Spectral Sequences

Fabian Hebestreit, Achim Krause, Thomas Nikolaus

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Abstract

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1 Preface

One of the big goals of homotopy theory is to compute homotopy classes of maps

$$[X, Y]_*$$

where X and Y are pointed CW complexes. Since CW complex are built out of spheres, its essential to first try to attack to easier problem of computing

$$\pi_n(S^k) = [S^n, S^k]$$

the homotopy classes of maps between spheres for positive integers k and n. These turn out to be groups and are called stable homotopy groups of spheres. Let review what we know about these groups so far:

- $\pi_n S^k = 0$ for $k < n$. This is a consequence of the cellular approximation theorem, since the k-sphere admits a model as a cell complex without cells of dimension $< k$.
- $\pi_n S^n \cong \mathbb{Z}$. This is a consequence of the Hurewicz theorem and the fact that $\mathrm{H}_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$.
- $\pi_k S^1 = 0$ for $k \geq 2$. This can be deduced using covering theory.
- $\pi_3 S^2 \neq 0$ since the attaching map of $\mathbb{C}P^2$ is a map $S^3 \to \mathbb{C}P^1 = S^2$ which is not nullhomotopic. Otherwise $\mathbb{C}P^2$ would be homotopy equivalent to $S^2 \vee S^4$ which is a contradiction to the ring structure on $\mathrm{H}^*(\mathbb{C}P^2; \mathbb{Z}) \simeq \mathbb{Z}[x]/x^3$. This is the Hopf-invariant argument.
- The sequence of homotopy groups

  $$\pi_k S^n \to \pi_{k+1} S^{n+1} \to \pi_{k+2} S^{n+2} \to ...$$

  eventually stabilizes by the Freudenthal suspension theorem. This colimit is called the $(k-n)$-th stable homotopy group of spheres. These are somewhat simpler than the unstable ones, but still very hard to understand.

One of the first goals of this lecture will be to prove the following theorem.

Theorem 1.1 (Serre). *The groups $\pi_n S^k$ are finite for $n \notin \{k, 2k-1\}$. We have $\pi_k S^k \cong \mathbb{Z}$ (as stated above) and $\pi_{2k-1} S^k \cong \mathbb{Z} \oplus M$ where M is a finite abelian group.*

The tool that Serre invented (and that we will use) to prove this theorem and related facts about the stable homotopy groups of spheres is the Serre-spectral sequence. Let us desribe more precisely what the Serre spectral sequence does. It will initially be unclear how this is related to stable homotopy groups of spheres, but we will come back to this point eventually.

Essentially the Serre spectral sequence is a tool to compute (co)homology of certain spaces, such as ΩS^3, $U(n)$ or $K(\mathbb{Z}, 3)$. We will explain what these spaces are soon, if you don't know that already. But note that while one can compute cohomology groups (or even rings) for a variety of spaces with rather elementary methods, but its rather tricky to approach a space like ΩS^3. One could try to analyse the cell structure of ΩS^3, but this is not that easy. The reason is that mapping spaces tend to have a lot of cells (they are typically infinity dimensional) even if the original space was very simple. In fact the only connected, finite CW complex X for which ΩX also has the homotopy type of a finite CW complex is $X \simeq \mathrm{pt}$. That is one of the facts that we will prove in the lecture. The key fact about ΩS^3 that we will exploit is the sequence

$$\Omega_x S^3 \to P_x S^3 \to S^3$$

where $P_x S^3$ is the space of path ending at some fixed point $x \in S^3$. The space $P_x S^3$ is contractible (by the obvious contraction). This sequence is a fibre sequence, a notion that we will review now:

Definition 1.2. Let $f : Y \to X$ be a (continuous) map of topological spaces and $x \in X$ a point. *The* homotopy fibre $\mathrm{hofib}_y(f)$ of f at x is defined to be the space

$$\mathrm{hofib}_x(f) := P_x X \times_X Y$$

where $P_x X = \{\gamma : [0, 1] \to Y \mid \gamma(1) = x\}$ is the based path space of X which comes with a map $P_x X \to X$ given by evaluation at 1.

In other words the homotopy fibre of f is the space of pairs (γ, y) where $y \in Y$ and $\gamma : [0, 1] \to X$ is a path in X with $\gamma(1) = x$ and $\gamma(0) = f(y)$.

Definition 1.3. A fibre sequence of topological spaces is a sequence $F \xrightarrow{i} Y \to X$ of topological spaces together with a homotopy $h : F \to X^{[0,1]}$ of the composition $F \to Y \to X$ to the constant map $c_x : F \to X$ for some $x \in X$ (which is part of the datum) such that the induced map

$$F \to \mathrm{hofib}_x(f) \qquad z \mapsto (hz, iz)$$

is a weak homotopy equivalence.

Examples 1.4. *1. For every pointed topological space (X, x) there is a fibre sequence $\Omega X \to \mathrm{pt} \xrightarrow{x} X$ where the homotopy $h : \Omega X \times [0, 1] \to X$ from the constant map c_x to itself is given by the evaluation map. In this case the map $\Omega X \to \mathrm{hofib}_x(f)$ is the identity.*

Warning: there is a different homotopy of the composition $\Omega X \to \mathrm{pt} \xrightarrow{x} X$ to the constant map, namely the constant homotopy. With this homotopy this is not a fibre sequence! The reason is that in this case the induced map $\Omega X \to \mathrm{hofib}_x(f) = \Omega X$ is given by the constant map at x. This map is not a weak homotopy equivalence, unless X is (weakly) contractible. This shows nicely that the homotopy, while usually not denoted, is an important part of a fibre sequence.

2. *For every pair of spaces F and X, and every basepoint $x \in X$ there is a fibre sequence $F \to F \times X \xrightarrow{f} X$. We refer to this as the trivial fibre sequence.*

   To see this we note that $\mathrm{hofib}_x(f) = F \times P_x X$ *so that the obvious inclusion map is a homotopy equivalence.*

3. *Let $p : E \to B$ be a fibre bundle with fibre $F = p^{-1}(b)$ for some $b \in B$. Then $F \to E \to B$ with the constant homotopy is a fibre sequence. This is a special case of the next example.*

4. *Let $p : E \to B$ be a Serre fibration with fibre $F = p^{-1}(b)$ for some $b \in B$. Then $F \to E \to B$ with the constant homotopy is a fibre sequence. This will be recalled in the exercises.*

5. *As a special case of the first example, the Hopf fibration gives us a fibre sequence $S^1 \to S^3 \to S^2$. To realize this we identify S^2 as $\mathbb{C}P^1$. Then the there is a tautological bundle $E \to \mathbb{C}P^1$ where $E \subseteq \mathbb{C}P^1 \times \mathbb{C}^2$ is the subset consisisting of pairs (L, z) where $z \in L$ and z has length 1 with respect to the standard metric on $\mathbb{C}^2$. Then its straithforward to check that L is a fibre bundle over $\mathbb{C}P^1$ whose fibre is S^1. Its a bit more tricky (but also true) that E is homeomorphic to S^3.*

6. *Similarily there are fibre sequences*

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

   and as the limiting case

$$S^1 \to \mathrm{pt} \to \mathbb{C}P^\infty.$$

   (where one really gets $S^\infty \simeq \mathrm{pt}$ in the middle).

In fact, one should always think of a fibre sequence as a twisted version of the trivial fibre sequence. This is true in a sense that we do not want to make precise here. Lets assume that we are given a fibre sequence $F \to Y \to X$. Then the following is the slogan for the Serre spectral sequence:

> **The Serre spectral sequence is a way to compute the cohomology of Y in terms of the cohomology of X and F.**

We already know how to do this in case of the trivial fibre sequence $F \to F \times X \to X$ by means of he Künneth theorem, namely that there is a split short exact sequence

$$0 \to H^*(F) \otimes H^*(X) \xrightarrow{\times} H^*(F \times X) \to \bigoplus_{p+q=n+1} \mathrm{Tor}(H^q F, H^p X) \to 0$$

provided that one of F and X is of finite type. Thus in some sense it can be considered as a generalization of the Künneth theorem for non trivial fibre sequences and it measures how far from the trivial fibre sequence a given fibre sequence is.

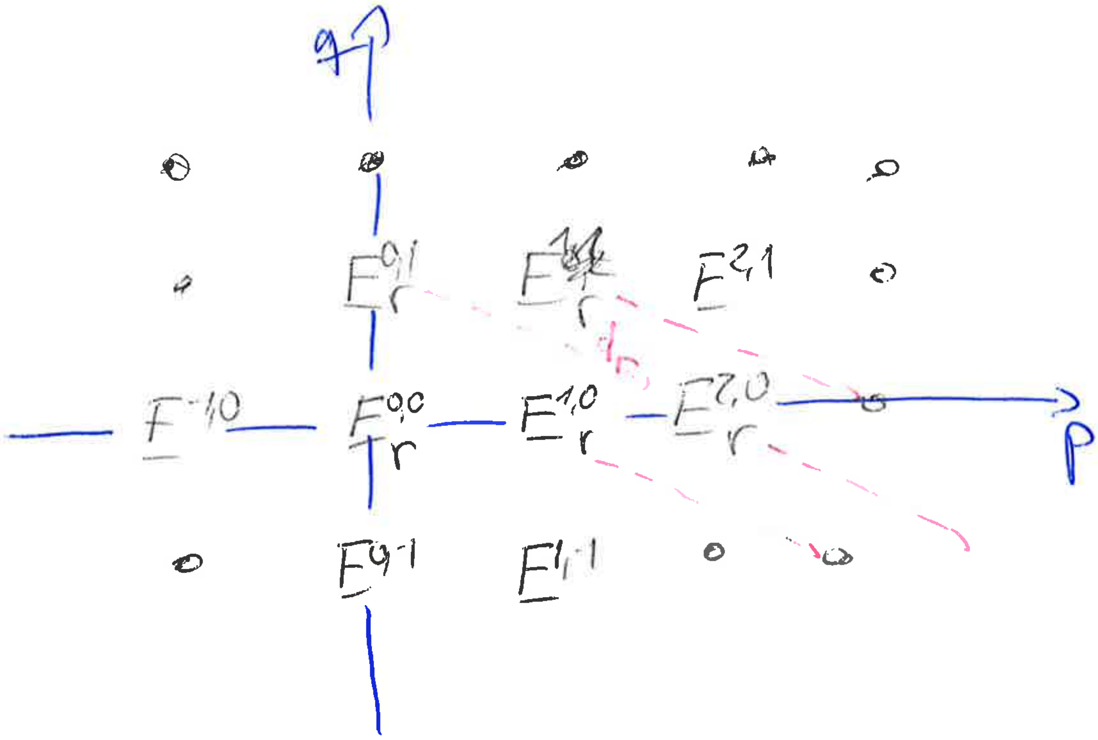
2 The Serre spectral sequence

We will start this lecture by stating the existence of the Serre spectral sequence (without proving it). Then we explain how it is applied to give the reader a feel for it. Eventually we will explain the construction and the more abstract setup of spectral sequences. Spectral sequence are a tool to organize complex computations. They are arguably the most important such tool in modern algebraic topology. The definition may at first seem very complicated and abstract, but the reader will hopefully get used to it eventually.

Definition 2.1. A (cohomologically, Serre-graded) spectral sequence is a triple $(E_\bullet, d_\bullet, h_\bullet)$, where

- $(E_r)_{r\geq 2}$ is a sequence of $\mathbb{Z}$-bigraded abelian groups. We write $E_r^{p,q}$ where the upper index is the bigrading $p, q \in \mathbb{Z}$. The bigraded group E_r is called the r-th page of the spectral sequence;
- $d_r : E_r \to E_r$ is a sequence of morphisms with d_r of bidegree $(r, 1-r)$ and $d_r \circ d_r = 0$. In other words we have morphisms $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that the composition $E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1} \xrightarrow{d_r} E_r^{p+2r,q-2r+2}$ is the zero morphism;
- $h_r : H_*(E_r) \to E_{r+1}$ is a sequence of bigrading preserving isomorphisms. Here the left hand side is the cohomology which inherits a bigrading from E_r.

We will picture every page of a spectral sequence in the coordinate plane, where p is the x-axis and q is the y-axis:



One gets from one page to the next by taking cohomology, only those elements survive that are cycles and we quotient by the differential. Its like turning pages of a book. Very often this process stabilizes as r tends to ∞, at least for every fixed entry (p, q).

Definition 2.2. We say that a spectral sequence is first quadrant if all abelian groups $E_2^{p,q}$ are trivial whenever either $p < 0$ or $q < 0$.

Lemma 2.3. *For a first quadrant spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ all groups $E_r^{p,q}$ for $r \geq 2$ and $p < 0$ or $q < 0$ are zero. In other words: all pages lie in the first quadrant. Moreover for fixed values of p, q the map h induces an isomorphism $E_r^{p,q} \to E_{r+1}^{p,q}$ for all $r > r_0 = \max(p, q+1)$.*

Proof. The first part immediately follows inductively since the (r+1)-th page is the cohomology of the r-th page. For the second part we note that if $r > r_0$ then there can not be any differential from $E_r^{p,q}$ to a non-trivial groups, since the r-th differential goes down by $r - 1 > q$ and thus leaves the first quadrant. Thus everything is a cycle. Similarly we see that there can not be a non-trivial differential with target $E_r^{p,q}$ since it has to start outside the first quadrant. Therefore the cohomology of d_r in degree (p, q) is equal to $E_r^{p,q}$. □

This means that the the abelian groups $E_r^{p,q}$ eventually stabilize for large values of r, more precisely the first stable group is $E_{r_0+1}^{p,q}$. These stable groups consitute a further page of the spectral sequence, the E_∞-page.

Definition 2.4. For a first quadrant spectral sequence we define the E_∞-page as the bigraded abelian group

$$E_\infty^{p,q} := \varinjlim E_r^{p,q}$$

where the colimit is taken for $r > r_0$ and tends to ∞.

The colimit is really harmless here, as it is eventually constant and thus isomorphic to the first stable group, its only there to make the Definition well-defined.

By a (bounded above, exhaustive) filtered object of any abelian category $\mathcal{A}$ we will mean an object H in $\mathcal{A}$ together with a sequence of inclusions (monomorphisms)

$$... \subseteq F_3 \subseteq F_2 \subseteq F_1 \subseteq F_0 = H.$$

We will use this in the case that $\mathcal{A}$ is the category of graded abelian groups and $H = \mathrm{H}^*(Y)$.

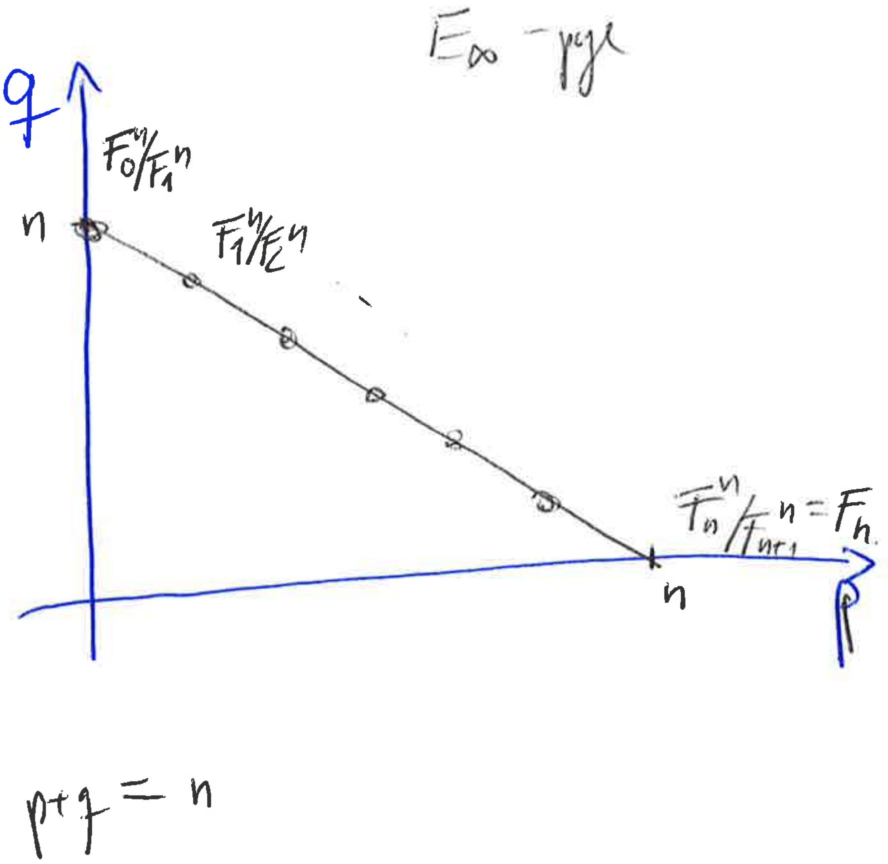
Definition 2.5. A first quadrant spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ is said to *converge* to a filtered object in graded abelian groups (H, F) if there is a chosen isomorphism

$$E_\infty^{p,q} \simeq F_p^{p+q} / F_{p+1}^{p+q}$$

for all p and q and if for all $p > n$ we have $F_p^n = 0$. In this case we write

$$E_2^{p,q} \Rightarrow H.$$

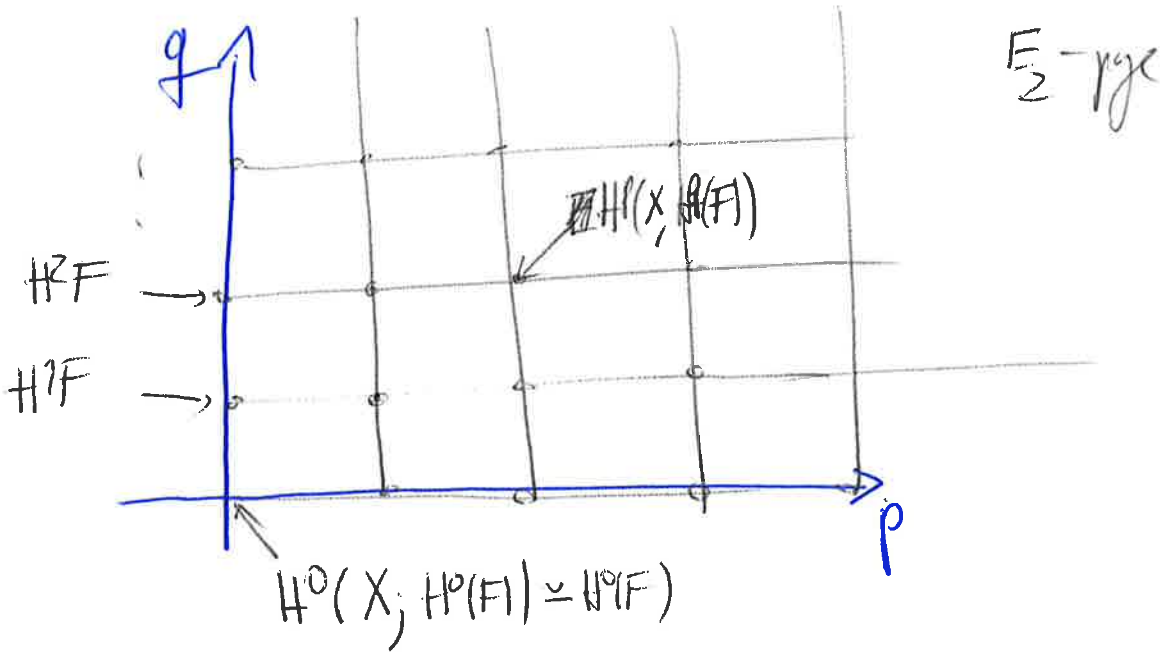
Note that as indicated in the definition, saying that a spectral sequence converges is really a datum, namely the choice of an isomorphism. Even worse, writing $E_2^{p,q} \Rightarrow H$ suppresses much more, namley all pages except for E_2, all the differentials and the filtration on H. So this should be read as saying that there are choices for all these data, but everything is suppressed in the notation. But it is common to suppress these choices and leave them implict. We will follow this convention to avoid awkward language and terminology but hope that it does not lead to confusion.



