The telescopic Galois, Picard, and Brauer groups

Joint work Tomer M. Schlank

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Arithmetic Pullback Square.

$$\begin{array}{ccc} X & \longrightarrow & \widehat{X}_p \\ \downarrow & & \downarrow \\ X[p^{-1}] & \longrightarrow & (\widehat{X}_p)[p^{-1}]. \end{array}$$

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Every $X \in \widehat{\mathrm{Sp}}_p$ has an “asymptotically defined” endomorphism

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Suffices for the definition of **localization** and **completion**:

$$X[v_1^{-1}] \in \mathrm{Sp}_{T(1)} \quad , \quad \widehat{X}_{v_1} \in \widehat{\mathrm{Sp}}_{(p,v_1)}.$$

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Chromatic Pullback square.

$$\begin{array}{ccc} X & \longrightarrow & \widehat{X}_{v_1} \\ \downarrow & & \downarrow \\ X[v_1^{-1}] & \longrightarrow & (\widehat{X}_{v_1})[v_1^{-1}] \end{array}$$

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This continues... Any $X \in \widehat{\mathrm{Sp}}_{(p,v_1,\dots,v_{n-1})}$ has an "asymptotically defined" endomorphism

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$$L_{T(n)}X := \widehat{X}_{(p,v_1,\dots,v_{n-1})}[v_n^{-1}] \quad \in \quad \mathrm{Sp}_{T(n)}.$$

Chromatic Picture

Prime ideals. A chain under specialization:

$$\overbrace{\underbrace{0}_{\mathrm{Sp}_{T(0)}}}^{\mathrm{Sp}_{\mathbb{Q}}} \to \overbrace{\underbrace{1}_{\mathrm{Sp}_{T(1)}} \to \underbrace{2}_{\mathrm{Sp}_{T(2)}} \to \dots \to \underbrace{n}_{\mathrm{Sp}_{T(n)}} \to \dots \to \underbrace{\infty}_{\text{“}\mathrm{Sp}_{T(\infty)}\text{”}}}^{\widehat{\mathrm{Sp}}_p}$$

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Residue fields. Morava K-theories:

$$\begin{array}{ccccccccccccc} K(0) & , & K(1) & , & K(2) & , & \dots & , & K(n) & , & \dots & , & K(\infty) \\ \| & & \wr\wr & & & & & & & & & & \| \\ \mathbb{Q} & & KU/p & & & & & & & & & & \mathbb{F}_p \end{array}$$

Coefficients. $\pi_* K(n) \simeq \mathbb{F}_p[v_n^{\pm}1] \quad , \quad |v_n| = 2p^n - 2.$

Telescope Conjecture

Comparison. For all $0 \leq n < \infty$,

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- $\mathrm{Sp}_{K(n)}$ – more computable, related to frormal groups and algebraic geometry.
- $\mathrm{Sp}_{T(n)}$ – less computable, related to unstable homotopy, redshift in algebraic K-theory.

Galois extensions

Definition (Rognes)

$(\mathcal{C}, \otimes, \mathbb{1})$ symmetric monoidal additive, $G \in \mathrm{Grp}$, $R \in \mathrm{CAlg}(\mathcal{C}^{\mathrm{B}G})$
R is a G-Galois extension if

1. $\mathbb{1} \xrightarrow{\sim} R^{hG}$.
2. $R \otimes R \xrightarrow{\sim} \prod_G R$ given informally as $x \otimes y \mapsto (x \cdot \sigma y)_{\sigma \in G}$.

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Example

- If R is a classical ring, Galois extensions in $\mathrm{Mod}_R(\mathrm{Ab})$ are the classical (not-connected) Galois extensions.
- $\mathrm{KO} \to \mathrm{KU}$ is a $\mathbb{Z}/2$-Galois extension.

Chromatic Galois extensions

Algebraic closure The Morava E-theory $E_n = E_n(\overline{\mathbb{F}_p}) \in \mathrm{Sp}_{K(n)}$.
Unit is $\mathbb{S}_{K(n)} = L_{K(n)}\mathbb{S}$.
Galois group The Morava stabilizer group $\mathbb{G}_n = S_n \rtimes \widehat{\mathbb{Z}}$.
$\det\colon S_n \to \mathbb{Z}_p^\times$.

