

# An intrinsic description of anomalies

## $\lambda$ -rings

with Thomas Nikolaus

### 1. $\lambda$ -rings

- introduced by Grothendieck in 1957 (Riemann-Roch-Theorem)
- realised:  $\exists$  integral poly's describing  $\Lambda^k(P \oplus P')$ ,  $\Lambda^k(P \otimes_R P')$  in terms of  $\Lambda^i(P), \Lambda^j(P')$ 's  
 $(P, P' \text{ Proj f.g. } R\text{-modules} / \sim)$

Def. ( $\lambda CR$ ): A  $\lambda$ -ring is a ring  $R \in CR$  tog. with maps of sets  $\pi^n: R \rightarrow R$  ( $n \geq 1$ ) s.t.

- $\pi^n(x+y) = S_n(\pi^1(x), \dots, \pi^n(x), \pi^1(y), \dots, \pi^n(y))$
- $\pi^n(xy) = P_n(\dots)$
- $\pi^n \pi^m(x) = P_{n,m}(\pi^1(x), \dots, \pi^{nm}(x))$

where  $S_n, P_n \in \mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_n]$ ,  $P_{n,m}$  integral poly in  $X_i$ 's.

Extract Frobenius lifts: Consider

$$\begin{array}{ccc}
 R & \xrightarrow{\lambda} & 1 + tR[\![t]\!]
 \\ 
 \cdot r \mapsto & 1 + \sum_{n=1}^{\infty} \pi^n(r)t^n & \xrightarrow{\text{lifts}}
 \end{array}
 \quad \begin{matrix} \text{lifts} \\ \text{of } \pi \end{matrix} \quad \begin{matrix} \pi \\ \text{on } R \end{matrix}$$

$=: (\psi^n)_{n \geq 1}$  Adams' operations

Prop.:  $R \in CR$ . Then the  $\psi^n$  satisfy the following properties:

i)  $\psi^n$  ring hom  $\forall n$

$$ii) \quad \varphi^n \circ \varphi^m = \varphi^{nm} \quad \forall n, m$$

$\rightsquigarrow \text{IN}_{\geq 0}$  action on  $R$   
det. by pairwise comm.  
Frobenius lifts  $V_p$

ii)  $y^p \equiv x^p \pmod{p} \quad \forall p \neq \text{prime}$

' naive struct. of Frobenius lifts '

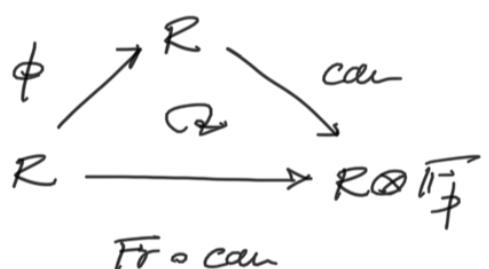
Thm (Boyer?):  $R \in CR$  torsion free. The  $\lambda$ -structures  
on  $R$  are equiv. to naive structures of  
Frobenius lifts on  $R$ .

Goal : Remove torsion-free assumption by interpreting  
'Frobenius lift' in a appropriately derived sense.

## Means :

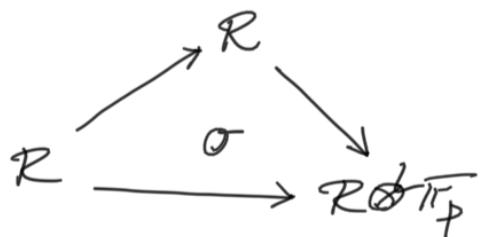
In CR:  $\phi$ -Frob lift on  $R$  given by  $R \xrightarrow{\phi} R$  s.t.

$$\phi \otimes \text{id}_{\overline{R}} = F_{\overline{R}} \circ R \otimes \overline{F_R}$$
, equiv:



derived: 2-simplex in  $CR^a := P_\Sigma(\text{Poly})$

(non-abelian  
derived category of  
 $\mathcal{CR}$  in sense of  
Quillen and Lurie)



- $\mathcal{CR}^a$  freely gen. under sifted colims by Poly
  - $\mathcal{CR} \hookrightarrow \mathcal{CR}^a$  fully faithful

Result : Structure of Frobenius lifts on a  $\mathbb{Z}$ -ring  $R$ : action by  $\text{IN}_{\geq 0}$  together with homotopies/  
 $\mathbb{Z}$ -simp. exhibiting each  $\phi^p$  as a  $p$ -Frob.  
 Left- s.t. the action restricted to  $\text{IN}_{\geq 0, \text{fp}}$   
 is in fact an action on  $\text{CR}^{\text{an}}$   $\mathbb{Z}$ -Frob. lift