Theorem 2.6 (Serre)**.** *For every fibre sequence $F \to Y \to X$ of spaces with X simply connected and every abelian group M there is a first quadrant spectral sequence*

$$E_2^{p,q} = H^p(X, H^q(F; M)) \Rightarrow H^{p+q}(Y; M) \ .$$

We refer to this spectral sequence as the Serre-spectral sequence. We advise the reader to not try to understand at this point how this spectral sequence is built, how the differentals are defined or even try to understand how the filtration on $H^{p+q}(Y)$ looks like. All this will be explain and discussed eventually. At this point we want to see what we can learn from the mere existence of the Serre spectral sequence. But let us again say that this innocent looking theorem really claims that there is a lot of data: a sequence of differentials on all successive pages of the spectral sequence, a filtration on $H^*(Y)$ and isomorphisms between the associated graded of the filtration and the E_∞-page.



Now we want to discuss example computation with the Serre-spectral sequnece. The first family of examples is the following:

Lemma 2.7. *There is a fibre sequence*

$$U(n-1) \xrightarrow{i} U(n) \to S^{2n-1}$$

where $U(n)$ denotes the (topological) group of unitary $n \times n$-matrices and i is the Standard inclusion which adds a trivial summand of $\mathbb{C}$.

Proof. We want to deduce this from (3) of Example 1.4, namely we claim that $U(n)$ can be equpped with the structure of a fibre bundle over S^{2n-1} whose fibre is $U(n-1)$. To this end we consider the defining action of $U(n)$ on $\mathbb{C}^n$ and observe that it restricts to an action of $U(n)$ on $S^{2n-1} \subseteq \mathbb{C}^n$. This action is transitive, since every vector can be extended to an orthonormal basis. As a result we get that S^{2n-1} is in bijection to the quotient of $U(n)$ by the stabilizer subgroup of any point. In fact we get a map $U(n)/\mathrm{Stab}_x \to S^{2n-1}$ and by compactness this is a homeomorphism. We consider the point $(0, ..., 0, 1) \in \mathbb{C}^n$ and see that the stabilizer subgroup is $U(n-1) \subseteq U(n)$. Therefore we obtain a homeomorphism

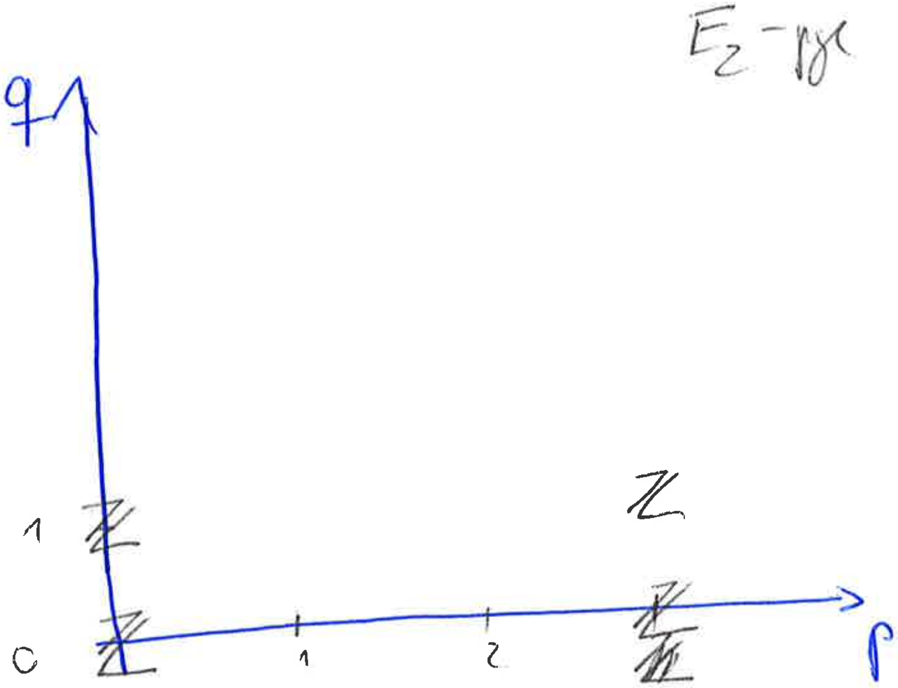
$$U(n)/U(n-1) \simeq S^{2n-1}.$$

Finally we use the fact that for a smooth, free action of a compact Lie group G on a smooth manifold M the map $M \to M/G$ is always a fibre bundle (in fact a G-principal bundle). This is standard fact from differential geometry (in fact this can be easily seen from the existence of action slices and holds much more general). □

Example 2.8. *Consider the case $n = 2$, i.e. the fibre sequence*

$$U(1) \to U(2) \to S^3.$$

We know the cohomology of S^3 and of $U(1) \simeq S^1$ and want to use the Serre-spectral sequence to compute the cohomology of $U(2)$. We get the following picture of the E_2-page:



recall that he entries are given by $H^p(S^3, H^q(S^1))$. We see that there can not be any differential for degree reasons! Thus this E_2-page is isomorphic to the E_∞-page. We deduce that there is a filtration on the groups $H^*U(2)$ where upper indices are the filtration quotients:

$$
\begin{aligned}
0 = F_1^0 \subseteq^{\mathbb{Z}} F_0^0 &= H^0U(2)\\
0 = F_2^1 \subseteq^0 F_1^1 \subseteq^{\mathbb{Z}} F_0^1 &= H^1U(2)\\
0 = F_3^2 \subseteq^0 F_2^2 \subseteq^0 F_1^2 \subseteq^0 F_0^2 &= H^2U(2)\\
0 = F_4^3 \subseteq^{\mathbb{Z}} F_3^3 \subseteq^0 F_2^3 \subseteq^0 F_1^3 \subseteq^0 F_0^3 &= H^3U(2)\\
0 = F_5^4 \subseteq^0 F_4^4 \subseteq^{\mathbb{Z}} F_3^4 \subseteq^0 F_2^4 \subseteq^0 F_1^4 \subseteq^0 F_0^4 &= H^4U(2)\\
\ldots \textit{all quotients } 0 \textit{ from now on}&
\end{aligned}
$$

All inclusions with a subindex 0 are equalities. Thus we easily deduce that

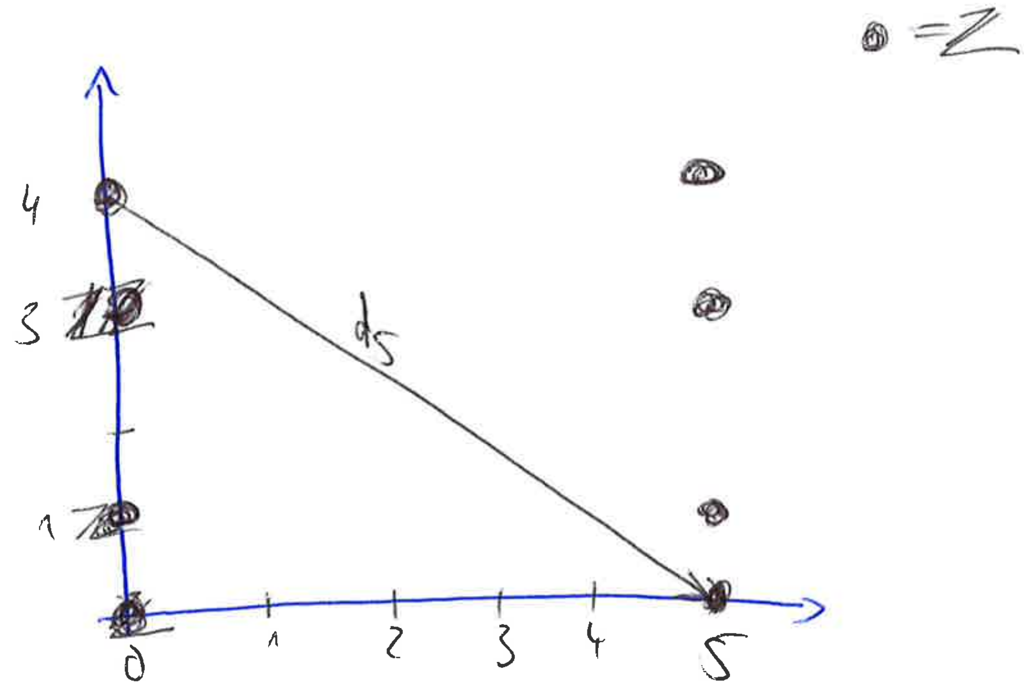
$$H^*U(2) \simeq \begin{cases} \mathbb{Z} & \textit{for } * = 0,1,3,4 \\ 0 & \textit{else} \end{cases}$$

(As a side note: one can also use the determinant map to show that $U(2)$ is homeomorphic to $U(1) \times S^3$ as a space and get the result using Künneth).

Example 2.9. *Consider the fibre sequence*

$$U(2) \to U(3) \to S^5$$

The E_2-page of the spectral sequence looks as follows:



For degree reasons there can not be any non-trivial differentials on pages 2,3 and 4. Thus the E_5-page is isomorphic to E_2. But then there is a homomorphism $d_5 : H^4(U(2)) \to H^5(S^5)$ (both groups are isomorphic to $\mathbb{Z}$, so its given by multiplication with some $n \in \mathbb{Z}$). At this point we can not decide if it is non-zero or not (the slang for this is: the differential exists or not). All other differentials are agains zero for degree reasons.

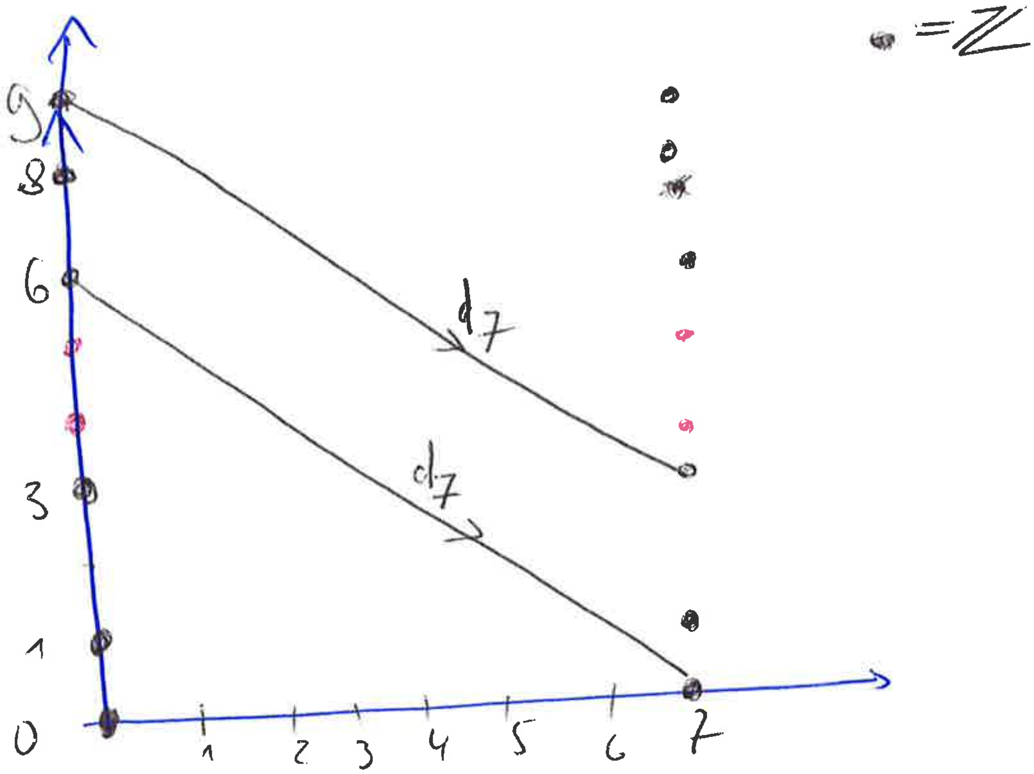
Again we get can deduce that in the filtration of $H^*(U(3))$ only one of the quotients is non-zero since there is only one diagonal entry. As a result we get that

$$H^*(U(3)) \simeq \begin{cases} \mathbb{Z} & \text{for } * = 0,1,3,6,8,9 \\ \ker d_5 & \text{for } * = 4 \\ \operatorname{coker} d_5 & \text{for } * = 5 \\ 0 & \text{else} \end{cases}$$

where $\ker d_5$ can be either 0 or $\mathbb{Z}$ and $\operatorname{coker} d_5 \simeq \mathbb{Z}/n$ (we will see soon that $n = 0$).

The last example illustrates very well a typical situation, namely that one can very often not fully determine all differentials in a SS purely formally but still deduce a lot about the abutment.

Example 2.10. *Consider $U(3) \to U(4) \to S^7$. Then the E_2-page looks as follows:*



These leaves two possible d_7 differentials (again all differentials before and afterwards vanish for degree reasons):

$$d_7\colon\ H^6(U(3)) \to H^7(S^7)$$
$$d_7\colon\ H^9(U(3)) \to H^7(S^7; H^3(U(3)))$$

which we can not decided at this point. Thus we can not fully decide the structure of the spectral sequence, but we can still deduce a lot, for example

$$H^0(U(4)) = \mathbb{Z} \quad H^1(U(4)) = \mathbb{Z} \quad H^2(U(4)) = 0 \quad H^3(U(4)) = \mathbb{Z}$$

and

$$H^{13}(U(4)) = \mathbb{Z} \quad H^{14}(U(4)) = 0 \quad H^{15}(U(4)) = \mathbb{Z} \quad H^{16}(U(4)) = \mathbb{Z}$$

as well as $H^k(U(4)) = 0$ for $k > 16$. Note that $U(4)$ is an orientable manifold of dimension 16, the dimension can (independently of our computation) be deduced noting that the the Lie algebra (i.e. the tangent space at the unit matrix) is $\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A + A^* = 0\}$ which has real dimension $6 \cdot 2 + 4$ (six off diagonal entries and four diagonal entries). Our computation reflects Poincaré duality in the groups that we have computed.

There are some groups which can not yet fully determine, but we want to focus on $H^8(U(4))$. From the E_∞-page we can deduce that there is a filtration

$$0 = F_9^8 = F_8^8 \subseteq^{\mathbb{Z}} F_7^8 = \ldots = F_1^8 \subseteq^{\mathbb{Z}} F_0^8 = H^8(U(4)).$$

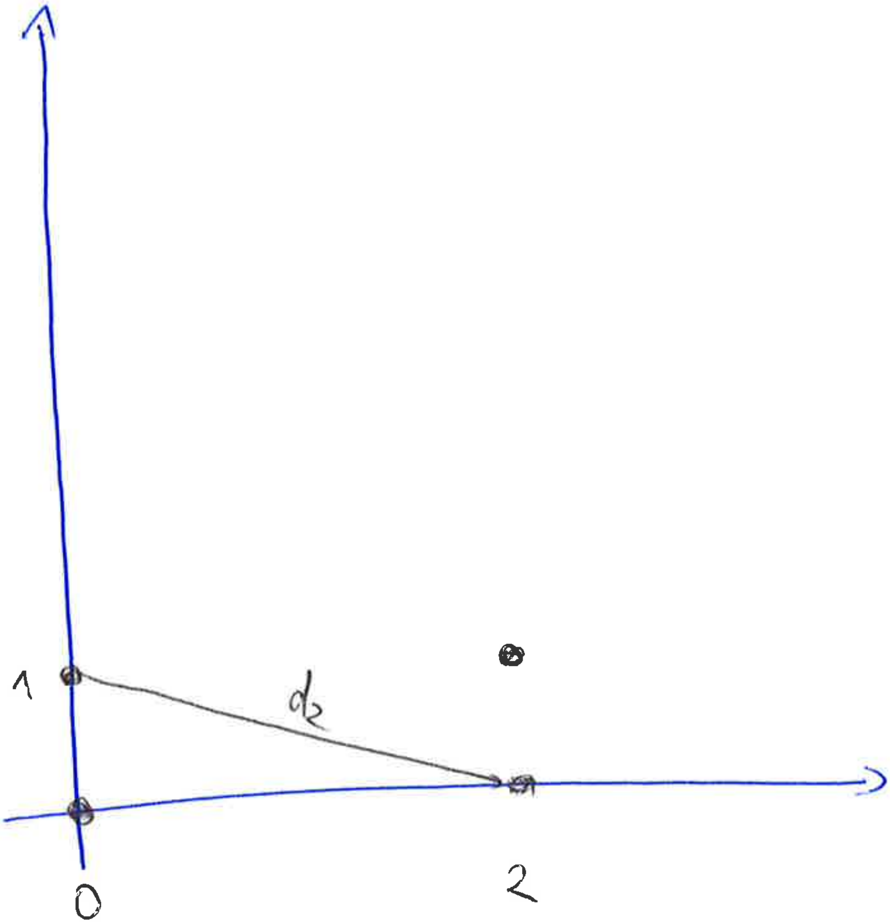
In other words we have a short exact sequence

$$0 \to \mathbb{Z} \to H^8(U(4)) \to \mathbb{Z} \to 0.$$

Every such sequence splits non-canonically, so that we have $H^8(U(4)) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

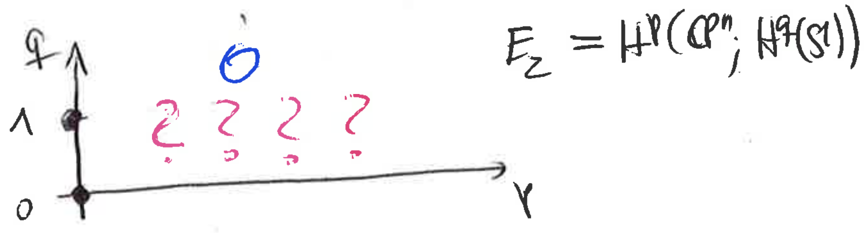
We will show soon that all differentials in the the above spectral sequences are trivial. Now we want to look at a different family of examples. The first one is rather useless for computations (since we already know everything) but only to illustrate a non-trivial differential.