Picard group

Definition

Let $\mathcal{C}$ be a monoidal category.

$$\mathrm{Pic}(\mathcal{C}) := \{X \in \mathcal{C}\ \otimes\text{-invertible}\}/\simeq$$

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Example

- $\mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Sp})$ given by $n \mapsto \mathbb{S}^n$.
- $\mathrm{Pic}(E_n) = \mathrm{Pic}(\mathrm{Mod}_{E_n}) \simeq \mathbb{Z}/2$.

Even Picard

The dimension map $\dim\colon \mathcal{C}^{\mathrm{dbl}} \to \pi_0(\mathbb{1})$ gives rise to a homomorphism $\dim\colon \mathrm{Pic}(\mathcal{C}) \to \mathbb{Z}/2$.

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Lemma (Carmeli, Schlank, Yanovski)

For odd primes,

$$\mathrm{Pic}^{\mathrm{ev}}(\mathrm{Sp}_{K(n)}) = \mathrm{Pic}_n^0 = \{X \in \mathrm{Pic}(\mathrm{Sp}_{K(n)}) \mid X \otimes E_n \in \mathrm{Pic}(E_n) \text{ is trivial}\}.$$

Kummer theory

Theorem (CSY)

Let $\mathcal{C}$ be a symmetric monoidal additive category with a primitive m-th root of unity. We have a split short exact sequence

$$0 \to \pi_0 \mathbb{1}^\times / (\pi_0 \mathbb{1}^\times)^m \to \pi_0 \operatorname{CAlg}^{\mathbb{Z}/m\text{-Gal}}(\mathcal{C}) \to \operatorname{Pic}^{\mathrm{ev}}(\mathcal{C})[m] \to 0$$

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Example (Classical Kummer theory)

For a field k with a primitive m-th root of unity and $\mathcal{C} = \operatorname{Vect}_k$, we have $\operatorname{Pic}(\mathcal{C}) = 0$, so $\mathbb{Z}/m$-Galois extensions of k are in bijection with ${}^{k^\times}/_{(k^\times)^m}$.

Chromatic cyclotomic extensions

Theorem (CSY)

*For every r $\exists \mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}] \in \mathrm{CAlg}^{(\mathbb{Z}/p^r)^\times\text{-Gal}}(\mathrm{Sp}_{K(n)})$ - **the p^r-th higher cyclotomic extensions**, and*

- *$\mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}]$ is dualizable and faithful as an object of $\mathrm{Sp}_{K(n)}$.*
- *$\mathbb{S}_{K(n)}[\omega^{(n)}_{p^r}] = E_n^{hN}$ where N is the kernel of the map $\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p^r)^\times$.*

Telescopic cyclotomic extensions

Theorem (CSY)

For every r there exists a lift $\mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}] \in \mathrm{CAlg}^{(\mathbb{Z}/p^r)^\times\text{-Gal}}(\mathrm{Sp}_{T(n)})$, *and*

- $\mathbb{S}_{K(n)}[\omega_{p^r}^{(n)}] = L_{K(n)}\mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}]$.
- $\mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}]$ *is dualizable and faithful as an object of* $\mathrm{Sp}_{T(n)}$.

Telescopic Picard

Definition

$\mathrm{Pic}_n^f := \mathrm{Pic}(\mathrm{Sp}_{T(n)})$, and $\mathrm{Pic}_n^{f,0}$ its subgroup of objects mapped to Pic_n^0 after $K(n)$-localization.

Theorem

$\mathrm{Pic}_n^0[p-1] \simeq \mathbb{Z}/_{p-1}$.

Theorem (CSY)

$\mathrm{Pic}_n^{f,0}[p-1]$ contains $\mathbb{Z}/_{p-1}$ as a direct summand.

The compact $T(n)$-local category

Notation: $|v_n| := 2p^n - 2$.

Definition

Let $X \in \mathrm{Sp}^{\omega}_{T(n)}$. A good v_n-self map is a map $w\colon \Sigma^{p^d|v_n|}X \to X$ satisfying

1. $\Sigma^{p^d|v_n|}X \otimes K(n) \xrightarrow{w\otimes \mathrm{id}} X \otimes K(n)$ is an equivalence.
2. $\Sigma^{p^d|v_n|}X \otimes K(m) \xrightarrow{w\otimes \mathrm{id}} X \otimes K(m)$ is nilpotent for all $m \neq n$.

