrightarrow{\ i\ } \mathrm{Sp}^{\hexagon}$$

$$X \longmapsto i(X)_n = X.$$

(2) $\quad \mathrm{THH}(R,M) \in \mathrm{Sp}^{\hexagon}$ w/

$$\mathrm{THH}(R,M)_n = \mathrm{THH}(R, \underbrace{M \otimes_R \cdots \otimes_R M}_{n}).$$

<u>Definition</u> $\quad \mathrm{TR}^{\hexagon} : \mathrm{Sp}^{\hexagon} \longrightarrow \mathrm{Sp}$ defined

$$\mathrm{TR}^{\hexagon}(X) \simeq \mathrm{map}_{\mathrm{Sp}^{\hexagon}}(i(\mathbb{S}), X).$$

$$\mathrm{TR}(R,M) = \mathrm{TR}^{\hexagon}(\mathrm{THH}(R,M)).$$

$\lrcorner$

There is an adjunction

$$\begin{tikzcd} \mathrm{CycSp} \arrow[r, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}^{\hexagon} \arrow[l, shift left, "R"] \end{tikzcd}$$

Theorem A (KMN)

If $X \in \mathrm{CycSp}$ bounded below, then

$$R\, i(X) \simeq \varprojlim \Omega\big(X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[t]/t^n)\big).$$

In particular, $\mathrm{TR}^{\mathcal{O}}(i(X)) \simeq \varprojlim \Omega \mathrm{TC}\big(X \otimes \widetilde{\mathrm{THH}}(\mathbb{S}[t]/t^n)\big)$

$$\simeq \mathrm{TR}(X).$$

Rest of the talk:

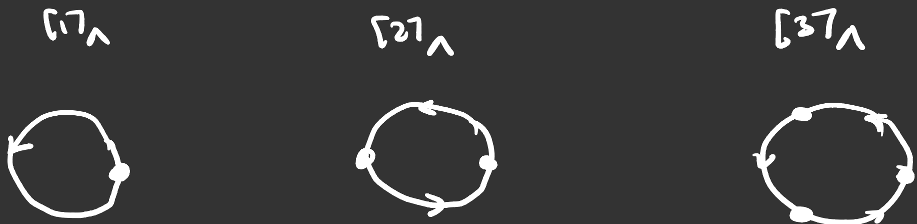
Theorem B (KMN)

$\mathrm{THH}(R;M) \in \mathrm{Sp}^{\mathcal{O}}$ w/ $\mathrm{THH}(R;M)_n = \mathrm{THH}(R, M^{\otimes_R n})$.

Idea: regard THH as a trace theory on cyclic graphs labelled by rings and bimodules.

(Kaledin, Nikolaus)

Formalized using the cyclic category Λ.

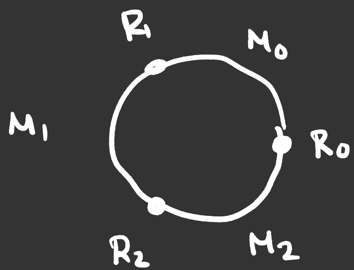


We construct a functor

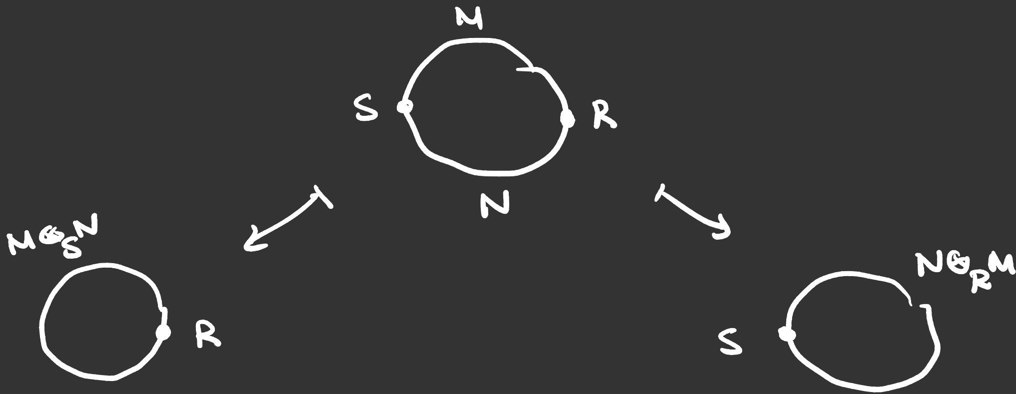
$$\Lambda^{op} \longrightarrow \mathcal{C}at_\infty$$

$$I \longmapsto \mathrm{CycBMod}_I .$$

e.g. if $I = [3]_\Lambda$, then an object of $\mathrm{CycBMod}_I$:



<u>Example</u> $[1]_\Lambda \rightrightarrows [2]_\Lambda \rightsquigarrow \mathrm{CycBMod}_2 \rightrightarrows \mathrm{CycBMod}_1$



This is packaged into a cocartesian fibration

$$\Lambda^{CycBMod} \downarrow \Lambda^{op}$$

Theorem C (KMN)

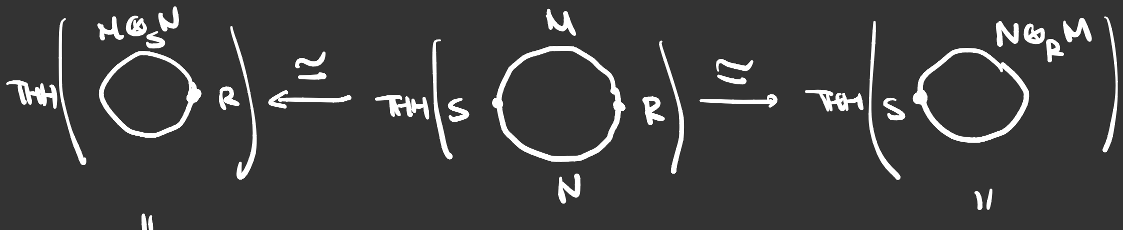
The construction $(R,M) \mapsto THH(R,M)$ refines to

$$THH : \Lambda^{CycBMod} \longrightarrow Sp$$

which sends cocartesian edges in $\Lambda^{CycBMod}$ to equivalences. (trace theory)

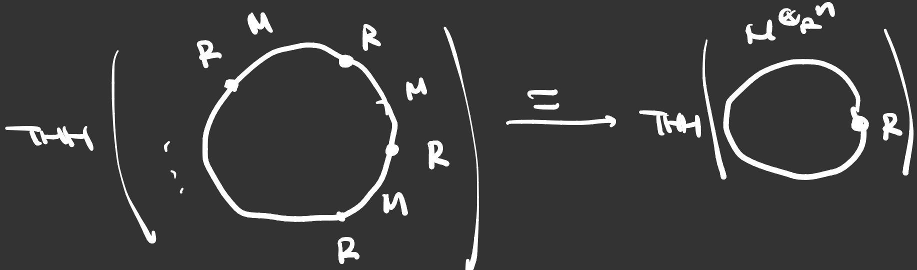
Consequences

(1) Cyclic invariance of $THH(R, M)$:



$$THH(R, M \otimes_S N) \xrightarrow{\simeq} THH(S, N \otimes_R M)$$

(2) C_n - action on $THH(R, M^{\otimes_R n})$.



$\circlearrowleft$

C_n by rotation.

$$THH(R, M^{\otimes_R n}) \longrightarrow THH(R, M^{\otimes_R pn})^{tC_p}.$$

C_n - equiv.