Example 2.11. *Consider the fibre sequence $S^1 \to S^3 \to S^2 = \mathbb{C}P^1$. Then the E_2-page looks as follows.*

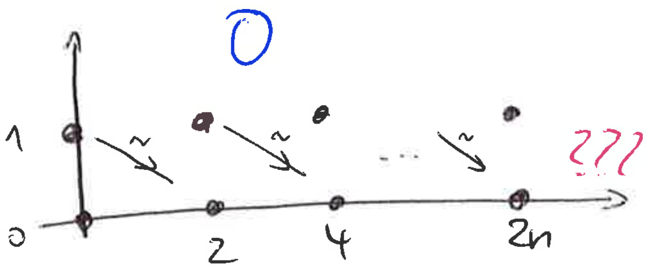


If we just know the E_2-page we see that there is a potential $d_2 : H^1(S^1) \to H^2(S^2)$ and we can not decided what it is from the E_2-page. But of course we already know the cohomology of S^3. Formally from the spectral sequence we get that the cohomology of S^3 has $\ker(d_2)$ *in degree* 1 *and* $\operatorname{coker}(d_2)$ *in degree 2. Thus both have to vanish which implies that d_2 is an isomorphism!*

Example 2.12. *Consider the fibre sequence $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ (the tautological bundle) for $n \leq 2$. We want to use this to compute the cohomologt of $\mathbb{C}P^n$, so lets pretend we do not know this cohomology. Since we do not know the cohomology of $\mathbb{C}P^n$ we do not know the full E_2-page, but we know something, namely the q-axis:*

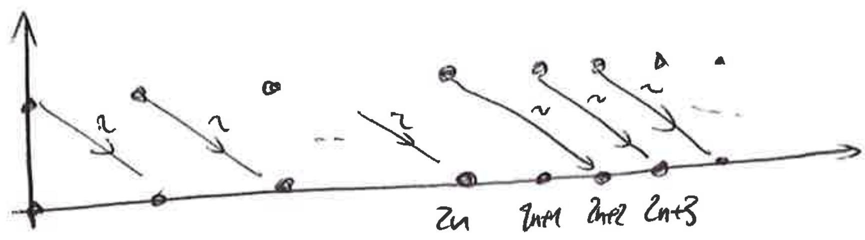


Now we know that the cohomology of S^{2n+1} does not have any class on degree 1, thus the E_∞-page can not have any non-trivial group along the first diagonal. Thus there has to be a differential out of the $\mathbb{Z}$ in degree $(0,1)$. For degree reasons it can only hit the entry $(2,0)$. Thus there has to be a group in degree $(2,0)$. It can only be a $\mathbb{Z}$ since the differential has to be injective (so that the $\mathbb{Z}$ in $(0,1)$ gets killed) and surjective, since there can also not be anything along the 2nd diagonal as $H^2(S^{2n+1}) = 0$. Thus we get that $H^2(\mathbb{C}P^n) \simeq \mathbb{Z}$. Thus we also have a $\mathbb{Z}$ in bidegree $(2,1)$. This again has to support a differential (since $H^3(S^{2n+1}) = 0$) which forces a $\mathbb{Z}$ in bidegree $(4,0)$. This can be iterated until we get the following picture of the E_2-page (not yet fully finished):



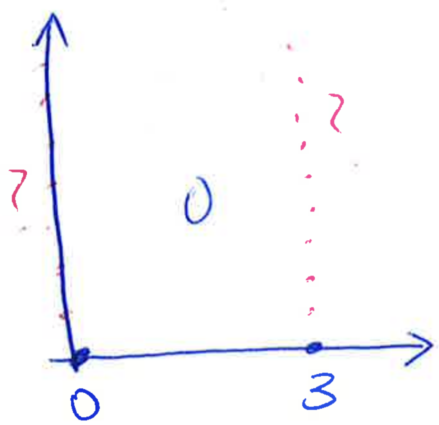
Now we don't know that the $\mathbb{Z}$ in degree $(2n,1)$ has to support a differnential, since $H^{2n+1}(S^{2n+1}) \simeq \mathbb{Z}$. There are two possibilities: namely that this $\mathbb{Z}$ contributes to the E_∞-page, which implies that the entry in bidegree $(2n+2,0)$ has to be zero and then the whole rest of the spectral sequence. Thus the picture above is complete (i.e. the question marks are zero). This leaves us with the cohomology of $\mathbb{C}P^n$ given by $\mathbb{Z}$ in degrees $0, 2, 4, ..., 2n$ and zero else.

The other possibility is that the group in bidegree $(2n,1)$ supports a differential. This can lead to patterns of differentials as in the following picture

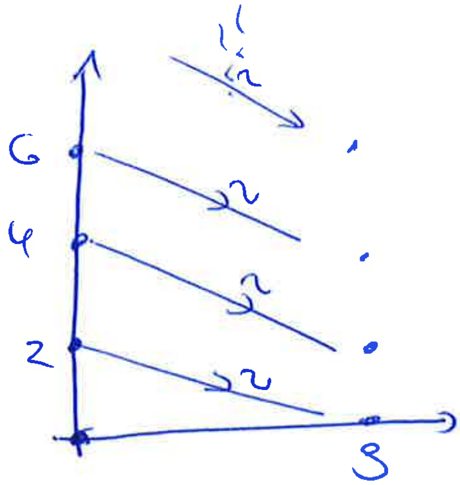


In order to rule this out we can input that $\mathbb{C}P^n$ is n-dimensional.

Example 2.13. *Consider the fibre sequence $\Omega S^3 \to \mathrm{pt} \to S^3$. The E_2-page of the Serre spectral sequence starts with the p-axis being the cohomology of S^3 (since ΩS^3 is connected) and we also already know that it is zero outside the $p = 0$ and $p = 3$ column:*



Since the cohomology of the total space is trivial there can not be anything in bidegree $(0, 1)$ but there has to be a differential killing the $\mathbb{Z}$ in bidegree $(3, 0)$. This forces a $\mathbb{Z}$ in bidgree $(0, 2)$ with the d_2 an iso. Then we also get a $\mathbb{Z}$ in bidgree $(3, 2)$. This again has to be killed etc. , so that we get the full E_2-page as



Thus we can deduce that

$$H^*(\Omega S^3) \simeq \begin{cases} \mathbb{Z} & * = 2k \\ 0 & else \end{cases} .$$

In particular we see that ΩS^3 is infinite dimensional!

Now we have made purely additive computations of cohomology groups. But of course we would like to understand the ring structure on cohomology. Therefore we will strengthen Theorem 2.6 a little bit to claim a comtaibility with the ring structure. It will turn out, that this compatibility will also be very useful to determine differentials.

Definition 2.14. A (commutative) multiplicative structure on a spectral sequence $(E_\bullet, d_\bullet, h_\bullet)$ is a bigraded (commutative) ring structure on the bigraded groups E_r [1] such that the differential $d_r : E_r \to E_r$ is a graded derivation, that is

$$d_r(xy) = d_r(x) \cdot y + (-1)^{p+q} x \cdot d_r(y)$$

for x in bidegree (p, q) [2]. As a result the cohomology $H^*(E_r)$ has the structure of a bigraded ring and we require the isomorphism $h : H^*(E_r) \xrightarrow{\simeq} E_{r+1}$ to be an isomorphism of bigraded rings.

The fact that the cohomology $H^*(E_r)$ admits the structure of a bigraded commutative ring as a result of the derivation property is a straightforward, still instructive little exercise for the reader. For a multiplicative, first quadrant spectral sequence the E_∞-page inherits also the structure of a bigraded commutative ring. A filtration

$$\ldots \subseteq F_2 \subseteq F_1 \subseteq F_0 = H$$

on a graded abelian group H which is equipped with a graded ring structure is said to be *multiplicative* or *compatible with the multiplicative strucutre* if $F_s \cdot F_t \subseteq F_{st}$. In this case

[1] Recall that a bigraded ring structure means that the product xy of elements x in bidegree (p, q) and y in bidegree (p', q') lands in bidgree $(p+p', q+q')$. It is called (bi)graded commutative if $xy = (-1)^{(p+q)(p'+q')} yx$

[2] Note that d_r has total degree 1, thus this is the standard sign rule

we say that (H, F) is a filtered graded ring. It follows that the associated graded of a filtered graded (commutative) ring, i.e. the bigraded abelian group F_p^{p+q}/F_{p+1}^{p+q}, is a bigraded (commutative) ring.

Definition 2.15. A multiplicative first quadrant spectral sequence $(E_\bullet, d, h)$ is said to *converge* to a filtered graded ring (H, F) if it converges additively and the chosen isomorphism

$$E_\infty^{p,q} \simeq F_p^{p+q}/F_{p+1}^{p+q}$$

is compatible with the bigraded ring structures. In this case we will also write s before

$$E_2^{p,q} \Rightarrow H.$$

and leave the ring structures implicit.

Theorem 2.16. *For every fibre sequence $F \to Y \to X$ of spaces with X simply connected and every (commutative) ring R the Serre spectral sequence admits the structure of a (commutative) multiplicative, first quadrant spectral sequence*

$$E_2^{p,q} = H^p(X, H^q(F; R)) \Rightarrow H^{p+q}(Y; R)\ .$$

DO WE HAVE TO INSERT A SIGN HERE? Explain what the E_2-page exactly is as a graded ring (this might not be super clear from what they know so far...)!!!!

Now we want to employ this multiplicativitiy to deduce all the missing differentials and the cup product structure in the examples that we have done before.

Example 2.17. *Lets follow up on Example 2.8 where we calculated the spectral sequence $U(1) \to U(2) \to S^3$. As a graded ring the E_2-page is given by $\Lambda(x_1, x_3)$ where x_1 is in bidegree $(0,1)$ and x_3 is in bidegree $(3,0)$. The $\mathbb{Z}$ in bidegree $(3,1)$ is spanned by x_1x_3. Again this agrees with the E_∞-page.*

Now what do we learn from this? We have the filtration (as extensively discussed in Example 2.8):

$$0 \subseteq^{x_1x_2} F_3 \subseteq^{x_2} F_2 \subseteq^{x_1} F_1 \subseteq H^*(U(2))$$

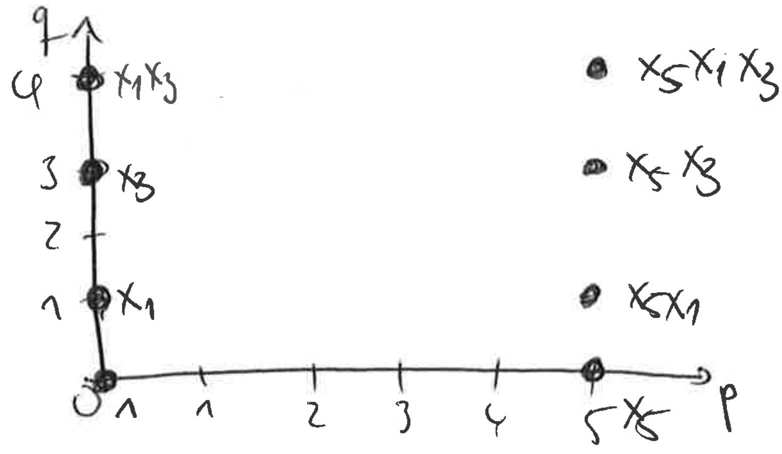
where the upper indices denote the element that spans the quotient (in this case the quotients are all isomorphic to a single $\mathbb{Z}$ in a single grade). Now we have an element $\overline{x}_1 \in H^*U(2)$ which lifts $x_1 \in H^*(U(2))/F_1$, an element $\overline{x}_2 \in F_2$ which lifts x_2 and an element $\overline{x_1x_2} \in F_3$ which lifts the product x_1x_2(the latter is actually already in F_3). Note that all these elements are unique!

Now what is $\overline{x_1}^2$. It lies in the zero filtration and lifts the element x_1^2. Thus it is zero. Similarly we see that $\overline{x_2}^2 = 0$ and that the product of $\overline{x_1x_2}$ lifts x_1x_2 which shows that it agrees with $\overline{x_1x_2}$. Thus in the end we see that this determines an isomorphism between the E_∞-page and $H^*U(2)$ which is thus given by $\Lambda(\overline{x_1}, \overline{x_2})$. with x_1 in degree 1 and x_2 in degree 2.

Example 2.18. *Now we use this computation as input for the spectral sequence*

$$H^p(S^5, H^q(U(2)) \Rightarrow H^{p+q}(U(3))$$

as in Example 2.9. Now the E_2-page is again an exterior algebra on generators x_1, x_2, x_3 which are located as pictured: (REDO this picture, because the labeling is wrong)



Now when we discussed this example before, it was unclear if there is a d_5 differential from the group spanned by x_1x_2 to the group spanned by x_3. Thus we want to decide what $d_5(x_1x_2)$ is. We do know that $d_5(x_1) = 0$ and $d_5(x_2) = 0$ for degree reasons. But then the product rule implies that

$$d_5(x_1x_2) = d_5(x_1)x_2 - x_1d_5(x_2) = 0.$$

This shows that there is no differential at all in the spectral sequence. Since there are also no extension problems we deduce that $H^*(U(3)) = \Lambda(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ (for the next computation we will not distinguish between x_i and $\bar{x}_i$ notationally).

As a side note: as soon as we understand the way the spectral sequence is built it will be clear that the inclusion $U(2) \to U(3)$ induces on cohomology the map that sends x_1 and x_2 to the respective elements in the cohomology of $U(2)$ and the element x_3 to zero.

Example 2.19. *We look at the spectral sequence $U(3) \to U(4) \to S^7$. We get the following picture of the E_2-page (with our definite knowledge of the cohomology ring of $U(3)$); As in the last example we argue that both possible d_7 differentials are zero (that were open in Example 2.10). Thus get that the E_2-page is isomorphic to the E_∞-page which is given by $\Lambda(x_1, x_2, x_3, x_4)$ with $|x_1| = (0,1), |x_2| = (0,3), |x_3| = (0,5)$ and $|x_4| = (7,0)$.*

Now recall that there was an extension Problem for $H^8(U(4))$. We concluded in Example 2.10 from the additive structure that there was a non-canonical isomorphism of this group with $\mathbb{Z} \oplus \mathbb{Z}$, but really we had a short exact sequence

$$0 \to \mathbb{Z}\langle x_1x_4\rangle \to H^8(U(4)) \to \mathbb{Z}\langle x_2x_3\rangle \to 0.$$

where we indicated the generators that span the groups on the E_∞-page of the respective groups. Now the point is that we have unique lifts $\bar{x}_i$ of all x_i's since in the respective degrees there are no choices involved. Then we do have a canonical lift of x_2x_3 to the group $H^8(U(4))$ as well, namely the product $\bar{x}_2\bar{x}_3$. Thus we have a canonical split of the sequence and therefore a canonical isomorphism

$$H^*(U(4)) \simeq \Lambda(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4).$$

This is a baby-instance of products (or other structures) solve extension problems.