Let $\mathcal{T}_d$ be the category of compact $T(n)$-local spectra with good v_n-self map of degree $p^d|v_n|$.

The compact $T(n)$-local category

Definition

Denote by $(-)^p\colon \mathcal{T}_d \to \mathcal{T}_{d+1}$ the power map given by $(X, w) \mapsto (X, w^p)$ where

$$w^p\colon \Sigma^{p^{d+1}|v_n|}X = \Sigma^{p\cdot p^d|v_n|}X \xrightarrow{w} \Sigma^{(p-1)\cdot p^d|v_n|}X \xrightarrow{w} \cdots \xrightarrow{w} X.$$

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Theorem (Periodicity, uniqueness)

There is an equivalence of categories

$$\operatorname*{colim}_d \mathcal{T}_d \xrightarrow{\sim} \mathrm{Sp}_{T(n)}$$

Picard as automorphism group

Proposition

$$\mathrm{Pic}(\mathrm{Sp}_{T(n)}) \simeq \mathrm{Aut}^{\mathrm{exact}}(\mathrm{Sp}_{T(n)}).$$

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Proof.

- $\mathrm{Sp}^{\mathrm{dbl}}_{T(n)} \to \mathrm{End}^{\mathrm{exact}}(\mathrm{Sp}_{T(n)})$ by $X \mapsto X \otimes -$ sends tensor prdoucts to compositions.

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- $\mathrm{Sp}^{\mathrm{dbl}}_{T(n)} \to \mathrm{End}^{\mathrm{exact}}(\mathrm{Sp}_{T(n)})$ by $X \mapsto X \otimes -$ sends tensor prdoucts to compositions.
- Restricting to invertible objects: $\mathrm{Pic}(\mathrm{Sp}_{T(n)}) \to \mathrm{Aut}^{\mathrm{exact}}(\mathrm{Sp}_{T(n)})$.

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- Write $\mathbb{S}_{T(n)} = \mathrm{colim}(X_1 \to X_2 \to \cdots)$ - a colimit of compact $T(n)$-local spectra.

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- Write $\mathbb{S}_{T(n)} = \mathrm{colim}(X_1 \to X_2 \to \cdots)$ - a colimit of compact $T(n)$-local spectra.
- Given an exact automorphism F we can evaluate it at the sphere

$$F(\mathbb{S}_{T(n)}) := \mathrm{colim}\, F(X_i).$$

Generalized suspension

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If $a' = a \mod p^{d-1}|v_n|$ then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{T}_{d-1} & \xrightarrow{\Sigma^{a'}} & \mathcal{T}_{d-1} \\
\downarrow{\scriptstyle (-)^p} & & \downarrow{\scriptstyle (-)^p} \\
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Definition

Given $a \in \mathbb{Z}_p \times \mathbb{Z}/|v_n| = \lim_d \mathbb{Z}/p^d|v_n|$ define $\Sigma^a\colon \mathrm{Sp}^{\omega}_{T(n)} \to \mathrm{Sp}^{\omega}_{T(n)}$ as the colimit of functors above.

Telescopic Picard elements

Corollary

There exists an embedding $\mathbb{Z}_p \times \mathbb{Z}/2p^n-2 \to \mathrm{Pic}^f_n$.

Proof.

$\Sigma^a \in \mathrm{Aut}^{\mathrm{exact}}(\mathrm{Sp}_{T(n)})$. □

Azumaya algebras

Definition

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a closed symmetric monoidal category.
$A \in \mathrm{Alg}(\mathcal{C})$ is called an Azumaya algebra if

1. A is dualizable and faithful as an object of $\mathcal{C}$.
2. $A \otimes A^{\mathrm{op}} \xrightarrow{\sim} \underline{\mathrm{End}}(A)$.

Proposition

If A_1, A_2 are two Azumaya algebras then so is $A_1 \otimes A_2$.

Morita equivalence

Definition (Morita equivalence)

Let A_1, A_2 be Azumaya algebras. Say that $A_1 \approx A_2$ if $\exists M_1, M_2 \in \mathcal{C}$ such that

$$A_1 \otimes \underline{\mathrm{End}}(M_1) \simeq A_2 \otimes \underline{\mathrm{End}}(M_2)$$

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as algebras.

