An intrinsic description of animated λ-rings

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1. λ-rings

- introduced by Grothendieck in 1957 (Riemann-Roch-Thm)
- realised: $\exists$ integral poly's describing $\underline{\Lambda}^k(P \oplus P')$, $\underline{\Lambda}^k(P \otimes_R P')$ in terms of $\underline{\Lambda}^i(P)$, $\underline{\Lambda}^j(P')$'s

  (P, P' proj fg R-modules$/\simeq$)

Def. (λCR): A λ-ring is a ring $R \in CR$ tog. with maps of sets $\lambda^n : R \longrightarrow R$ ($n \geq 1$) s.th.

$$\cdot\ \lambda^n(x+y) = S_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y))$$

$$\cdot\ \lambda^n(xy) = P_n(\dots)$$

$$\cdot\ \lambda^n \lambda^m(x) = P_{n,m}(\lambda^1(x), \dots, \lambda^{nm}(x))$$

where $S_n, P_n \in \mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_n]$, $P_{n,m}$ integral poly in X_i's.

Extract Frobenius lifts: Consider

$$\begin{tikzcd} R \arrow[r, "\lambda"] \arrow[rr, bend right, "=:(\psi^n)_{n\geq 1}"'] & 1 + tR[[t]] \arrow[r, "t\frac{d}{dt}\log"] & \prod_{n \geq 1} R \\[-20pt] r \arrow[r, mapsto] & 1 + \sum_{n=1}^{\infty} \lambda^n(r) t^n & \end{tikzcd}$$

$=: (\psi^n)_{n \geq 1}$ Adams operations

Prop.: $R \in \lambda CR$. Then the ψ^n satisfy the following properties:

i) ψ^n ring hom $\forall n$

ii) $\psi^n \circ \psi^m = \psi^{nm} \quad \forall\, n,m$

iii) $\psi^p \equiv x^p \mod p \quad \forall\, p$ prime

⇝ $\mathbb{N}_{\geq 0}$ action on R det. by pairwise comm. Frobenius lifts $\forall p$

$\underbrace{\hspace{6cm}}$

'naive struct. of Frobenius lifts'

Thm (Borger?): $R \in CR$ torsion free. The λ-structures on R are equiv. to naive structures of Frobenius lifts on R.

Goal: Remove torsion-free assumption by interpreting 'Frobenius lift' in an appropriately derived sense.

Means:

In CR: p-Frob lift on R given by $R \xrightarrow{\phi} R$ s.t. $\phi \otimes \mathrm{id}_{\mathbb{F}_p} = \mathrm{Fr}_p$ on $R \otimes \mathbb{F}_p$, equiv.:

$$\begin{tikzcd} & R \arrow[dr, "\mathrm{can}"] & \\ R \arrow[ur, "\phi"] \arrow[rr, "\mathrm{Fr} \circ \mathrm{can}"'] \arrow[rr, phantom, shift left=3, "\circlearrowleft"] & & R \otimes \mathbb{F}_p \end{tikzcd}$$

derived: 2-simplex in $CR^{a} := \mathcal{P}_{\Sigma}(\mathrm{Poly})$ (non-abelian derived category of CR in sense of Quillen and Lurie)

$$\begin{tikzcd} & R \arrow[dr] & \\ R \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "\sigma"] & & R \otimes^{L} \mathbb{F}_p \end{tikzcd}$$

- CR^{a} freely gen. under sifted colims by Poly
- $CR \hookrightarrow CR^{a}$ fully faithful

Result: Structure of Frobenius lifts on an animated ring R: action by $\mathbb{N}_{>0}$ together with homotopies / 2-simp. exhibiting each φ^p as a p-Frob. lift s.t. the action restricted to $\mathbb{N}_{>0, \neq p}$ is in fact an action on $CR^a \,\mathcal{Q}\, p$-Frob. lift

Formally:

Def: Define the category $FrCR^{an}$ of animated rings with Frobenius lifts as the pb

$$\begin{tikzcd}
FrCR^{an} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p Fun(\mathbb{N}_{>0,\neq p}, CR^{a}_{\pi_p}) \arrow[d, "Fr"] \\
Fun(\mathbb{N}_{>0}, CR^{an}) \arrow[r, "\otimes \pi_p"'] & \prod_p Fun(\mathbb{N}_{>0}, CR^{a}_{\pi_p})
\end{tikzcd}$$

in Cat_∞.

Thm (Borger, BS, EH): The following diag is a pb square in Cat_∞:

$$\begin{tikzcd}
\lambda CR \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & FrCR^{an} \arrow[d] \\
Fun(\mathbb{N}_{>0}, CR) \arrow[r] & Fun(\mathbb{N}_{>0}, CR^{an})
\end{tikzcd}$$

'λ-rings are animated rings with a struct of Frob lifts whose underlying ring is discrete.'

Work in progress: $FrCR^{an} \simeq P_\Sigma(\lambda Poly) = \lambda CR^{an}$.

Idea of proof of thm:

Bhatt–Scholze (BS):

$\exists$ pb square $W_{p^2}^{a}(R) \xrightarrow{\omega_1} R$ in CR^a

$$\begin{tikzcd} {} \arrow[d, "w_0"'] & {} \arrow[d, "\mathrm{can}"] \\ R \arrow[r, "\mathrm{Fr}\circ\mathrm{can}"'] & R\hat{\otimes}\prod_p \end{tikzcd}$$

BS, Bhatt–Lurie: $W^{a}_{p,\mathbb{Z}}|_{CR}$ is $W_{p,\mathbb{Z}}$

$\leadsto$ For $R \in CR$ TFAE:

i) p-Frobenius lift on R in CR^{a}

ii) section $s : R \longrightarrow W_{p,\mathbb{Z}}(R)$ of w_0 in CR

Note: ii) $\iff$ R p-typical δ-ring, i.e. $\exists\, \delta_p : R \longrightarrow R$ satisfying xyz

$\left(x \mapsto (x, \delta_p(x)) \longleftarrow\!\!\shortmid\, \delta_p \right)$

$s \longmapsto pr_1 \circ s$

$R \in \delta_p CR$ has assoc p-Frob. lift $\phi^p(x) := x^p + p\,\delta_p(x)$.

Find: Coherences in struct of Frob lifts on animated rings reduce to commutation conditions for δ_p's and ϕ^q's, i.e.

$R \in CR$. TFAE:

i) struct of Frob lifts on R in CR^{a}

ii) $\exists\, \delta_p : R \longrightarrow R$ $\forall$ primes p s.th. $\delta_p \circ \phi^q = \phi^q \circ \delta_p$ $\forall p \neq q$

$\leadsto$ global δ-rings

Statement of the Thm: $\mathcal{X}CR \simeq$ global δ-rings

Why?

· $\lambda(R) = \mathrm{Coalg}(W^{\wedge})$ (by def.)

$$\begin{tikzcd}
 & W^{\wedge}(R) \arrow[d, "\partial"', "\cong"] & \text{Witt vectors, power series form} \arrow[l] \\
R \arrow[ur, "\lambda"] \arrow[r, dashed, "\delta"] & W(R) \underset{\text{set}}{=} \prod_{n\geq 1} R & \text{based on prime factor decomposition} \arrow[l]
\end{tikzcd}$$

Have δ_p $\forall p$, coherence relations precisely those cond. needed to define remaining δ_n's from δ_p's.