It turns out, that this is more generally true:

Theorem 2.20. *$H^*(U(n)) \cong \Lambda(x_1, \ldots, x_n)$ with $|x_i| = 2i - 1$.*

Proof. We argue inductively, the start is $U(1)$ in which case the result is obvious (or one of the cases discussed above). Thus asume that the result is true for some n. Then we get that the E_2-page of the Serre spectral sequence for the fibre sequence $U(n) \to U(n+1) \to S^{2n-1}$ is isomorphic as a bigraded ring to

$$\Lambda(x_1, x_2, ..., x_n, x_{n+1})$$

where $|x_i| = (0, 2i-1)$ and $|x_{n+1}| = (2n+1, 0)$. For degree reasons the elements $x_1, ..., x_n$ can not support any diffferentials since the smalles possible differentials is a d_{2n+1}. Thus the product rule implies that there are also not further differentials. There are also no extension problems for these elements, so that we get unique elements $\bar{x}_i \in H^*(U(n+1))$ lifting the elements on the E_∞-page. This may be a good moment to point out that it is not clear from the SS that their squares of the elements $\bar{x}_i$ are 0, since for the higher up generators the squares no longer live in lowest filtration, all you obtain is that they live in lower filtration and are therefore multiples of x_{n+1}. But alas they all live in odd degree and the cohomology groups are torsionfree (from the analysis of the additive structure), so they have to square to zero anyway (they can only be two torsion). Thus one obtains a map from the exterior algebra into the cohomology

$$f : \Lambda(x_1, x_2, x_3, .., x_{n+1}) \to H^*U(n+1)).$$

If we define a filtration on $\Lambda(x_1, .., x_{n+1})$ for which all x_i are in degree 0 and the element x_{n+1} is in degree $2n+1$, then this map is filtration preserving. Moreover it is an isomorphism on the associated graded pieces. To see this one checks easily that the associated graded of the source is exterior on the x_i's as well and then the map is an isomorphism to the associated graded of the target which is the E_∞-page of the spectral sequence. Then it follows from the next Lemma that f is an isomorphism, which shows the claim. □

Lemma 2.21. *Let A and B be graded abelian groups equipped with filtrations*

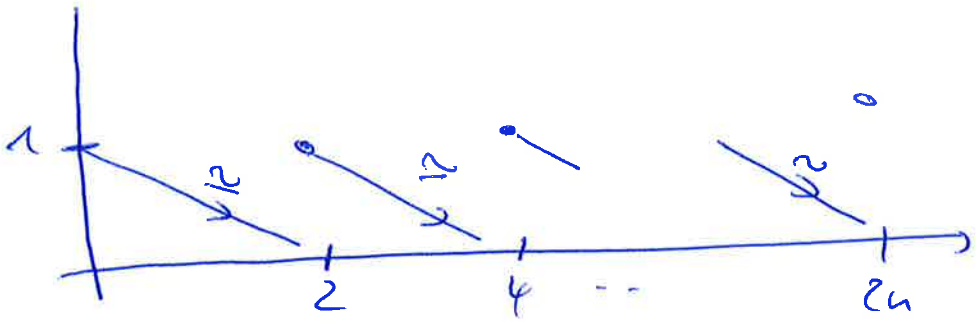
$$... \subseteq F_2 \subseteq F_1 \subseteq F_0 = A$$
$$... \subseteq G_2 \subseteq G_1 \subseteq G_0 = B$$

which are in every degreewise eventually zero. If we have a grading and filtration preserving morphism $f : A \to B$ that induces an isomorphism on all associated graded pieces $F_i/F_{i+1} \to G_i/G_{i+1}$. Then f is an isomorphism.

Proof. This is an 'iterated 5-lemma' argument. □

Remark 2.22. One should actually be aware of the definition of $\Lambda(x_1, ..., x_{n+1})$ over $\mathbb{Z}$. While in characteristic different from 2 this is the free graded commutative ring on generators $x_1, .., x_{n+1}$, this is not true in characteristic 2 and thus also not over $\mathbb{Z}$. In fact we define $\Lambda(x_1, .., x_{n+1})$ to be the free algebra on $x_1, .., x_{n+1}$ subject to the relations $x_i x j = -x_j x_i$ for all $i \neq j$ and the relation $x_i^2 = 0$ for all i. The second relation is different from the obvious generalization of the first relation to the case $i = j$ which would be $2x_i^2 = 0$. In the presence of 2-torsion this makes a difference. For example there is graded commutative ring $H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[x]$ for x in degree 1. But this does not receive a map from $\Lambda(x)$ which sends x to x since in the exterior algebra $x^2 = 0$. In fact the free graded commutative ring on a generator x in degree 1 is given by $\mathbb{Z}[x]/2x^2$ which has a $\mathbb{Z}$ in degrees 0 and 1 and $\mathbb{Z}/2$ in all other degrees.

Example 2.23. *Now we revisit the computation of $\mathbb{C}P^n$ given in Example 2.12 using the fibre sequence $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. But this time we want to determine the product structure of $H^*(\mathbb{C}P^n)$. We did get (just from the knowledge of the additive structure the following pattern of differentials):*



Lets pick a generator e of the group in bidegree $(0,1)$ (which is $H^1(S^1)$) and denote the generator $d_2(e)$ in bidegree $(2,0)$ by x. Then the element in bidegree $(2,1)$ is given by ex (since multiplication by e induces the isomorphism between the bottom line and the 1-line in the picture). As a result we get from the product rule that $d_2(ex) = xd_2(x) + d_2(e)x = x^2$. Thus the group in degree $(4,0)$ is generated by x^2. Similarly we get that the next groups are generated by $x^3, ..., x^n$. Thus $H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}$.

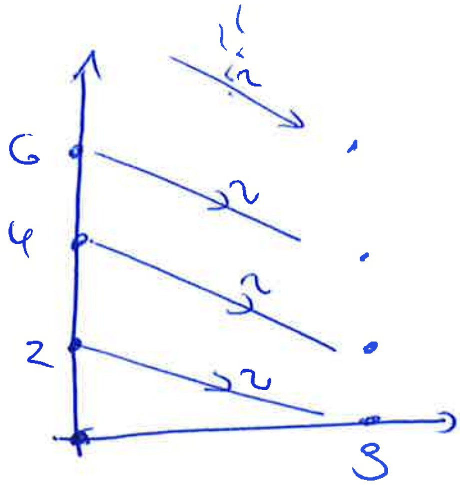
In the limit (or using the spectral sequence of $S^1 \to \mathrm{pt} \to \mathbb{C}P^n$) this shows that $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ for x in degree 2.

Recall that we showed in Example 2.13 the cohomology of $H^*\Omega S^3$ is given by $\mathbb{Z}$ in every even degree, thus additively isomorphic to the cohomology of $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$. An interesting question (that was asked in the lecture) is if this is an accident. Let us be more precise: the fact that $H^2\Omega S^3 = \mathbb{Z}$ implies that there is a map $\Omega S^3 \to K(\mathbb{Z}, 2)$ which pulls back the universal class of $K(\mathbb{Z}, 2)$ to a generator of $H^2(\Omega S^3)$. This map can alternatively be described by looping of the canonical map $S^3 \to K(\mathbb{Z}, 3)$ and is therefore the second Postnikov section. Thus the question is, what this map induces on cohomology groups:

$$H^*\mathbb{C}P^\infty \to H^*\Omega S^3.$$

This can be answered by considering the ring structure as will be done in the next example.

Example 2.24. *We revisit the computation of ΩS^3 from Example 2.13 using the Serre spectral sequence for $\Omega S^3 \to \mathrm{pt} \to S^3$. Recall that we found the pattern of differentials*



We pick a generator $x \in H^2(\Omega S^3)$ i.e. in bidegree $(0,2)$. Then $e := d_3(x)$ generates the group in bidegree $(3,0)$ (which is $H^3(S^3)$). Then the group in bidgree $(3,2)$ is generated by ex and we have that $d_3(x^2) = d_3(x)x + xd_3(x) = 2ex$. Thus the element x^2 is twice the generator of $H^4(\Omega S^3)$ which maps to ex. We denote this generator by $x^2/2$. Similarly we find that $d_3(x^3) = 3xd(x^2) = 3!ex^2$. Thus we conclude that x^3 is $3!$ times the generator of $H^6(\Omega S^3)$. Inductively we get that x^n is $n!$ times the generator of $H^{2n}(\Omega S^3)$. As a result we get an isomorphism

$$H^*\Omega S^3 \simeq \mathbb{Z}\Big[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots\Big]$$

The notation is not meant to be a polynomial algebra, but to indicate the relations that hold between the generators. This is the free divided power algebra on a generator x in degree 2.

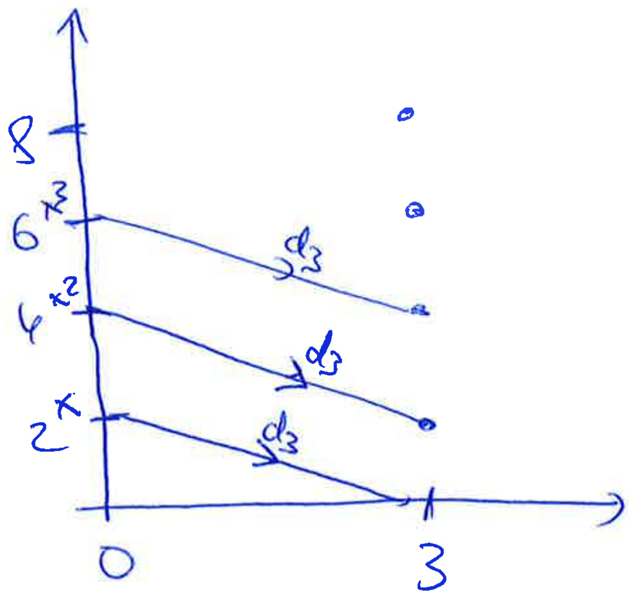
As a result we see that the map $H^*\mathbb{C}P^\infty \to H^*\Omega S^3$ sends a generator of $H^{2k}\mathbb{C}P^\infty \simeq \mathbb{Z}$ to $k!$ times a generator of $H^{2k}\Omega S^3 \simeq \mathbb{Z}$.

Now we finally want to describe the easiest example of a series of computations of homotopy groups that we will discuss eventually.

Example 2.25. *Consider the map $S^3 \to K(\mathbb{Z},3)$ which classifies the generator of $H^3(S^3) = \mathbb{Z}$ (or maybe better that classifies one of the generators). Let X denote the homotopy fibre, so that $X \to S^3 \to K(\mathbb{Z},3)$ is a fibre sequence. The long exact sequence of homotopy groups easily shows that X is 3-connected, i.e. $\pi_0(X) = \pi_1(X) = \pi_2(X) = \pi_3(X) = 0$. Taking a further fibre we obtain a fibre sequence*

$$K(\mathbb{Z},2) \to X \to S^3.$$

We know that $K(\mathbb{Z},2)$ is $\mathbb{C}P^\infty$ and thus that $H^*(K(\mathbb{Z},2)) \simeq \mathbb{Z}[x]$. The E_2-page of the Serre spectral sequence for this second fibre sequence looks as follows



By the fact that X is 3-connected we conclude that it can also not have cohomology in degrees $1, 2$ and 3. In particular the differential $d_3 : H^2(K(\mathbb{Z}, 2)) \to H^3(S^3)$ has to be an isomorphism (since we otherwise get a nontrivial H^2 or H^3 for X). As a result we get that $d_3(x^2)$ is twice the generator and more generally that $d_3(x^n)$ is n times the generator. In particular we get that

$$\tilde{H}^*(X) \simeq \begin{cases} \mathbb{Z}/k & * = 2k+1 \\ 0 & else \end{cases}$$

By the universal coefficients theorem we conclude that

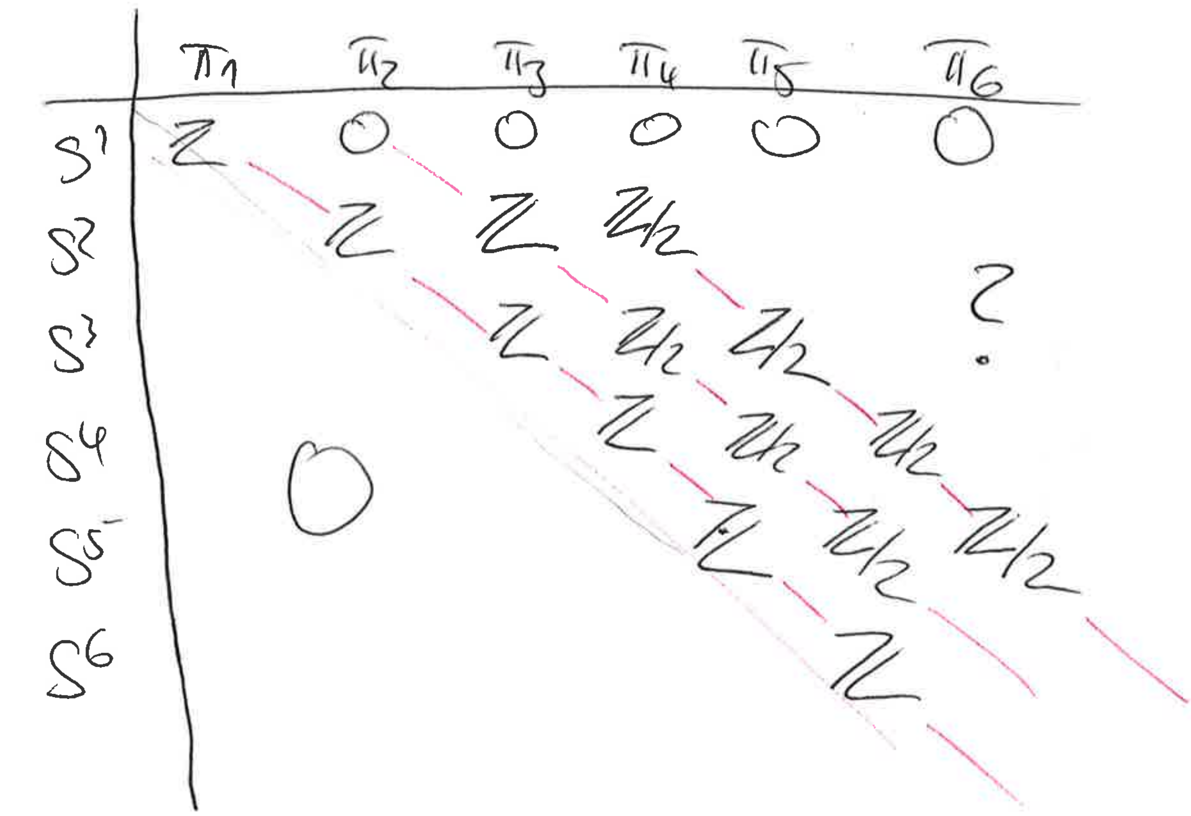
$$\tilde{H}_*(X) \simeq \begin{cases} \mathbb{Z}/k & * = 2k \\ 0 & else \end{cases}$$

Corollary 2.26. *We have that $\pi_4(S^3) = \mathbb{Z}/2$ and $\pi_4(S^2) = \mathbb{Z}/2$.*

Proof. The space X from Example 2.25 is 3-connected and $H_4(X) = \mathbb{Z}/2$. Thus we find that $\pi_4(X) = \mathbb{Z}/2$ by the Hurewicz theorem. From the long exact sequence of homotopy groups associated to $X \to S^3 \to K(\mathbb{Z}, 3)$ we get that $\pi_4(S^3) = \mathbb{Z}/2$ as well.

For the second part we consider the fibre sequence $S^1 \to S^3 \to S^2$ and deduce immeditely from the associated long exact sequence that $\pi_4(S^2) = \pi_4(S^3) = \mathbb{Z}/2$. □

In fact the long exact sequence of the Hopf fibration shows that $\pi_i(S^3) = \pi_i(S^2)$ for all $i \geq 3$. Using the Freudenthal suspension theorem we get that $\pi_4(S^2) \simeq \pi_5(S^3) \simeq \ldots$ and thus we know the following homotopy groups of spheres:



In particular we have computed the first 3-stable homotopy groups of spheres $\pi_0^s = \mathbb{Z}$, $\pi_1^s = \mathbb{Z}/2$ and $\pi_2^s = \mathbb{Z}/2$.

3 Spectral Sequences in general

Let us set up a bit of notational conventions that we will from now on follow. We saw that there are two types of degree in play in the Serre spectral sequence, the cohomological degree and the filtration index (therefore giving each page three indices (namely, page number, filtration index (along the vertical axis) and (!) the difference between cohomological degree and filtration index (along the horizontal axis)). Only the first and last are actually relevant to the development of spectral sequences so we will suppress the cohomological one entirely in what follows. This will hopefully allow you to keep the indices from swimming in front of your eyes and points to the conceptual features that go into the *construction* and inner workings of a spectral sequence. For *applications* the cohomological degree is, however, equally important as you hopefully have already seen.

Definition 3.1. A cohomologically graded spectral sequence in an abelian category $\mathcal{A}$ is a triple (E, d, h), where E_n is a sequence of $\mathbb{Z}$-graded objects of $\mathcal{A}$, $d_n : E_n \to E_n$ are a sequence of morphisms with d_n of degree n and $d_n \circ d_n = 0$ and finally $h_n : H^*(E_n) \to E_{n+1}$ is a sequence of (degree 0) isomorphisms.

Example 3.2. *For the Serre spectral sequence associated to a fibre sequence $F \to Y \to X$ we consider the E_2-page $H^p(X, H^q(F))$ as graded by p, i.e. the p-th graded component is* *given by $\bigoplus_q H^p(X, H^q(F))$. Then the r-th differential has degree r. This way we consider it as a cohomological spectral sequence in the category of abelian groups.*

More generally, every cohomologically, Serre graded spectral sequence $E_\bullet^{p,q}$ in the sense of Definition 2.1 can be considered as a cohomological spectral sequence in the category of abelian groups in sense of Definition 3.1 by considering only the q degree as the filtration degree. Note that in these examples the cohomological degree of the differentials is always 1 *independent of the page number. To reiterate: This an (admittedly important) extra datum to be kept in mind, but suppressed in the general constructions of this section.*

The standard category $\mathcal{A}$ to think about is modules over some ring, but it may well happen that all terms in the spectral sequence carry a natural grading or even bi-grading giving a tri-graded spectral sequence (especially if it is used to compute an early page of another spectral sequence) that is respected by the differentials in some way. It is therefore not unhealthy to think of $\mathcal{A}$ as graded module with arbitrary degree homomorphism, although these do not form an abelian category.

We left it a bit vague where our indexing sequences start. So far we started at $r = 2$, but we will later see that the Serre spectral sequence really starts at $r = 1$. Of course formally this can always be extended even to $r = 0$ by setting differentials zero.

Definition 3.3. A morphism of spectral sequences $f : (E, d, h) \to (E', d', h')$ is a sequence of morphisms $f_n : E_n \to E'_n$ (usually of degree 0), that commute with both differentials and the identifications, i.e.

$$f_n \circ d_n = d'_n \circ f_n$$

and

$$f_{n+1} \circ h_n = h'_n \circ H^*(f_n)$$

where the latter makes sense because of the former.

From the second condition we immediately find that the entire morphism f is determined by its first term (i.e f_1 if the spectral sequence starts on the first page). But its a rather elaborate condition that a morphism of the first pages induce a morphism of spectral sequences.

We now set out to define the limit page of a spectral sequence (E, d, h) in this generality, i.e. where the terms need not stabilise (which does not even make sense anymore since we have lobbed the columns of a bigraded spectral sequence into one object which only stabilises degree wise in a first quadrant spectral sequence).

