On $K(\mathbb{Z}/p^n)$ (jt w/ B. Antieau, T. Nikolaus)

Thm: There is an explicit chain complex for each p, n, i

$$\mathbb{Z}_p^{ni-1} \to \mathbb{Z}_p^{2(ni-1)} \to \mathbb{Z}_p^{ni-1}$$

with $H^0 \simeq 0$, $H^1 \simeq K_{2i-1}(\mathbb{Z}/p^n)^\wedge_p$, $H^2 \simeq K_{2i-2}(\mathbb{Z}/p^n)^\wedge_p$

$\boxed{\text{more generally: } \mathcal{O}_K/\varpi^n}$

Thm: For $i > \left(\frac{p}{p-1}\right)^2 \cdot p^n$

$$K_{2i-2}(\mathbb{Z}/p^n) = 0$$

$$|K_{2i-1}(\mathbb{Z}/p^n)| = p^{(n-1)\cdot i}.$$

$K(R) \to TC(R)$ is an iso on p-completed homotopy groups in nonnegative degrees.

$$TC(R) \to TC^-(R) \xrightarrow{\mathrm{can}-\varphi} TP(R)$$

$$\Big\downarrow \text{BMS filtration}$$

$$\mathbb{Z}_p(i)(R) \to N^{\geq i}\Delta_R\{i\} \xrightarrow{\mathrm{can}-\varphi} \Delta_R\{i\}$$

"Absolute-to-relative descent"

Inspired by

K.-Nikolaus: $THH(R) \to THH(R/\mathbb{S}[z]) \rightrightarrows THH(R/\mathbb{S}[z_0, z_1])$

Wang-Liu: $TP(R) \to TP(R/\mathbb{S}[z]) \rightrightarrows TP(R/\mathbb{S}[z_0,z_1])$

Relative prismatic cohomology:

$\Delta_{R/A}$ is defined for a pair:

- prism (A, I)
- A/I-algebra R

In good cases $R = A/(I, r_1, \dots, r_n)$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \Delta_{R/A} = B \arrow[d] \\ A/I \to R \arrow[r] & B/J \end{tikzcd}$$

$\leftarrow$ is actually itself a prism (B, J) such that $r_1, \dots, r_n$ land in J

$$\Delta_{R/A} = A\left\{\frac{r_1}{I}, \dots, \frac{r_n}{I}\right\}.$$

Observation: If $I, I' \subseteq \ker(A \to R)$ define prism structures, then

$$\Delta_{R/(A,I)} \cong \Delta_{R/(A,I')}.$$

Prop: Prismatic cohomology is functorial in pairs R/A where A is a delta-ring.

Thm: (e.g. $R = \mathbb{Z}/p^n$)

$$\widehat{\Delta}_R \to \widehat{\Delta}_{R/\mathbb{Z}_p[[z]]} \rightrightarrows \widehat{\Delta}_{R/\mathbb{Z}_p[[z_0, z_1]]} \cdots$$

$(\mathbb{Z}_p[[z]], (z+p))$

is a limit diagram.

(has "cohomological dimension 1")

$$\begin{tikzcd} \mathbb{Z}_p(i)(R) \arrow[d] & & \\ N^{\geq i}\widehat{\Delta}_R\{i\} \arrow[r] \arrow[d, "\mathrm{can}-\varphi"] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \widehat{\Delta}_R\{i\} \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$K_{2i-1}(\mathbb{Z}/p^n)$

$K_{2i-2}(\mathbb{Z}/p^n)$

$$\Big\downarrow \text{descale}$$

$$\begin{tikzcd} \mathbb{Z}_p^{ni-1} \arrow[r] \arrow[d] & \mathbb{Z}_p^{ni-1} \arrow[d] \\ \mathbb{Z}_p^{ni-1} \arrow[r] & \mathbb{Z}_p^{ni-1} \end{tikzcd}$$

$$K(\mathbb{Z}) \to K(\mathbb{Z}_p) \to \lim K(\mathbb{Z}/p^n)$$

computed by Tsalidis

$R:$	$\mathbb{Z}/4$	$\mathbb{Z}/8$	$\mathbb{Z}/16$	$\mathbb{Z}/32$
K_1	2^1	$2^1, 2^1$	$2^1, 2^2$	$2^1, 2^3$
K_2	2^1	2^1	2^1	2^1
K_3	2^3	$2^3, 2^2$	$2^3, 2^4$	$2^3, 2^6$
K_4	0	2^1	2^2	2^3
K_5	2^3	$2^1, 2^6$	$2^1, 2^1, 2^9$	$2^1, 2^2, 2^{12}$
K_6	0	0	2^1	2^1
K_7	$2^1, 2^3$	$2^4, 2^4$	$2^1, 2^4, 2^8$	$2^1, 2^1, 2^4, 2^{11}$
K_8	0	0	2^1	2^2
K_9	$2^1, 2^1, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^4$	$2^1, 2^1, 2^2, 2^{12}$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^{17}$
K_{10}	0	0	0	2^1
K_{11}	$2^1, 2^5$	$2^1, 2^1, 2^1, 2^2, 2^2, 2^5$	$2^3, 2^3, 2^{12}$	$2^1, 2^3, 2^5, 2^{16}$
K_{12}	0	0	0	2^1
K_{13}	$2^1, 2^2, 2^4$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^1, 2^3, 2^{15}$	$2^1, 2^1, 2^1, 2^1, 2^3, 2^{22}$
K_{14}	0	0	0	2^1
K_{15}	$2^1, 2^1, 2^1, 2^5$	$2^1, 2^1, 2^1, 2^1, 2^2, 2^2, 2^3, 2^5$	$2^1, 2^1, 2^2, 2^5, 2^{15}$	$2^1, 2^1, 2^2, 2^3, 2^5, 2^{21}$
K_{16}	0	0	0	2^1
K_{17}	$2^1, 2^1, 2^1, 2^3, 2^3$	$2^1, 2^1, 2^2, 2^2, 2^3, 2^9$	$2^1, 2^2, 2^2, 2^2, 2^3, 2^{17}$	$2^1, 2^1, 2^2, 2^2, 2^2, 2^3, 2^{26}$
K_{18}	0	0	0	0
K_{19}	$2^2, 2^3, 2^5$	$2^1, 2^3, 2^4, 2^{12}$	$2^3, 2^3, 2^4, 2^{20}$	$\cancel{2^3, 2^3, 2^3, 2^4, 2^{27}}$
K_{20}	0	0	0	0