Formally :

Def : Define the category  $\text{FCR}^{\text{an}}$  of animated rings with  
 Frobenius lifts as the  $\mathbb{Z}$

$$\begin{array}{ccc} \text{FCR}^{\text{an}} & \longrightarrow & \overline{\prod} \text{Fun}(\text{IN}_{\geq 0, \text{fp}}, \text{CR}^{\text{an}}_{\mathbb{Z}_p}) \\ \downarrow & \rightarrow & \downarrow \text{FF} \\ \text{Fun}(\text{IN}_{\geq 0}, \text{CR}^{\text{an}}) & \xrightarrow{\otimes \mathbb{Z}_p} & \overline{\prod} \text{Fun}(\text{IN}_{\geq 0}, \text{CR}^{\text{an}}_{\mathbb{Z}_p}) \end{array}$$

in  $\text{Cat}_{\infty}$ .

Then (Borger, BS, EH) : The following diag is a  $\mathbb{Z}$ -square

in  $\text{Cat}_{\infty}$ :

$$\begin{array}{ccc} \text{NCR} & \longrightarrow & \text{FCR}^{\text{an}} \\ \downarrow & \rightarrow & \downarrow \\ \text{Fun}(\text{IN}_{\geq 0}, \text{CR}) & \longrightarrow & \text{Fun}(\text{IN}_{\geq 0}, \text{CR}^{\text{an}}) \end{array}$$

'  $\mathbb{Z}$ -rings are animated rings with a struc- of  
 Frob lifts whose underlying ring is discrete.'

Work in progress :  $\text{FCR}^{\text{an}} \cong P_{\Sigma}(\mathbb{Z}\text{Poly}) = \text{NCR}^{\text{an}}$ .

Idea of proof of this :

Bhatt-Scholze (BS) :

$$\exists \mathbb{Z}$$
-square  $(\mathbb{Z}_{\geq 0}^{\text{an}}(R)) \xrightarrow{\omega_1} R \quad \perp \text{-} \text{CR}^{\text{an}}$

$$\begin{array}{ccc}
 & \pi & \\
 & \downarrow w_0 & \downarrow \text{can} \\
 R & \xrightarrow{\quad} & R \otimes_{\mathbb{Z}_p}^{\text{can}} \\
 & & \text{Fr} = \text{ca}
 \end{array}$$

BS, Bhattacharya:  $\omega_{\mathbb{Z}/p^2}^a|_{\text{ca}}$  is  $\omega_{\mathbb{Z}/p^2}$

$\sim$  For  $R \in CR$  TFAE:

- i)  $p$ -Frobenius lift on  $R$  in  $CR^a$
- ii) section  $s: R \rightarrow \omega_{\mathbb{Z}/p^2}(R)$  of  $w_0$  in  $CR$

Note: ii)  $\iff R$   $p$ -typical  $\delta$ -ring, i.e.  $\exists \delta_p: R \rightarrow R$   
 satisfying  $xyz$   
 $(x \mapsto (x, \delta_p(x)) \leftrightarrow \delta_p)$   
 $s \mapsto \text{pr}_1 \circ s$

$R \in CR$  has assoc  $p$ -Frob. Lift  $\phi^p(x) := x^p + p\delta_p(x)$ .

Find: Coherences in struc of Frob lifts on anisotropic rings reduce to commutative conditions for  
 $\delta_p'$ 's and  $\phi^q \circ \delta_p$ , i.e.

$R \in CR$ . TFAE:

- i) struc of Frob lifts on  $R$  in  $CR^a$
- ii)  $\exists \delta_p: R \rightarrow R$   $\forall$  primes  $p$  s.t.  $\delta_p \circ \phi^q = \phi^q \circ \delta_p$   
 $\forall p \neq q$   
 $\rightsquigarrow$   
 global  $\delta$ -rings

Statement of the Thm:  $CR \cong$  global  $\delta$ -rings

why?

$$\cdot \pi CR = \text{Coalg}(\mathcal{O}^{\mathfrak{d}}) \quad (\text{by def.})$$

$$\begin{array}{ccc}
 & \omega^{\wedge}(R) & \\
 R \xrightarrow{\pi} & \downarrow \cong \partial & \text{Witt vectors, power series form} \\
 & \omega(R) = \varprojlim_{n \geq 1} \overline{\pi^n R} & \text{based on prime} \\
 & & \text{factor decomposition}
 \end{array}$$

Have  $\delta_p \nmid V_f$ , coherence relations precisely those cond.  
 needed to define remaining  $\delta_i$ 's from  
 $\delta_p'$ 's.

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