Definition 3.4. Fix a number k and consider the following subobjects of E_k:

$$B_k^k \subseteq \dots \subseteq B_k^n \subseteq B_k^{n+1} \dots \subseteq \dots Z_k^{n+1} \subseteq Z_k^n \subseteq \dots \subseteq Z_k^k$$

where $Z_k^k = ker(d_k), B_k^k = im(d_k)$ and for $n > k$

$$Z_k^n = \pi_k^{-1}(Z_{k+1}^n)$$

$$B_k^n = \pi_k^{-1}(B_{k+1}^n)$$

where $\pi_k : Z_k^k \to E_{k+1}$ is the composition

$$Z_k^k \to H^*(E_k) \to E_{k+1}$$

of the projection and the identification h_k.

This definition may be somewhat difficult to process on first reading. To understand what is going on suppose that $\mathcal{A}$ is a concrete abelian category, e.g. the category of R-modules for some ring R or its graded analogue. Then Z_k^k are simply the cycles for d_k, i.e. the elements $x \in E_k$ with

$$d_k(x) = 0$$

and similarly B_k^k consists of the boundaries. Any element $x \in Z_k^k$ gives rise to a new element $\pi_k(x) \in E_{k+1}$, let us denote it by $[x]_{k+1}$. The group Z_k^{k+1} consists of all those $x \in Z_k^k$ with

$$d_{k+1}([x]_{k+1}) = 0$$

Similarly B_k^{k+1} consists of those elements in $x \in Z_k^k$ for which $[x]_{k+1}$ lies in the image of d_{k+1}. Continuing in this fashion we obtain

Observation 3.5. *If for concrete $\mathcal{A}$ and $x \in Z_k^n$ we set*

$$[x]_{n+1} = \big(\pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_k\big)(x) \in E_{n+1}$$

then

$$Z_k^{n+1} = \{x \in Z_k^n \mid [x]_{n+1} \text{ is a cycle for } d_{n+1}\}$$
$$B_k^{n+1} = \{x \in Z_k^n \mid [x]_{n+1} \text{ is a boundary for } d_{n+1}\}$$

We obtain the following slogan: $Z_k^n \subseteq E_k$ consists of those elements that make it to the n-th page and are cycles there, and $B_k^n \subseteq Z_k^n$ consists of those that become boundaries there. When $k = 1, 2$ we will usually suppress the lower index.

Proposition 3.6. *The map $\pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_k$ induces an isomorphism*

$$Z_k^n / B_k^n \to E_{n+1}$$

whenever $n \geq k$.

Proof. Excercises! □

Given these formulas the following definitions hopefully seems reasonable:

Definition 3.7. The subobject $Z_k^\infty \subseteq E_k$ of *permanent cycles* is defined to be $\varprojlim_{n \geq k} Z_k^n$ and the subobject $B_k^\infty \subseteq E_k$ of *eventual boundaries* is defined to be $\varinjlim_{n \geq k} B_k^n$.

Clearly we have $B_k^\infty \subseteq Z_k^\infty$ and for concrete $\mathcal{A}$, $Z_k^\infty = \bigcap_n Z_k^n$ and $B_k^\infty = \bigcup_n B_k^n$.

Corollary 3.8. *The map $\pi_{l-1} \circ \pi_{l-2} \circ \cdots \circ \pi_k$ induces isomorphisms*

$$Z_k^\infty / B_k^\infty \to Z_l^\infty / B_l^\infty$$

whenever $l \geq k$.

Proof. Excercises! □

Definition 3.9. We denote the common quotient from the previous corollary by E_∞. It is called the *limit page* of E (or abusively the E_∞-page).

Example 3.10. *Assume that we have a first quadrant spectral sequence in the sense of Definition 2.1 and Definition 2.2. Then it is easy the see that the E_∞-page as discussed there is isomorphic to our new E_∞-page, since all the terms stabilize levelwise.*

Now for convergence in this abstract setting: Because of the cohomological conventions, all filtrations are *descending*! In particular, from now on, we allow very general filtrations of objects T in a category $\mathcal{A}$ of the form:

$$... \subseteq F_2 \subseteq F_1 \subseteq F_0 \subseteq F_{-1} \subseteq ... \qquad \subseteq T$$

We recall

Definition 3.11. The associated graded of the filtered object (C, F) is

$$\mathrm{Gr}_F C = \bigoplus_i F_i/F_{i+1}$$

with the indicated grading. Clearly this constitutes the object part of a functor from filtered objects in $\mathcal{A}$ (with filtration preserving morphisms) to graded objects in $\mathcal{A}$ (with degree 0 morphisms).

Before we go on, a few quick remarks about this construction: A *quasi-split* for a filtration is a collection of complements $B_i \oplus F_i = F_{i-1}$. Note that such data always exists for modules over a field, but need not exist in general, for example for

$$\dots 8\mathbb{Z} \subset 4\mathbb{Z} \subset 2\mathbb{Z} \subset \mathbb{Z} \subseteq Z \subseteq Z \dots$$

the B_i exist precisely for the inclusion which are equalities (and then $B_i = 0$), so this filtration does not split.

Observation 3.12. *For a filtration (C, F) quasi-split by a choice B, the inclusions induce an isomorphism*

$$\bigoplus_i B_i \longrightarrow \mathrm{Gr}_F C.$$

The inclusions also induce a map $\bigoplus_i B_i \longrightarrow F_{-\infty}/F_\infty$, which is readily checked to be injective. Very often (for example for a finite filtration as in the lecture, though unfortunately not always) it is an isomorphism. If this is the case we say that the filtration is *split*. A simple counterexample is

$$\prod_{i\in\mathbb{Z}} \mathbb{Z} \text{ filtered by } F_j = \prod_{i\geq j} \mathbb{Z}$$

which is split by $\mathbb{Z} \cdot e_{j-1} \oplus F_j = F_{j-1}$. In this filtration $F_\infty = 0$ and $F_{-\infty}$ consists of all those sequences f with $f(-n) = 0$ for large enough $n \in \mathbb{N}$ (i.e. the sequence is bounded below in its domain). However, the image of $\oplus_j \mathbb{Z} \cdot e_{j-1}$ in $F_{-\infty}$ clearly consists only of those sequences for which both $f(n)$ and $f(-n)$ are zero for large enough $n \in \mathbb{N}$ (i.e. sequence bounded in their domain in both directions). This example can also be truncated above zero, say, to produce a slightly smaller one with exhaustive filtration.

Observation 3.13. *Let $\varphi : (C, F) \to (C', F')$ be a morphism of filtered objects, quasi-split by B and B', respectively. Consider the diagram:*

$$\begin{tikzcd}
\bigoplus_i B_i \arrow[r] \arrow[d, "\tilde{\varphi}"'] & F_{-\infty}/F_\infty \arrow[d, "\varphi"] \\
\bigoplus_i B'_i \arrow[r, "\cong"] & F'_{-\infty}/F'_\infty
\end{tikzcd}$$

where $\tilde{\varphi}$ is defined by the other three arrows. Represent it by an infinite block matrix Φ. The map φ being filtration preserving translates into Φ being lower triangular (whence it has finite rows) and it also has finite columns, since the columns are elements of $\oplus_i B'_i$. The diagram

$$\begin{tikzcd}
\bigoplus_i B_i \arrow[r, "\cong"] \arrow[d, "\widetilde{\mathrm{Gr}\varphi}"'] & \mathrm{Gr}_F C \arrow[d, "\mathrm{Gr}\varphi"] \\
\bigoplus_i B'_i \arrow[r, "\cong"] & \mathrm{Gr}_{F'} C'
\end{tikzcd}$$

defines a map $\widetilde{\mathrm{Gr}\varphi}$. It is easily checked that it is represented by the matrix $D(\Phi)$ consisting of only the diagonal blocks of Φ.

Thus applying the functor Gr to a split filtration does not really change the underlying object. However, induced maps between split filtration are changed quite drastically.

The matrix Φ can still be defined as an infinite, lower triangular matrix with infinite columns when F' is only quasi-split: A map $\Phi_{i,j} : B_i \to B_j$ for $j \geq i$ is obtained from φ by considering the composite

$$B_i \longrightarrow F_{i-1} \xrightarrow{\varphi} F'_{i-1} \longrightarrow F'_{i-1}/F'_j \xleftarrow{\cong} B'_i \oplus B'_{i+1} \oplus \cdots \oplus B'_j \longrightarrow B'_j.$$

It is hopefully clear that this agrees with $\Phi_{i,j}$ defined in the above observation whenever it applies. In the general case Φ represents a map $\bigoplus_i B_i \to \prod_i B'_i$ and at this point it is unclear, how this relates to the map φ. Indeed, as we will show with an example below, the image of the composition

$$\bigoplus_i B_i \longrightarrow F_{-\infty}/F_\infty \xrightarrow{\varphi} F'_{-\infty}/F'_\infty$$

need not be contained in (the image of) $\bigoplus_i B'_i$ in general. We will later see that the map

$$\bigoplus_i B'_i \longrightarrow F'_{-\infty}/F'_\infty$$

extends to an isomorphism

$$\prod_i B'_i \longrightarrow \widehat{F'}_{-\infty}/\widehat{F'}_\infty$$

where $(\widehat{C'}, \widehat{F'})$ denotes the *completion* of (C', F'), a concept we have yet to discuss, and in general Φ represents left horizontal arrow in the diagram

$$\begin{tikzcd} \bigoplus_i B_i \arrow[r] \arrow[dd] & F_{-\infty}/F_{\infty} \arrow[d, "\varphi"] \\ & F'_{-\infty}/F'_{\infty} \arrow[d] \\ \prod_i B'_i \arrow[r, "\cong"] & \widehat{F'}_{-\infty}/\hat{F'}_{\infty} \end{tikzcd} .$$

With this definition of Φ (or just the explicit construction above) the claim about the second diagram of the observation still makes sense and indeed remains true: On the associated graded $\mathrm{Gr}\varphi$ is represented by the diagonal part of Φ!

Example 3.14. *Finally let us give the announced counterexample to the factorisation of the observation, if (C', F') is only quasi-split by B': To this end let*

$$C = \bigoplus_{i\in\mathbb{Z}} \mathbb{Z} \text{ and } C' = \prod_{i\in\mathbb{Z}} \mathbb{Z}$$

filtered as above by

$$F_j = \bigoplus_{i\geq j} \mathbb{Z} \text{ and } F_j = \prod_{i\geq j} \mathbb{Z},$$

respectively, and quasi-split by $\mathbb{Z}\cdot e_{j-1} \oplus F_j = F_{j-1}$ and $\mathbb{Z}\cdot e_{j-1} \oplus F'_j = F'_{j-1}$. Let furthermore φ be the map defined by sending the unit vector $e_j \in C$ to $f_j \in C'$, by which we denote the sequence with $f_j(n) = 1$ whenever $n \geq j$ and $f(n) = 0$ otherwise. This map clearly preserves the filtrations and yet the image of $e_{j-1} \in B_j$ is not contained in

$$\bigoplus_j B'_j = \bigoplus_j \mathbb{Z}\cdot e_{j-1} = \bigoplus_j \mathbb{Z}.$$

We return to the convergence of a spectral sequence:

Definition 3.15. An *abutment* for a spectral sequence E is a filtered object (T, F) of $\mathcal{A}$ (T is then usually called the *target*) together with a degree zero map $w : Gr_F T \to E_\infty$ of graded objects in $\mathcal{A}$. We write

$$E \Rightarrow T$$

in this case. By abuse of notation, a spectral sequence E with an abutment is said to *weakly converge* to T if w is an isomorphism.

Note that there are obvious categories of spectral sequences and spectral sequences with abutment. We will not give them specific names but use them to make sense of notion like naturality of spectral sequences etc.

Let us also recall the introduce/recall the following notions:

Definition 3.16. A filtration F of an object T is called *exhaustive* if $\mathrm{colim} F_i = C$ and *hausdorff* if $\lim F_i = 0$.

Clearly these are desirable properties for a filtration to have if $\mathrm{Gr}_F T$ is to tell us something about T. We will later see another important property, namely *completeness* that will turn out to be extremely important and lead to *strong convergence*.

Example 3.17. *The notion of convergence for first quadrant spectral sequences as introduced in Definition 2.5 in particular implies weak convergence in the sense of the previous definition. In particular the Serre spectral sequence for a fibre sequence of spaces $F \to Y \to X$ with X simply connected weakly converges to $H^*(Y)$.*

We will later introduce stronger notions of convergences and see that the Serre spectral sequence convergences in the strongest possible sense.

4 The spectral sequence of a filtered complex

We will now set out to construct our first example of a spectral sequence, in particular showing the inner workings of spectral sequences. To this end consider a chain complex C in $\mathcal{A}$ together with a filtration by subcomplexes $C \supseteq F_i \supseteq F_{i+1}$ (again this is the cohomological convention). The main goal of this section is to construct a spectral sequence

$$\mathcal{L}_1^{p,q} = H^*(Gr_F C) \Rightarrow H^*(F_{-\infty}C)$$

(recall that $F_{-\infty}C = \cup_i F_i$) and discuss its convergence properties. For now we shall not require the filtration to be bounded in either direction (i.e. it is indexed by the integers), nor that it stabilises in any way. These matters will not be relevant for the construction and we shall introduce them only when needed during the discussion of convergence.

4.1 Idea

Let us first give a description of the idea: We are interested in the cohomology of C, but maybe that is too hard to compute directly. But maybe things can be arranged so that the associated graded complex is easier to understand. To understand how this might happen consider a bounded chain complex of vector spaces with a finite filtration :

$$V_n \xrightarrow{d_n} V_{n+1}$$

together with a filtration by subcomplexes. If we pick bases adapted to the filtration, then the differential preserving filtration translates exactly into the the matrix D_n associated to d being block triangular. The idea of the spectral sequence now is the following: Instead of taking cohomology with respect to d directly, we can first take it with respect to the diagonal piece of D_n only; it is readily checked that these pieces still form differentials (i.e. that composition of successive pieces are 0). This may seem like a strange thing to do until one realises the following (which really is the heart of spectral sequences): The first line off the diagonal of the D_n defines a differential on the arising cohomology (and it certainly did not do so before)! So we can take cohomology with respect to this new differential. On this iterated cohomology group, the second line off the diagonal now defines a differential and so on and so forth. Clearly, taking homology in this iterative manner produces some sort of approximation (formalised in the concept of convergence) to the cohomology of C we were originally interested in. This is explained well in Ausoni's secret notes and we have little to add (though eventually we should copy his notes here)!

The spectral sequence we are about to construct is nothing more than the generalisation of the above idea, when the filtration is not finite, the modules involved not free and so on. So in general, how do we separate out the diagonal piece of a differential? We saw in the at the end of the previous chapter that this is precisely what happens by passing to the associated graded of a filtration!

Consider therefore the $\mathbb{Z}$-graded object

$$\mathcal{L}(C,F)_0 = \mathrm{Gr}_F C = \bigoplus_i F_i/F_{i+1},$$

which will form the first page of the spectral sequence associated to (C,F). Forming associated graded objects is an additive functor from the category of filtered objects to that of $\mathbb{Z}$-graded objects, whence the differential d on C induces a degree 0 map

$$\mathrm{Gr}(d) = d^0 : \mathcal{L}(C,F)_0 \longrightarrow \mathcal{L}(C,F)^0,$$

the first differential of the spectral sequence. Note that this does precisely what we advertised before and we can set

$$\mathcal{L}_1(C,F) \cong H^*(\mathcal{L}(C,F), d^0).$$

How to implement the next step? By definition an element z of $\mathcal{L}_1^i$ is represented by an element y of $\mathcal{L}_0^i$ (i.e $[y]_1 = z$), such that $d^0(y) = 0$. If in turn y is represented by $x \in F_i$ (i.e. we have $[x]_0 = y$ and thus $[x]_1 = z$), this translates into $d(x) \in F_{i+1}$ (a condition independent of the choice of x representing z). If we let $p_i : H^*(F_i) \to H^*(F_i/F_{i+1})$ denote the map induced by the projection, we might therefore try to set

$$d^1([x]_1) = [p_{i+1}d(x)]_1.$$

This attempt at a definition raises a few questions: is $p_{i+1}d(x)$ a cycle in $\mathcal{L}_0^{i+1}$, so that it even represents an element in $\mathcal{L}_1^{i+1}$ and if so does it really not depend on any of the choices involved in its construction? The answer to both questions turns out to be yes and one can set

$$\mathcal{L}_2 \cong H^*(\mathcal{L}_1, d^1)$$

Checking in detail that this construction precisely implements the first off-diagonal line of an adapted matrix representation of the differential is somewhat cumbersome and will be an important exercise. But by the same observations as before one can now see that the elements $z \in \mathcal{L}_2^i$ all admit a representative $x \in F_i$ (i.e $[x]_2 = z$) with $d(x) \in F_{i+2}$. We can set try to set

$$d^2([x]_2) = [p_{i+2}d(x)]_2$$

and so on and so forth. Checking the well-definedness of such an iterative definition gets ever more messy. A better i.e. more direct description of the $\mathcal{L}_i$ is needed.

To this end consider the 'diagram':

$$\begin{tikzcd}
{} \arrow[r, "i"] & H^*(F_{i+1}) \arrow[r, "i"] \arrow[d, "p_{i+1}"] & H^*(F_i) \arrow[r, "i"] \arrow[d, "p_i"] & H^*(F_{i-1} \arrow[r, "i"] \arrow[d, "p_{i-1}"'] & {} \\
& H^*(F_{i+1}/F_{i+2}) \arrow[ul, "\delta"] & H^*(F_i/F_{i+1}) \arrow[ul, "\delta"] & H^*(F_{i-1}/F_i) \arrow[ul, "\delta"] &
\end{tikzcd}$$

Note that it makes no sense to ask it to commute! It is instead each triangle is exact in each of its three spots. Such a triangle of two $\mathbb{Z}$-graded objects is called an *exact couple* and spectral sequences can in fact formally be derived from exact couples, i.e. there is a functor from the category of exact couples to that of spectral sequences, which we will essentially construct now.

4.2 Construction

Here now is the direct definition of the spectral sequence:

Definition 4.1. Let
$$Z^n = \delta^{-1}(im(i^n)) \subseteq \bigoplus_i H^*(F_i/F_{i+1})$$
and
$$B^n = p(ker(i^n)) \subseteq \bigoplus_i H^*(F_i/F_{i+1}).$$

Clearly, Z^n consists of precisely those elements of $\mathcal{L}_1 = H^*(\mathrm{Gr}_F C, d^1)$ whose boundary admits a representative from n filtration steps up, whereas by exactness B^n consists of those elements that become boundaries n filtration steps steps down.

