Seminar “Sheaves in Geometry and Logic”

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In this seminar we want to learn the concept of an elementary topos and how it unifies sheaf theory, which belongs to topology and algebraic geometry, and certain aspects of set theory and logic. We will learn techniques out of both areas that play an important role in modern mathematics (topology, algebraic geometry, logic).

An (elementary) topos is a category with certain nice properties. More precisely, it is a category that has finite limits, exponentials, and a subobject classifier. These properties mean that this category looks very similar to the category of sets. Examples of topoi are given by categories of sheaves on topological spaces or sheaves on a Grothendieck site (we will learn what all of that means). Such categories initially arose in studying cohomology theories in algebraic geometry, like étale cohomology, and are now widely used to understand cohomology in geometric contexts.

We will adopt the perspective (initially due to Lawvere) to consider a topos as a generalized ‘universe of sets’ and use it to study set theory. This is from the geometric perspective a very suprising idea and we will carefully explain the interplay. In particular we will see that every topos has an ‘internal logic’ and we will study this logic. For example there are topoi in which the axiom of choice does not hold, i.e. not every epimorphism has a section. Also, there is the topos of finite sets which encodes finitary logic.

Our main application will be a topos theoretic treatment of Cohen’s proof [Coh63, Coh64] that the continuum hypothesis is independent of the usual axioms of set theory (Zermelo-Fraenkel). This method, which is known as ‘forcing’, can very naturally be interpreted in the language of topoi, as observed by Lawvere and Tierney. The idea is to construct a new topos from a given one such that in this new topos there is an object lying strictly in between $\mathbb{N}$ and the power set $P(\mathbb{N})$, thus violating the continuum hypothesis. For the construction of this new topos we have to introduce some tools to manipulate topoi. More precisely the new topos will be given by the topos of double negation sheaves on a poset of forcing conditions with values in the old topos.

In the seminar we closely follow the book [MM96]. The seminar is aimed at master students, but apart from some basic concepts in category theory and the notion of a topological space we will not assume anything, so that Bachelor students should also be able to follow. The modus of the seminar will be a bit experimental as some of the talks will cover a lot of material and are more designed as ‘overview’ talks. This means that the main goal is to give a summary of the important aspects of a chapter of the book rather than detailed proofs of everything, so you should be prepared to deviate from the usual way you learned how to give seminar talks.

In case of further questions please do not hesitate to write an e-mail to nikolaus@uni-muenster.de. There will be a preparatory meeting to distribute talks next monday (July 8th) at 16:00 in Room SR0. If you want to participate in the seminar but don't have time to show up for the meeting (or if you missed it), you can also write an e-mail. The seminar will take place Wednesdays 14:00 – 16:00, but the time can also be changed if necessary.

List of talks

Talk 1 (Stern)**.** This talk covers Sections I.1-6. These introduce important categorical concepts, not all of which we have seen before. You should discuss these in relation to (some, not all of) the examples in Section I.1, with focus on the presheaf category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Specifically, introduce subobjects and subobject classifiers as in Sections I.3-4 (including the description in the presheaf category by sieves), discuss the Yoneda Lemma and the perspective on the presheaf category from Section I.5, and explain the notion of exponentials as in Section I.6.

Talk 2 (Westerhoff)**.** Recall Boolean algebras as in Section I.7. Then define Heyting algebras and discuss their properties following Section I.8, with an emphasis on the categorical perspective taken there. Also cover how to treat quantifiers in this picture (Section I.9).

Talk 3 (Janssen)**.** Section II is rather long, but sheaves are going to be discussed in a more general formal setting in later talks as well. So the purpose of this talk is to familiarize the audience with the notion of sheaves by discussing sheaves on topological spaces, with the goal of establishing the results of II.8. Motivate and introduce the definition of sheaves and discuss Proposition II.1, which characterizes sheaves equivalently through sieves. Feel free to mention some of the examples from II.3, but do not go into detail. Explain the perspective of bundles (II.4-6). Section II.8 is the key part of the talk, try to explain these constructions in more detail. End by discussing the functoriality of sheaves as in Section II.9.

Talk 4 (Walendzik)**.** Introduce the definition of a Grothendieck topology, and of a basis for it, as in Section III.2, and explain the relation between the two notions. Discuss examples (a)-(f) at the end of the section; in particular, discuss the dense topology and the atomic topology in detail. Define sheaves on a site and prove Proposition III.4.1. Define canonical and subcanonical topologies. Characterize sheaves for the atomic topology. Give the definition of a Grothendieck topos. Give the construction of the associated sheaf functor, and prove Theorem III.5.1.

Talk 5 (Kesting)**.** Prove that every Grothendieck topos is an elementary topos, following Section III.6, to show that it has limits, colimits, and exponentials (leaving out some details if necessary), and Section III.7 to show that it has a subobject classifier. Discuss constant sheaves. Prove Proposition III.8.1, giving an explicit description of the implication operator. Prove Proposition III.8.2, discussing carefully the adjoints of the pullback functor. Conclude saying a few words on the examples at the end of Section III.8.