Theorem (Johnson)

$A \approx \mathbb{1}$ if and only if it is isomorphic to a matrix algebra $\underline{\mathrm{End}}(M)$ in $\mathrm{Alg}(\mathcal{C})$ for some $M \in \mathcal{C}$.

The Brauer group

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The Brauer groups of $\mathcal{C}$ is the set of Azumaya algebras up to Morita equivalence, with product given as tensor product.

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Remark

$\mathrm{Br}(R) := \mathrm{Br}(\mathrm{Mod}_R) = \mathrm{Pic}(\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^L))$.

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Remark

There exists Picard and Brauer spectra $\mathrm{pic}(\mathcal{C}), \mathrm{br}(\mathcal{C})$ *satisfying:*

$$\pi_0(\mathrm{pic}(\mathcal{C})) \simeq \mathrm{Pic}(\mathcal{C}),\ \Omega\mathrm{pic}(\mathcal{C}) \simeq \mathbb{1}^\times$$
$$\pi_0(\mathrm{br}(\mathcal{C})) \simeq \mathrm{Br}(\mathcal{C}),\ \Omega\mathrm{br}(\mathcal{C}) \simeq \mathrm{pic}(\mathcal{C})$$

Cyclic Azumaya algebras

Definition

Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$. Define the group of units functor

$$(-)^\times \colon \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{S}) \to \mathrm{CAlg}(\mathcal{S})^{\mathrm{gp}} \simeq \mathrm{Sp}_{\geq 0}$$

as the composition of

- Right adjoint to the unit.
- Right adjoint to the inclusion.

Definition

Given $R \in \mathrm{CAlg}(\mathcal{C})$ define its group of strict units as

$$\mathbb{G}_m(R) := \hom_{\mathrm{Sp}}(\mathbb{Z}, R^\times)$$

Cyclic Azumaya algebras

Definition

Let G be a cyclic group of order m with generator σ, $A \in \mathrm{CAlg}^{[\text{-Gal}}\mathcal{C}]G$ a G-Galois extension and $u \in \mathbb{G}_m(\mathbb{1})$. Define a twisted G-action on $\mathrm{Mat}_m(A) = \underline{\mathrm{End}}(A^{\oplus m})$ by

$$(\sigma.x)_{i,j} = u^{\delta_{i,0} - \delta_{j,0}} \sigma.x_{i-1,j-1}$$

Cyclic Azumaya algebras

Definition

$A[\sigma, u] := \mathrm{Mat}_m(A)^{hG}$.

Example

For a classic ring R and a classic Galois extension A, the cyclic Azumaya algebra is given as

$$A[\sigma, u] = A\left\langle e \;\middle|\; \begin{array}{l} ea = (\sigma.a)e \;\forall a \in A \\ e^m = u \end{array} \right\rangle$$

Telescopic Brauer elements

Theorem

$A[\sigma, u]$ is an Azumaya algebra.

Theorem

Let $u \in \mathbb{F}_p^\times \subseteq \mathbb{G}_m(\mathbb{S}_{T(n)})$ be a non-square, $H \leq (\mathbb{Z}/p^r)^\times$, σ a generator of $\mathbb{Z}/p^r$ and $A := \mathbb{S}_{T(n)}[\omega_{p^r}^{(n)}]^{hH} \in \mathrm{CAlg}^{\frac{(\mathbb{Z}/p^r)^\times}{H}\text{-Gal}}(\mathrm{Sp}_{T(n)})$. Then $A[\sigma, u]$ is a non-trivial Brauer element.

Telescopic Brauer elements

Idea of proof.

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- Use the HFPSS to calculate $\pi_0 A [= \mathbb{Z}_p$ up to nilpotents.

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- $A = E_n^{hN'}$ for $S_n \leq N' \leq \mathbb{G}_n$ of finite index.
- Use the HFPSS to calculate $\pi_0 A [= \mathbb{Z}_p$ up to nilpotents.
- Deduce $\pi_0 A[u, \sigma] \approx \mathbb{Z}_p[\sqrt[m]{u}]$, but $\pi_0 \mathrm{Mat}_m(\mathbb{S}_{K(n)}) = \mathrm{Mat}_m(\mathbb{Z}_p)$.

□