Observation 4.2. *We have*
$$0 = B^0 \subseteq B^1 \subseteq \cdots \subseteq B^n \subseteq \mathrm{im}(p) = \ker(\delta) \subseteq Z^n \subseteq \cdots \subseteq Z^1 \subseteq Z^0 = \bigoplus_i H^*(F_i/F_{i+1})$$
for all $n \geq 1$.

Definition 4.3. Set $\mathcal{L}_i = Z_{i-1}/B_{i-1}$ with its grading induces from the filtration degree.

Now for the differential: Given $x \in Z_i^n \subseteq H^*(F_i/F_{i+1})$, we have $\delta(x) = i^n(y)$ for some y and we set
$$d^{n+1}([x]_{n+1}) = [p(y)]_{n+1}.$$

Proposition 4.4. *The map $d^{n+1} : \mathcal{L}_{n+1} \to \mathcal{L}_{n+1}$ is well-defined and satisfies $d_{n+1} \circ d_{n+1} = 0$.*

Proof. $d^{n+1} : Z^n \to Z^n/B^n$ is independent of the choice of y: if both y and y' satisfy $i^n(y) = \delta(x) = i^n(y')$, then $p(y) - p(y') = p(y - y') \in B^n$ since $i^n(y - y') = 0$.

$d^{n+1}(B^n) = 0$: If $x \in B^n$, there exists y with $i^n(y) = 0$ and $p(y) = x$. But then
$$\delta(x) = \delta(p(y)) = 0 = i^n(0)$$
so
$$d^{n+1}(x) = p(0) = 0.$$

$(d^{n+1})^2 = 0$: $d_{n+1}(x) = p(y)$ where $\delta(x) = i^n(y)$. But again $\delta(p(y)) = 0 = i^n(0)$ so
$$d_{n+1}d_{n+1}(x) = d_{n+1}(p(y)) = p(0) = 0.$$

□

The final pieces of data are the necessary isomorphisms $h_n : H^*(\mathcal{L}_n, d^n) \to \mathcal{L}_{n+1}$. This is clear by definition for the passage from the first to the second page. For the higher ones we have:

Proposition 4.5.
$$\ker(d_{n+1}) = Z^{n+1}/B^n \textit{ and } \mathrm{im}(d_{n+1}) = B^{n+1}/B^n$$

Proof. $\subseteq$: Let $d_{n+1}(x) = 0$. That means $p(y) \in B^n$ for some y with $i^n(y) = \delta(x)$. By definition of B^n this in turn means $p(y) = p(y')$ for some y' with $i^n(y') = 0$. But then $p(y - y') = 0$, so there exists a z with $i(z) = y - y'$. But now $i^{n+1}(z) = i^n(y - y') = \delta(x)$ which shows the claim. Now let $x = d_{n+1}(z)$. This means $x = p(y)$ for some y with $i^n(y) = \delta(z)$. But then $i^{n+1}(y) = i(i^n(y)) = i(\delta(z)) = 0$, which shows $p(y) \in B^{n+1}$. $\supseteq$: Let $x \in Z^{n+1}$. That means $\delta(x) = i^{n+1}(y)$. But then $d_{n+1}(x) = p(i(y)) = 0$. Now let $x \in B^{n+1}$. That means $x = p(y)$ for some y with $i^{n+1}(y) = 0$. In particular, $i(i^n(y)) = 0$ so there exists z with $\delta(z) = i^n(y)$. But then $d_{n+1}(z) = p(y) = x$. □

One of the isomorphism theorems now yields the desired isomorphism. So we indeed have a spectral sequence.

4.3 Convergence

Let us turn to convergence, first we identify the groups of permanent cycles and eventual boundaries.

Proposition 4.6.

$$Z_1^n = Z^n \text{ and } B_1^n = B^n$$

holds for all $n \geq 2$.

Here the left hand sides denote the groups that are defined intrinsically through the spectral sequence $\mathcal{L}(C, F)$ whereas the right hand side denotes the groups we used to construct it.

Proof. Induction on n: For $n = 1$ we have $Z_1^1 = ker(d^1)$ by definition but by the previous proposition we have $ker(d^1) = Z^1/B^0 = Z^1$, since $B^0 = 0$. For the induction step note that for $pi : Z_1^n \longrightarrow E_{n+1}$ we have $Z_1^{n+1} = \pi^{-1} ker(d^{n+1})$ by a slight reformulation of the original definition. But by induction $Z_1^n = Z^n$, by definition $E_{n+1} = Z^n/B^n$ and π is just the projection by construction. Thus we find

$$Z_1^n = \pi^{-1}(ker(d^{n+1})) = \pi^{-1}(Z^{n+1}/B^n) = Z^{n+1}.$$

The argument for the boundaries is similar. □

Corollary 4.7.

$$Z_1^\infty = \delta^{-1}(\bigcap_n \mathrm{im}(i^n)) \text{ and } B_1^\infty = p(\bigcup_n \ker(i^n))$$

Moreover,

$$(B_1^\infty)^i = p_i(\ker(i^\infty\colon H^*(F_i) \longrightarrow H^*(F_{-\infty})$$

or in short (and slightly abusive) $B^\infty = p(\ker(i^\infty))$.

Proof. The first statements are clear from the definitions of Z^n and B^n and the previous proposition. The second statement follows since sequential (filtered?) colimits commute with kernels, so

$$p(\bigcup_n \ker(i^n\colon H^*(F_i) \longrightarrow H^*(F_{i-n}))) = p(\ker(i^\infty\colon H^*(F^i) \longrightarrow \mathrm{colim}_n H^*(F_{i-n})))$$

and the sequential (again filtered?) colim functor is exact, so

$$p(\ker(i^\infty\colon H^*(F^i) \longrightarrow \mathrm{colim}_n H^*(F_{i-n}))) = p(\ker(i^\infty\colon H^*(F_i) \longrightarrow H^*(F_{-\infty})$$

□

Remark 4.8. Neither of the two reformulations of B_1^∞ of the proof hold for the cycles in general (which is the source of all convergence trouble for the present spectral sequence!): In general limits do not commute with images as we shall see momentarily and the sequential lim-functor is only left exact (by adjointness). Examples of non-trivial right derived functors of lim we will see left and right in the chapter on convergence, so let us refrain from providing counterexamples here. A simple counterexample to the first statement at the level of groups is given by (p a prime, $p \neq 2$)

$$\ldots 8\mathbb{Z} \subset 4\mathbb{Z} \subset 2\mathbb{Z} \subset \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0 \subseteq 0 \subseteq 0 \ldots$$

where all composition with target $\mathbb{Z}/p$ are surjective, since 2 is a unit in $\mathbb{Z}/p$, so the intersection of their images is $\mathbb{Z}/p$ in its entirety, whereas the intersection (a.k.a. the limit) of the sequence is 0 so its image in $\mathbb{Z}/p$ vanishes as well. This long exact sequence implemented as H^1 for example by the following filtration of chain complexes with at most 2 non-zero groups:

$$\begin{array}{cccccccc}
\ldots & 0 & \subseteq 0 & \subseteq 0 & \subseteq p\mathbb{Z} & \subset \mathbb{Z} & \cdots = & C^0 \\
 & \downarrow & \downarrow & \downarrow & \downarrow{\scriptstyle i} & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle d^0} \\
\ldots & 4\mathbb{Z} & \subset 2\mathbb{Z} & \subset \mathbb{Z} & \subseteq \mathbb{Z} & \subseteq \mathbb{Z} & \cdots = & C^1
\end{array}$$

Now we want to relate these groups to the obvious filtrations on the cohomology of C (or rather $F_{-\infty}$).

Definition 4.9. Let (C, F) be a filtered chain complex. Define a filtration on the graded group $H^*(C)$ by $H(C; F)_i = \mathrm{im}(i^\infty : H^*(F_i) \longrightarrow H^*(C))$. Furthermore, let $HF_i = H(F_{-\infty}, F)_i \subseteq H^*(F_{-\infty})$.

Clearly the groups $H(C, F)_i$ form a filtration of $H^*(C)$. Note also that the groups really depend on C and not just on the filtration itself: The map obvious map $HF_i \to H(C, F)_i$ is always surjective, but may fail to be injective (unless the filtration is exhaustive of course). Usually we will be interested in the $H^*(C)$ filtered by $H(C, F)$ but the spectral sequence only interacts with $H^*(F_{-\infty})$ filtered by HF.

We can now construct the abutment, which will take the form

$$p\colon \mathrm{Gr}_{HF} H^*(F_{-\infty}) \to \mathcal{L}_\infty(C, F).$$

Note that the inclusion induces a surjective map $\mathrm{Gr}_{HF} H^*(F_{-\infty}) \to \mathrm{Gr}_{H(C,F)} H^*(C)$ which may fail to be injective and thus does not directly relate to $\mathcal{L}_\infty(C, F)$ unless the abutment p is an isomorphism.

To construct the abutment we have

Proposition 4.10. *We have:*

1. *The map $p_i : H^*(F_i) \longrightarrow H^*(F_i/F_{i+1}) = \mathcal{L}_2^i(C, F)$ takes values in the subgroup $(Z^\infty)^i$.*

2. $p_i : H^*(F_i) \longrightarrow (Z^\infty)^i \longrightarrow \mathcal{L}^i_\infty(C, F)$ *factors over* $HF_i \subseteq H^*(F_{-\infty})$.

3. *The restriction of* $p_i : HF_i \longrightarrow \mathcal{L}^i_\infty(C, F)$ *vanishes on* $HF_{i+1} \subseteq H^*(F_{-\infty})$.

Proof. The first assertion follows from $ker(\delta) \subseteq Z^\infty$ apparent from the previous proposition and $\delta \circ p = 0$. For the second assertion note that the kernel of the map $H^*(F_i) \longrightarrow HF_i \subseteq H^*(F_{-\infty})$ consists by definition of $ker(i^\infty) : H^*(F_i) \longrightarrow H^*(F_{-\infty})$ and $p_i(ker(i^\infty)) = (B^\infty)^i$ by the previous proposition. For the third assertion note that $H^*(F_{i+1}) \longrightarrow HF_{i+1}$ is surjective, so that it suffice to check that the composition

$$H^*(F_{i+1}) \longrightarrow HF_{i+1} \longrightarrow HF_i \xrightarrow{p_i} \mathcal{L}^i_\infty(C, F)$$

vanishes. But this composite clearly equals

$$H^*(F_{i+1}) \xrightarrow{i} H^*(F_i) \xrightarrow{p_i} (Z^\infty)^i \longrightarrow \mathcal{L}^i_\infty(C, F)$$

and already the composition of the first two maps vanishes. □

Theorem 4.11 (Leray-Serre)**.** *The assignment* $\mathcal{L}$ *constitutes a functor* $\mathcal{L} : \mathfrak{FCC} \to \mathfrak{SS}$, *together with a (tautological) natural isomorphism*

$$\mathcal{L}_1(C, F) \cong H^*(\mathrm{Gr}_F C)$$

and natural abutment

$$p : \mathrm{Gr}_{HF} H^*(F_{-\infty}) \to \mathcal{L}_\infty(C, F)$$

Furthermore, the filtration HF *always exhausts* $H^*(F_{-\infty})$, *the map* p *is always injective and surjective iff*

$$\delta^{-1}(0) = \delta^{-1}(\bigcap_n \mathrm{im}(i^n))$$

or in other words $\mathrm{im}(p) = Z^\infty_2$.

Only the injectivity and surjectivity statements remain to be checked, but first the most common application:

Corollary 4.12. *If in each degree an exhaustive filtration* F *of a chain complex* C *eventually reaches* 0 *(that is at a finite stage!), we have a weakly convergent spectral sequence*

$$H^*(Gr_F C) \Rightarrow H^*(C)$$

and the induced filtration HF *on* $H^*(C)$ *is both exhaustive and Hausdorff.*

Proof. By the theorem we only need to check the condition on the cycles for convergence and this can be done degreewise. But in a single degree i^n is zero for large enough n by assumption. Hausdorffness of HF is clear since in each degree HF reaches zero whenever F did. □

Proof of the theorem. Everything follows from:

$$\frac{HF_j}{HF_{j+1}} \xleftarrow{\cong} \frac{H^*(F_j)}{\mathrm{im}(i) + \ker(i^\infty)} \xrightarrow{p_j} \frac{\mathrm{im}(p_j)}{p_j(\ker(i^\infty))} = \frac{\mathrm{im}(p_j)}{(B^\infty_2)^j} \subseteq \frac{(Z^\infty_2)^j}{(B^\infty_2)^j} = \mathcal{L}^j_\infty(C, F)$$

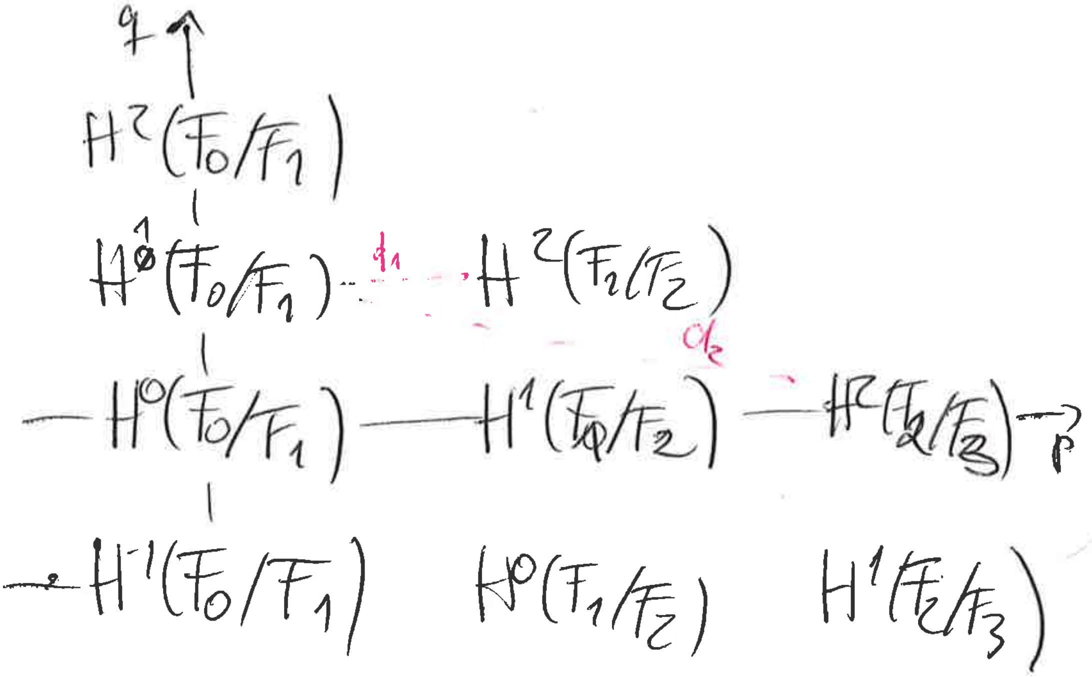
since the map labelled p_j is an isomorphism. □

4.4 Examples

Next we want to give some instructive examples to see the inner workings of the spectral sequence in action. In the last section we developed everything suppress the internal grading. If we do not supress this and use the Serre grading as we are used to, the spectral sequence takes the form

$$\mathcal{L}_1^{p,q} = H^{p+q}(F_p/F_{p+1}) \Rightarrow H^{p+q}(F_{-\infty})$$

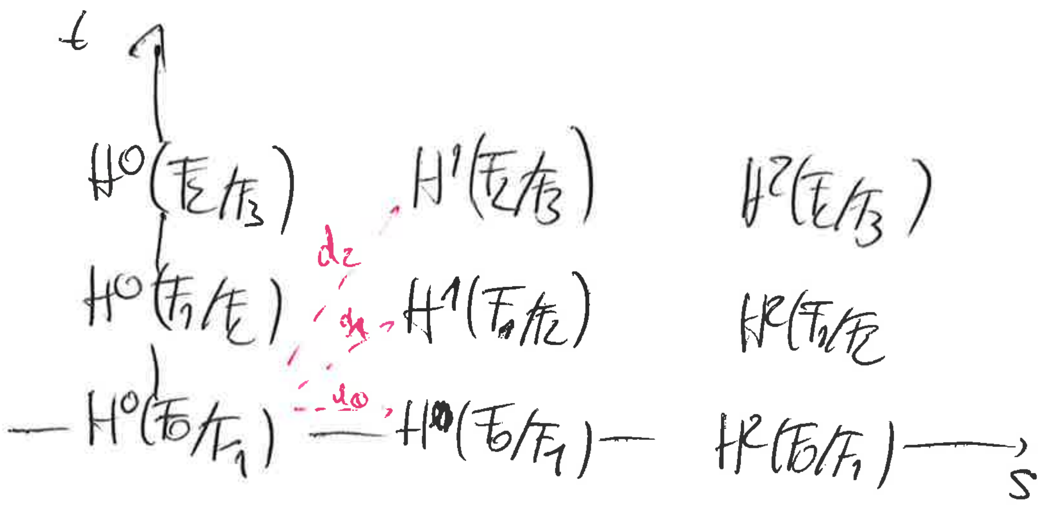
and could be pictured as before on a plane with p on the x-axis and q on the y-axis But note that it can now be arbtritrarly placed and is not first quadrant. This would result in the following picture:



In this picture the r-th differential has bidegree $(r, 1-r)$ as before. We realize that it is a bit more convienent to have the x axis be $p+q =: s$ and the y-axis be $p =: t$. Then the spectral sequence takes the form

$$\mathcal{L}_1^{s,t} = H^s(F_t/F_{t+1}) \Rightarrow H^s(F_{-\infty})$$

and the axis are given by s and t:



in which the r-th differential has bidegree $(1, r)$. This is the *Adams grading* which will be more convenient for us now. It is a good excercise for the reader to do the examples that we will give now also in Serre-grading.