Talk 6 (Felix)**.** Give the definition of an (elementary) topos and explain that it is possible to define topoi without any reference to the category of sets whatsoever (as in section IV.1). Explain that the categories of sheaves on a topological space or sheaves on a site give examples of topoi. Discuss some categorical similarities of topoi and the category of sets. Examples of these similarities are given by Proposition IV.1.2 and Theorem V.2.1, which states the existence of exponentials in topoi. Note that in the latter the exponential B^A in a topos is constructed as a certain subobject of the power object of the direct product $A \times B$; as is the case in the category of sets. Finally cover the content of section IV.6. Important results here are Proposition 1 (existence of mono-epi-factorizations in topoi), Proposition 3 (Sub(A) forms a lattice and pulling back subobjects has a left adjoint) and Proposition 5. In section IV.6 proofs could be left out if there's not enough time.

Talk 7 (Bröring)**.** Explain that the property of being a topos is preserved under taking the slice over an object (this is Theorem IV.7.1). Furthermore dicuss Theorem IV.7.2 which states that base-change functors between slice topoi are logical morphisms and admit left and right adjoints. Continue by defining the notions of internal Lattice- and internal Heyting algebra objects in a category (section IV.8). Explain that for an object A in a topos $\mathrm{Sub}(A)$ (resp. $\mathrm{P}(A)$) carries the structure of a Heyting algebra (resp. of an internal Heyting algebra object) and that pulling back subobjects defines a homomorphism of Heyting algebras (resp. of internal Heyting algebra objects).

Talk 8 (Huebner)**.** Recall from Proposition IV.6.3 that pulling back subobjects $\mathrm{Sub}(B) \to \mathrm{Sub}(A)$ along a morphism $A \to B$ admits a left adjoint and explain that it also admits a right adjoint (Proposition IV.9.3). Follow the proof of Theorem IV.9.2 to show that this implies the existence of internal left and right adjoints of the pullback morphism $\mathrm{P}(B) \to \mathrm{P}(A)$ between power objects (also see Proposition IV.9.4). Then explain and prove the Beck-Chevalley conditions for all of these (internal) adjoints (Proposition IV.9.1 and Theorem IV.9.2; also see before Proposition IV.9.4). Proceed by proving that the subobject classifyer and power objects in a topos are injective objects (Proposition IV.10.1 and Corollary IV.10.2).

Talk 9 (Riedel)**.** Motivate and introduce the concept of a Lawvere–Tierney topology on a topos as in Section V.1. Especially, discuss carefully the example of presheaves on open sets in a topological space X. Explain the relation to 'closure operators' and how a Grothendieck topology on a category $\mathcal{C}$ gives rise to such an operator on presheaves. Introduce the notion of a sheaf for a Lawvere-Tierney topology (as in Section V.2) and spell it out explicitly in the case of a topological space. Follow Section V.2 and explain (roughly) how to see that the category of sheaves for a given Lawvere–Tierney topology is itself a topos.

Talk 10 (Bals)**.** Give the topos theoretic construction of the 'sheafification' functor as in Section V.3. Try to relate this sheafification functor to the constructions we had earlier in the seminar (using the ++-construction). Also cover the content of Section V.4 proving that for presheaf-categories Lawvere–Tierney topologies are in 1-1 correspondence with Grothendieck topologies and that the notions of sheaves agree. If there is time then discuss several instance of sheaves in this language, for example for the dense and the atomic topology on a presheaf category to illustrate the different viewpoints.

Talk 11 (Tönies)**.** Explain the interplay between internal and external working in a topos as explain in Section V.5. Only briefly state the result of Section V.6 (Theorem 1) and Section V.7 (possibly without giving any details of proof of time does not allow it). Proceed with Section V.9 to explain the 'filter quotient' construction to improve properties of a given topos or more precisely its Heyting Algebra.

Talk 12 (Fiedler)**.** Study properties of the topos of sets: it has a natural numbers object, it is well-pointed and Boolean (following VI.1). Explain how considering double negation sheaves (i.e. dense sheaves) makes every topos Boolean (Theorem 3) and how to make it two-valued by a filter quotient construction (Proposition 6). If there is time left then indicate how to use these methods to build a topos in which the Axiom of Choice fails following Section VI.4.

Talk 13 (Krisam)**.** Start with an overview of what Cohen proves (and we will do) and how this is roughly done (following the Introduction to Chapter VI and Section VI.2). In particular introduce the Cohen topos of double negation sheaves on the poset of forcing conditions. Then establish some of its features following section VI.2 and indicate how to prove that the continuum hypothesis fails in it (Section VI.3). For the last part it will be important not to get lost in technical details but to keep an overview what the goal is that we want to achieve.

References

[Coh63] Paul Cohen, *The independence of the continuum hypothesis*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1143–1148. MR 0157890

[Coh64] Paul J. Cohen, *The independence of the continuum hypothesis. II*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 105–110. MR 0159745

[MM96] S. MacLane and I. Moerdijk, *Topos theory*, Handbook of algebra, Vol. 1, Handb. Algebr., vol. 1, Elsevier/North-Holland, Amsterdam, 1996, pp. 501–528. MR 1421810