Example 4.13. *Let C be the cochain complex given by $C^0 = \mathbb{Z}, C^1 = \mathbb{Z}^2, d^0 = \begin{pmatrix} a & b \end{pmatrix}$ and $C^i = 0$ otherwise, i.e.*

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{(a,b)} \mathbb{Z}^2 \to 0 \to \ldots$$

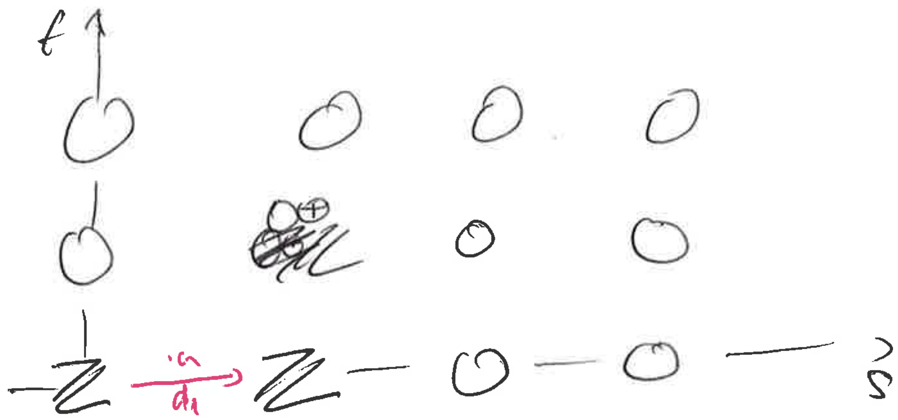
filtered by $F_2 = 0, F_1^0 = 0, F_1^1 = 0 \oplus \mathbb{Z}$ and $F_0 = C$ (which forces everything else). Thus we have

$$\begin{aligned} F^1 &= \qquad \ldots \to 0 \to 0 \to 0 \oplus \mathbb{Z} \to 0 \to \ldots \\ F^0/F^1 &= \qquad \ldots \to 0 \to \mathbb{Z} \xrightarrow{a} \mathbb{Z} \to 0 \to \ldots \end{aligned}$$

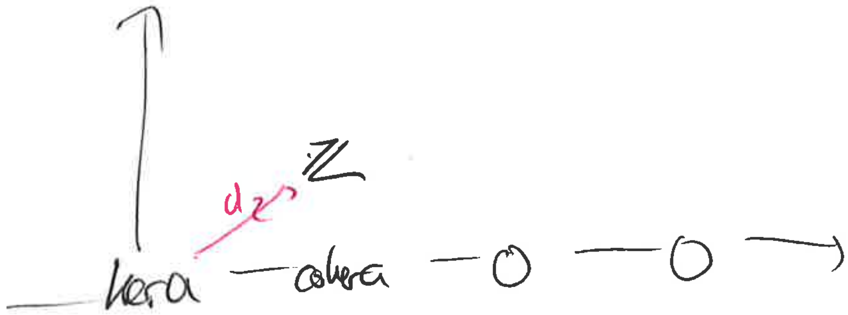
with the obvious inclusion into $F^0 = C$. This is clearly a filtration that is degree-wise split (by $\mathbb{Z} \oplus 0$, in the only position where there is a choice, namely $F_1^1 \subseteq F_0^1$) and the associated matrix for d_1 is

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

Lets look at the associated spectral sequence in Adams grading. The 0-th page is given as follows:



Since we know the differentials explicitly we obtain that the first page is given as



Let us now assume that $a \neq 0$. Then $\ker a = 0$ and no further differentials are possible. In particular, the spectral sequence is entirely independent of b in the sense that, $\mathcal{L}(C, F)$ are canonically isomorphic for any two choices of b (but the same choice of a!).

From these considerations we can deduce that $H^1(C)$ sits in an extension

$$\mathbb{Z} \to H^1(C) \to \mathbb{Z}/a \ . \tag{1}$$

with the subgroup generated by a cycle $f \in F_1^1$ with $p_1([f]) = [p_1(e_2)]_\infty$ and quotient by a cycle $e \in F_0^1$ with $p_0([f]) = [p_0(e_1)]_\infty$. Since $F_2 = 0$ this determines f uniquely by this, but e is far from well-defined: $e + f$ for example is a perfectly valid choice! This is a fundamental observation: From neither the spectral sequence nor its abutment witness can we determine the element e, there will always be a choice involved, if we define it using the spectral sequence!

This is a good example of the 'propagation of information and missing information'-principle which makes spectral sequences so useful: Even if we did not know the value of b for whatever reason, the spectral sequence still let us derive a lot of information about $H^*(C)$. Of course we can easily compute $H^1(C)$ from the chain complex C: it is given by $\mathbb{Z} \oplus \mathbb{Z}/(a, b)$ and the filtration HF is given by

$$0 \subseteq 0 \oplus \mathbb{Z} \subseteq \mathbb{Z}^2/(a, b).$$

Using the basis

$$(m, -n) \quad \tfrac{1}{\gcd(a,b)}(a, b)$$

for $C^1 = \mathbb{Z}^2$, *where* $na + mb = \mathrm{gcd(a, b)}$ *gives an isomorphism*

$$H^1(C) = \mathbb{Z}^2/(a, b) \simeq \mathbb{Z} \oplus \mathbb{Z}/\mathrm{gcd(a, b)} \ .$$

Under this isomorphism the short exact sequence (1) *can be also easily expressed*

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/\mathrm{gcd(a, b)} \to \mathbb{Z}/a \ .$$

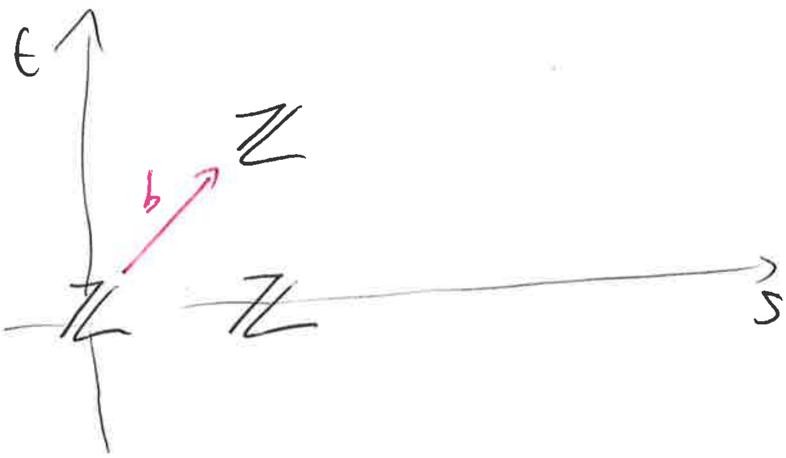
which we leave to the reader. In particular the extension is for some b *trivial and for some* b *non-trivial.*

This shows that spectral sequence really do not 'know' about their extension problems. However, the spectral sequence does tell us where to look in the original chain complex: We had $a \cdot [p_0(e_1)]_2 = 0$ *since* $d^1([p_0(1)]_1) = a[p_0(e_1)]_1$. *So to find out whether* $0 = a[e_1] \in H^1(C)$, *we have to compute* $d(1)$ *in the original chain complex Again note, that the actual value* $d(1)$ *is not contained anywhere in the spectral sequence, only in the original complex.*

When $a = 0$, *the spectral sequence does depend on* b *(which is unlucky if one knew* a *but not* b *for whatever reason). Its second page is isomorphic to the first and generally by definition*

$$d^1([p_i(z)]_1) = [p_{i+1}d(z)]_1.$$

Thus $d^1([p_0(1)]_1) = b[p_1(e_2)]_1$ *i.e. the first page is given by*



and (unless $b = 0$ *as well) the second page is given as*

Again we can deduce that $H^1(C)$ sits in an extension

$$\mathbb{Z}/b \to H^1(C) \to \mathbb{Z}$$

with the subgroup generated by a cycle $f \in F_1^1$ with $p_1([f]) = [p_1(e_2)]_\infty$ and quotient by a cycle $e \in F_0^1$ with $p_0([f]) = [p_0(e_1)]_\infty$. And again this determines f uniquely, but not e. However, this time the sequence splits, any choice of e gives an isomorphism $H^*(C) \cong \mathbb{Z} \oplus \mathbb{Z}/b$. In both examples we see the philosophy about differentials being off-diagonals!

Example 4.14. *Let $C^0 = \mathbb{Z}^2$ and $C^1 = \mathbb{Z}^2$ as well, with all other groups 0 with differential*

$$\begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}.$$

The filtration shall be given by $F_3 = 0, F_2^0 = 0, F_2^1 = 0 \oplus \mathbb{Z}, F_1^0 = 0 \oplus \mathbb{Z}, F_1^1 = C^1$ and $F_0 = C^1$:

$$\begin{array}{ll} F_2 & \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \oplus \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots \\ \\ F_1 & \ldots \longrightarrow 0 \longrightarrow 0 \oplus \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \ldots \\ \\ F_0 & \ldots \longrightarrow 0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \ldots \end{array}$$

Again the filtration is split and the matrix associated to the obvious choice is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & a & 0 \end{pmatrix}$$

The 0-th page of the spectral sequence looks as follows:

We find that the second page to consist of a single $\mathbb{Z}$ in the 0-column generated by $[p_0(e_1)]_1$ and another in the 1-column given by $[p_2(e_2)]_1$ [3]*. There is no room for differentials, so the third page looks the same. Now following the pattern*

$$d^2([p_i(z)]_2) = [p_{i+2}(d(z))]_2$$

so $d^2([p_0(e_1)]_2) = [p_2(e_1)]_2$, which (given the picture) surely means that d^2 is an isomorphism. However, upon closer inspection one finds $e_1 \neq F_2^1$, whence $p_2(e_1)$ has no meaning.

What has gone wrong? The answer is that the formula for d^2 is only valid on elements of Z^1! Since $e_1 \in F_0^0$ and $d(e_1) = e_1 \notin F_2^1$ there is no reason it should apply to $p_0(e_1)$ in the 0-column.

Why does this problem occur? The fact that $e_1 \in F_0^0$, whereas $d(e_1) \in F_1^1$ tells us that $d^1([p_0(e_1)]_1) = [p_1(d(e_1))]_1$ and the fact that $d(e_1) \neq F_2^1$ ought to tell us that this differential is non-trivial. However, an earlier differential (namely $d^0([p_1(e_2)]_0) = [p_1(e_1)]_0$) took out the target already, thus letting $[p_0(e_1)]_1$ make it to the third page somewhat undeservedly.

There are two ways out of this misery: One is to take a closer look at the actual definition of d^2: First the boundary map δ is applied to the class $[p_0(e_1)] \in H^0(F_0/F_1)$ giving (by definition of δ) the element $[d(e_1)] = [e_1] \in H^1(F_1)$ and then we search for a preimage

[3] Recall that $p_i : H^*(F_i) \to H^*(F_i/F_{i+1})$

(which is guaranteed to exist since $\ker(d^2) = \mathbb{Z}^2$*) of this in* $H^1(F_2)$ *to which we apply* p_2*. Now* $H^1(F_2) = \mathbb{Z}^2/(1,a)$ *and* $H^1(F_2) = 0 \oplus \mathbb{Z}$*. Thus* $i(-a[e_2]) = -a[e_2] = [e_1]$*, so in truth the differential is given by*

$$d^2([p_0(e_1)]_2) = -a[p_2(e_2)]_2$$

This approach required quite detailed knowledge of the original complex and its filtration, however, something we are desperately trying to avoid. It cannot of course not be avoided entirely (what else should the differential be determined by, after all), but one can also let the spectral sequence guide us once more: What kept us from applying the general formula was that $d(e_1) \notin F_2^1$*, which meant that* $[p_0(e_1)]_1$ *only survived to the third page because its differential had already been killed by an earlier differential.*

This means that there exists an element e in a strictly smaller (i.e higher) filtration with $d(e) - d(e_1) \in F_3^1$ *representing the source of that earlier differential. But then the formula applies to* $e_1 - e$ *and* $[p_0(e_1 - e)]_2 = [p_0(e_1)]_2$ *as* $p_0(e) = e$ *living in higher filration. If we look for it we find* $d^0([p_1(e_2)]_0) = [p_1(e_1)]_0$ *in the spectral sequence making* e_2 *the obvious candidate. Indeed* $e_2 \in F_1^0$ *and* $d(e_1 - e_2) = (1,0) - (1,a) = -ae_2 \in F_2^1$*. Thus we can apply the formula and find*

$$d^2([p_0(e_1)]_2) = d^2([p_0(e_1 - e_2)]_2) = [p_2(d(e_1 - e_2))]_2 = -a[e_2]_2$$

Correcting representatives by elements in smaller filtration, when these representatives only survived accidentally, to compute differentials is an important organising technique for calculations with the spectral sequence of a filtered complex (as hopefully you noticed while solving exercise 4 on the third sheet).

Example 4.15. *Loss of hausdorffness: Filter* $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$ *by* $q^i\mathbb{Z}$ *on both sides. In case* $p \neq q$ *the filtration* HF *is no longer Hausdorff, but the spectral sequence still converges (though this requires explicit checking which ought to be done!) I'll hopefully write this out tmr, bu it's simple enough. Something similar happens when one filters only the target this way (that's probably a good exercise). We don't yet have an example where even weak convergence fails. If you also do double complexes tmr we can also let them show balancing of Tor.*

Example 4.16. *Here we give a first indication of a proof of Theorem 2.6: We start with a fibre sequence. We can assume without loss of generality that* $Y \to X$ *is a Serre-fibration over a CW basis* X *with skeleta* $X_n \subseteq X$ *and filter the singular chain complex* $C^*(Y)$ *by setting*

$$F_pC^*(Y) = \{x \in C^*(Y) \mid i_n^* x = 0\}$$

where i_n *also denotes the map* $i_n^* Y \times_X X_n = p^{-1}(X_n) \to Y$*. This clearly defines a filtration that exhausts* $C^*(Y)$ *and is Hausdorff (even finite degreewise). We get a spectral sequence*

$$E_1^{p,q} = H^{p+q}(F_pC^*(Y)/F_{p+1}C^*(Y) \Rightarrow H^{p+q}(Y;M)$$

The claim is that this leads to the Serre spectral sequence. We will show this soon, even more generally stated then in Theorem 2.6. But since we will need to talk about local coefficients there this has to wait for a bit.

4.5 Double complexes

Recall that a double complex $C = C^{\bullet,\bullet}$ is family of abelian groups $C^{i,j}$ for $i, j \in \mathbb{Z}$ together with differentials $d^h : C^{i,j} \to C^{i+1,j}$ and $d^v : C^{i,j} \to C^{i,j+1}$ such that $d^h \circ d^h = d^h \circ d^v + d^v \circ d^h = d^v \circ d^v = 0$ (be aware that there are sometimes slightly differing sign conventions).

Definition 4.17. For a double complex C we define its total complex $\mathrm{Tot}C$ as the cochain complex with k-th summand

$$(\mathrm{Tot}C)^k := \bigoplus_{i+j=k} C^{i+j}$$

and with differential $d = d^h + d^v$.

We check that that this really defines a cochain complex, i.e. that the differential squares to zero:

$$(d^h + d^v)^2 = (d^h)^2 + d^h d^v + d^v d^h + (d^v)^2 = 0.$$

For all i, the i-th column of the double complex C is a cochain complex

$$(C^{i*}, d^v): \qquad \ldots \xrightarrow{d^v} C^{i,j-1} \xrightarrow{d^v} C^{i,j} \xrightarrow{d^v} C^{i,j+1} \xrightarrow{d^v} \ldots$$

and we can take its cohomology $H^*(C^{i*}, d^v)$. Then the horizontal differential d^h induces maps

$$d^h : H^*(C^{i*}) \to H^*(C^{i+1,*})$$

since it sends cycles to cycles and boundaries to boundaries (it is not a chain map though since there are sign issues). We get that this new d^h also square to zero, so that we can take homology again. We denote the resulting groups by

$$H^*(H^*(C, d^v), d^h)$$

We want to relate those groups to the cohomology of $\mathrm{Tot}C$.

Warning 4.18. *Similarly we have groups $H^*(H^*(C, d^h), d^v)$. These groups are not the same as the example of the following double complex shows*

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r, "2"] \arrow[u, "\mathrm{id}"] & \mathbb{Z} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

where the lower left $\mathbb{Z}$ is in bidegree $(0,0)$. Then we have $H^*(H^*(C, d^v), d^h)$ a single $\mathbb{Z}$ in bidegree $(1,0)$ and $H^*(H^*(C, d^h), d^v)$ a $\mathbb{Z}/2$ in bidegree $(1,0)$ and a $\mathbb{Z}$ in bidegree $(0,1)$. The cohomology of the total complex is a $\mathbb{Z}$ in degree 1.

Now we want to consider a filtration of the complex $\mathrm{Tot}C$. This will come from a filtration of double complexes defined as

$$(F_pC)^{ij} = \begin{cases} C^{i,j} & \text{for } i \geq p \\ 0 & \text{else} \end{cases}$$

and such that the inclusion $F_pC \subseteq C$ is a map of double complexes. In other words, we are just taking the part of the double complex which lies to the right of the vertical line through

$(p,0)$. Now we look at the spectral sequence $\mathcal{L}$ induced from the filtration $\mathrm{Tot}F_pC \subseteq \mathrm{Tot}C$. We have by construction that (in Serre grading)

$$\mathcal{L}_0^{p,q} = \mathrm{Tot}(F_p/F_{p+1})^{p+q} = C^{p,q}$$

i.e. $\mathcal{L}_0 = C$ and the differential d_0 is the vertical differential in the double complex. Thus $\mathcal{L}_1 = H^*(C, d^v)$. By construction the first differential in the spectral sequence is given by the differential in the total complex applied to cycles for the vertical differential. Since the differential in the total complex is $d^h + d^v$ this shows that the differential d_1 on the first page is induced by the horizontal differential. As a result we find that $\mathcal{L}_1^{p,q} \simeq H^*(H^*(C,d^v),d^h)$. We will write the p,q term of this double graded abelian group as $H^p(H^q(C,d^v),d^h)$ but note that this notation does strictly speaking not make any sense. We also observe that the filtration $\mathrm{Tot}F_pC$ is exhaustive and Hausdorff, since every element in the total complex is given by a finite sum of elements in homogenous degrees. Thus we have proven:

Theorem 4.19. *For every double complex C There are spectral sequences*

$$H^p(H^q(C,d^v),d^h) \Rightarrow H^{p+q}(\mathrm{Tot}C)$$
$$H^q(H^p(C,d^h),d^v) \Rightarrow H^{p+q}(\mathrm{Tot}C)$$

If C is first or third quadrant, then they are weakly convergent (in fact convergent in the sense of our first lectures).

Note that in the theorem the second spectral sequence is the dual of the first. The way we have stated the differentials would also go the other way around (i.e. of bidegree $(1-r,r)$). We will just take the convention that we reflect the double complex for the second one on the middle, i.e. think of the q-coordinate as the x-axis and the p-coordinate as the y-axis. Then it is just the same as the first.

5 Local coefficients and Dress' construction

We now want to quickly recall (co)homology with local coefficients to construct the Serre spectral sequence for a fiber sequence $F \to Y \to X$ without the assumption that X is simply connected. In fact we will first construct a homological version of the Serre spectral sequence, which will (for X simply connected) be a spectral sequence

$$H_p(X, H_q(F)) \Rightarrow H_{p+q}(Y)$$

where this is in homological grading, which means that the r-th differential is of degree $(-r, r-1)$. The best way to remember this, if you are used to the cohomological grading, to which we want to stick now, is to draw this in the first quadrant. Then the filtration as well as the differential go as we are used to

picture

The construction can then easily be dualized to give the cohomological version.

5.1 (Co)homology with local coefficients

Definition 5.1. A local system M of abelian groups on a topological space X consists of the assignment of an abelian group M_x for each $x \in X$ and for every path $\gamma : I \to X$ a map $M(\gamma) : M_{\gamma(0)} \to M_{\gamma(1)}$ such that $M(\mathrm{id}) = \mathrm{id}$, $M(\gamma * \gamma') = M(\gamma) \circ M(\gamma')$ and for homotopic paths $\gamma \simeq \gamma'$ we have $M(\gamma) = M(\gamma')$.

The following proposition is essentially a reformulation of the definition.

Proposition 5.2. *The (obviously defined) category of local systems on a space X is equivalent to functors $\Pi_1 X \to \mathrm{Ab}$ where $\Pi_1 X$ is the fundamental groupoid of X and Ab is the category of abelian groups.*

As a result we see that for a connected space X local systems on X are essentially the same as abelian groups with an action of the fundamental group $\pi_1(X)$. Moreover for simply connected spaces the category of local systems on X is equivalent to the category of abelian groups. More precisely for every space X and every abelian group M there is the constant local system with value M. This will also be denoted by M and we get that all local system over a simply connected space are up to equivalence of this form.

Example 5.3. *Let $Y \to X$ be a Serre fibration and $q \in \mathbb{Z}$. Then define a local system by considering for $x \in X$ the abelian group $H_p(Y_x)$ where $Y_x \subseteq Y$ is the preimage of the point $x \in X$. For a path γ in X we get a map $Y_{\gamma(0)} \to Y_{\gamma(1)}$ in the homotopy category $\mathrm{Ho(Top)}$. This maps is well defined up to homotopy by the lifting properties of $Y \to X$. Moreover for a composition of paths in the base we get that the lifted maps are the compositions. More precisely we get for every fibration $Y \to X$ a functor*

$$\Pi_1 X \to \mathrm{Ho(Top)} \ .$$

Concretly one can describe the map $Y_{\gamma(0)} \to Y_{\gamma(1)}$ as the zig-zag $Y_{\gamma(0)} \leftarrow Y_\gamma \to Y_{\gamma(1)}$ which gives a map in $\mathrm{Ho(Top)}$ as both maps are weak equivalences. Now we can define a map $H_q(Y_{\gamma(0)}) \to H_q(Y_{\gamma(1)})$ by pushforward along this map. This will define a local system on X.

For an arbitrary map of spaces $Y \to X$ (not necessarily a fibration) there is a local system $x \mapsto H_q(\mathrm{hofib}_x)$. This can be obtained by replacing by a fibration.

Example 5.4. *Let M be a topological manifold of dimension n. Then we have the local system $M_x = H_n(M, M\backslash\{x\})$. For a path γ we get an induced map $M_{\gamma(0)} \to M_{\gamma(1)}$ by covering the path γ by contractible open sets U_i and using that $H_n(M, M \setminus \{x\}) \cong H_n(M, M \setminus U_i)$ iteratively. This twist will be denoted by $\mathbb{Z}^{w_1 M}$ (can you think of a reason?).*

Now we want to define (co)homology with coefficients in a local system M on X. To this end we define the singular chains

$$C_n(X; M) := \bigoplus_{\sigma:\Delta^n \to X} M_{\sigma_0}$$

where σ_0 is the point obtain by the 0-th vertex of Δ^n. Definition (co)homology with local coefficients. Now lets define a differential

$$d : C_n(X; M) \to C_{n-1}(X; M)$$

on this chain complex as follows. We will write an element $m \in M_{\sigma_0}$ as a pair (σ, m) for simplicity. Also recall that the i-th face maps $d^i : \Delta^{n-1} \to \Delta^n$ is given by the unique affine linear map which sends the vertices $v_0, ..., v_{i-1}$ to $v_0, ..., v_{i-1}$ and $v_i, .., v_{n-1}$ to $v_{i+1}, .., v_n$. With this terminology the differential in the chain complex is defined as

$$d(\sigma, m) = \big(\sigma \circ d^0, (\sigma_{0,1})_*(m)\big) + \sum_{i=1}^{n} (-1)^i (\sigma \circ d^i, m)$$

where $d^i : \Delta^{n-1} \to \Delta^n$ is the i-th face map and $\sigma_{0,1}$ is the composition $\Delta^1 \to \Delta^n \to X$ where the first map is the straight line connecting the 0-th vertex of Δ^n and the first vertex of Δ^n. Note that all the faces $\sigma \circ d^i$ have the same 0-th vertex as σ.

Lemma 5.5. *This actually defines a chain complex, i.e. the differential squares to zero.*

Proof. TODO □

Moreover its obvious that in the case of the constant local coefficient system this chain complex is just the ordinary singular chain complex of X. Clearly $C_*(X, M)$ is functorial in both variables: if there is a map of local systems $M \to N$ then we get an induced map $C_*(X, M) \to C_*(X, N)$. For a map of spaces $f : X \to X'$ we get a pullback local system f^*M and a map $C_*(X, f^*M) \to C_*(X, M)$.

Definition 5.6. For a local system M on X we define $H_*(X, M)$ as the homology of the chain complex $C_\bullet(X, M)$.

We now similary want to define the cochain complex $C^\bullet(X; M)$ for a local system M. This is defined as

$$C^n(X; M) = \prod_{\sigma : \Delta^n \to X} M_{\sigma_0} \ .$$

We think of an element in this product as a map f which assigns to every simplex $\sigma : \Delta^n \to X$ an element in $f(\sigma) \in M_{\sigma_0}$. Then we can define the differential $d : C^n(X; M) \to C^{n+1}(X; M)$ as

$$(df)(\sigma) = M(\sigma_{0,1})^{-1}\Big(f(d^0\sigma)\Big) + \sum_{i=1}^{n} (-1)^i f(d^i\sigma)$$

Definition 5.7. Cohomology with local coefficients is defined as $H^*(X, M) := H^*C^\bullet(X, M)$.

Now this has similar properties as ordinary cohomology which we leave to the reader to work out. Let us note that one very nice feature of homology with local coefficients is that it lets us formulate a version of Poincaré duality without orientation assumtion (which we will not prove here):

Theorem 5.8. *Let M be a compact topological manifold without boundary. Then there is a fundamental class $[M] \in H_n(M, \mathbb{Z}^{w_1 M})$ such that capping induces isomorphisms*

$$\begin{aligned} \cap[M]\colon \quad & H^*(M, \mathbb{Z}) \to H_{n-*}(M, \mathbb{Z}^{w_1 M}) \\ \cap[M]\colon \quad & H^*(M, \mathbb{Z}^{w_1 M}) \to H_{n-*}(M, \mathbb{Z}) \ . \end{aligned}$$

To formulate this we have used ring and module structures on twisted cohomology. More specifically there are maps

$$\begin{aligned} \cup\colon \quad & H^*(X, M_1) \otimes H^*(X, M_2) \to H^*(X, M_1 \otimes M_2) \\ \cap\colon \quad & H^*(X, M_1) \otimes H_{-*}(X, M_2) \to H^*(X, M_1 \otimes M_2) \end{aligned}$$

where M_1, M_2 are local systems on X and where $\otimes$ is the pointwise tensor product of local systems. These maps are defined with the exact same formulas as there non-twisted formulas. Moreover we have $\mathbb{Z}^{w_1 M} \otimes \mathbb{Z}^{w_1 M} \simeq \mathbb{Z}$. Is this correct or did I mess up were the local system is?

5.2 Construction of the Serre spectral sequence

As already explained earlier, we want to construct the homological version of the Serre spectral sequence in this section. Again remember that a homological first quadrant spectral sequence is the same as a cohomological third quadrant spectral sequence. In fact there is even an equivalence for the category of chain complexes to the category of cochain complex which sends a chain complex $C_\bullet$ to the cochain complex $C^\bullet$ with $C^n := C_{-n}$. We will implicitly use this equivalence. Thus for example if we have a double chain complex, then it can also be considered as a double cochain complex. In particular it has an associated spectral sequence.

The homological Serre spectral sequence that we are going to construct exists for every map $Y \to X$ of spaces. We assume without loss of generality that $Y \to X$ is a Serre fibration. Then it will take the form

$$H_p(X, H_q(Y_x)) \Rightarrow H_{p+q}(Y)$$

where $H_q(Y_x))$ is the local system on X considered in Example 5.3. Again, this is a homological spectral sequence, which means we can think of it as a cohomological spectral sequence concentrated in the third quadrant.

In this situation above, we consider the double complex $C_{\bullet,\bullet}$ which is given in bidegree p, q by the free abelian group generated by all maps $\Delta^p \times \Delta^q \to Y$ that fit into a square of the form

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^p \arrow[r] & X \end{tikzcd} .$$

Both differentials in this double complex are just given by alternating sums of face maps. The horizontal one in the p direction and the vertical one in the q-direction. As a result we obtain two spectral sequences

$$\begin{aligned} H_p(H_q(C, d^v), d^h) &\Rightarrow H_{p+q}(\mathrm{Tot} C) \\ H_q(H_p(C, d^h), d^v) &\Rightarrow H_{p+q}(\mathrm{Tot} C) \end{aligned}$$

which we want to analze now. The first one will lead to the identification of the homology of the total complex with the homology of Y and the second one will be the Serre spectral sequence.

Let us start with the second one. Recall that we want to use as a convention that we rotate the double complex for this spectral sequence.Since we do not really want to interchange p

and q we will just think of q as the x-axis and p as the y-axis temporarily (so that we again first take the vertical and then the horizontal differential). We fix q and then we consider a square of the form

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^p \arrow[r] & X \end{tikzcd} .$$

by the exponential law equivalently as a square

$$\begin{tikzcd} \Delta^p \arrow[r] \arrow[d] & Y^{\Delta^q} \arrow[d] \\ X \arrow[r] & X^{\Delta^q} \end{tikzcd}$$

where the right hand map is given by postcomposition with $p : Y \to X$ and the lower horizontal map is the inclusion of constant maps, i.e. induced from the projection $\Delta^q \to \mathrm{pt}$. We denote by Y' the pullback of the lower part of the diagram. Thus this is the same as a map $\Delta^p \to Y'$. As a result we get that for fixed q the vertical chain complex

$$... \to C_{q,p-1} \to C_{q,p} \to C_{q,p+1} \to ...$$

is equivalent to the singular chain complex of Y'. This is the 0-th page of the spectral sequence. The first page is thus the singular homology of Y' as the differential is the alternating differential in the Δ^p. Now by construction as a pullback

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y^{\Delta^q} \arrow[d] \\ X \arrow[r] & X^{\Delta^q} \end{tikzcd} .$$

we get that Y' is homotopy equivalent to Y. Thus the first page of the spectral sequence is given as follows:

Picture: columns are the homology of Y.

What are the differentials? The next differential goes to the right. It is given by the horizontal differential in the total complex. Since we have identified Y^{Δ^q} with Y this is given by the alternating sums of the identity map. As a result it alternates between the identity and the zero map. Thus the E_2-page is concentrated in the Y-axis and agrees with the homology of the space Y. Thus we get an isomorphism $H_*\mathrm{Tot}C_{\bullet,\bullet} \simeq H^*(Y)$.

Now let us work out the other spectral sequence. No we again use our usual convention concerning p and q. We fix a map $s : \Delta^p \to X$ and consider the square

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^p \arrow[r, "\sigma"] & X \end{tikzcd}$$

equivalently as a square

$$\begin{tikzcd} \Delta^q \arrow[r] \arrow[d] & Y^{\Delta^p} \arrow[d] \\ \mathrm{pt} \arrow[r] & X^{\Delta^p} \end{tikzcd}$$

where the lower map is given by the point $\sigma \in X^{\Delta^p}$. We again consider this as a map $\Delta^q \to F_\sigma$ where F_σ is the pullback

$$\begin{tikzcd} F_\sigma \arrow[r] \arrow[d] & Y^{\Delta^p} \arrow[d] \\ \mathrm{pt} \arrow[r] & X^{\Delta^p} \end{tikzcd} .$$

Thus the columns of the double complex are isomorphic to a sum over all maps $\sigma : \Delta^p \to X$ of the singular chain complex of F_σ (the differential only goes into the q-direction. The space F_σ is weakly homotopy equivalent to the space F_{σ_0} defined as the fibre of $Y \to X$ over σ_0 which is the value of σ on the 0-vertex:

$$\begin{tikzcd} F_\sigma \arrow[r] \arrow[d] & Y^{\Delta^p} \arrow[r] \arrow[d] & X^{\Delta^p} \arrow[d] \\ F_{\sigma_0} \arrow[r] & Y \arrow[r] & X \end{tikzcd} .$$

Therefore the singular chain complex of F_σ is quasi-isomorphic to the one of F_{σ_0}. Thus we find that the columns of the spectral sequence are given by the direct sum of the homologies

$$\bigoplus_{\Delta^p \to X} H^q(F_{\sigma_0}) \simeq \bigoplus_{\Delta^p \to X} H^q(F_\sigma)$$

in vertical directions. The next differential is the horizontal differential, which is in this basis given by taking $\sigma : \Delta^p \to X$ to the alternating sum of the faces $\sigma \circ d^i$ and on the summands by the map $H^q(F_\sigma) \to H^q(F_{\sigma \circ d^i})$ given by the obvious map. Now this map corresponds under the equivalences $H^q(F_\sigma) \to H^q(F_{\sigma_0})$ and $H^q(F_\sigma) \to H^q(F_{\sigma \circ d^i(0)})$ to either the identity map (if $i \neq 0$) or to the map in the local system by definition. This proves that the E^2-page of the spectral sequence is given by homology $H_p(X, H_q(Y_x))$. Thus we have proven:

Theorem 5.9. *For every map of spaces $Y \to X$ there is a convergent first quadrant, homological spectral sequence*

$$H_p(X, H_q(Y_x)) \Rightarrow H_{p+q}(Y)$$

Proof. We just replace $Y \to X$ without loss of generality by a Serre fibration and apply the construction as above. □

6 Serre Class theory

6.1 Serre classes

In this section we want to cover the basics of what Serre did to attack homotopy groups algebraic topology.

Definition 6.1. Let $\mathcal{D} \subseteq \text{AbGr}$ be a non-empty full subcategory of the category of abelian groups. Then $\mathcal{D}$ is called a *Serre class* if it satisfies the following property: Given a short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups. Then $B \in \mathcal{D}$ iff $A, C \in \mathcal{D}$.

Note that by considering the short exact sequence $0 \to 0 \to A \xrightarrow{f} A'$ for an isomorphism we see that a Serre class is closed under isomorphisms. It also by definition is closed under subobjects, extensions and quotients.

Examples 6.2.

- *The category of all abelian groups and the category of all trivial groups are Serre classes.*
- *The category of all finitely generated abelian groups is a Serre class.*
- *The category of all torsion abelian groups is a Serre class. Similarly the category of all p-power torsion abelian groups is a Serre class (why not p-torsion abelian groups?).*
- *The category of all finite abelian groups is a Sere class.*
- *The category of all rational abelian groups (i.e. uniquely divisible) is not a Serre class (Why? Think of $\mathbb{Z} \subseteq \mathbb{Q}$).*

Lemma 6.3. *Let $\mathcal{D}$ be a Serre class. Then the following are true:*

1. *Given an exact sequence $0 \to A_0 \to A_1 \to \ldots \to A_n \to 0$. If all but one object belong to $\mathcal{D}$, then so does the last.*
2. *If $C_\bullet$ is a chain complex of abelian groups with $C_n \in \mathcal{D}$ for all n, then $H_n(C_\bullet) \in \mathcal{D}$ for all n.*
3. *Let*
$$0 = A_0 \subseteq A_1 \subseteq \ldots. \subseteq A_{n-1} \subset A_n = A$$
be a finite filtration of an abelian group A. The filtration quotients A_k/A_{k-1} belong to $\mathcal{D}$ iff A belongs to $\mathcal{D}$.

Proof. Clear (TODO). □

Definition 6.4. We say that a Serre class $\mathcal{D}$ is good if it has the following two properties:

1. If A and B belong to $\mathcal{D}$, then so do $A \otimes B$ and $\text{Tor}(A, B)$.
2. If A belongs to $\mathcal{D}$, then so does $H_k(K(A, 1)) = H_k^{\text{gp}}(A)$ for all $k > 0$.

Proposition 6.5. *The Serre classes of finitely generated abelian groups is good.*

Proof. Assume that A and B are finitely generated. Then $A \otimes B$ is also finitely generated (by tensor products of the generators). Now we choose a presentation $0 \to P_2 \to P_1 \to A \to 0$ with P_1, P_2 finitely generated free. Then $\text{Tor}(A, B)$ is the kernel of the map

$$P_2 \otimes B \to P_1 \otimes B$$

which is finitely generated as a subgroup of $P_2 \otimes B$. If $A = \mathbb{Z}$ or $A = \mathbb{Z}/n$ then the group homology $H_*^{\mathrm{gp}}(A)$ is given by

$$H_*^{\mathrm{gp}}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 1 \\ 0 & * > 1 \end{cases} \qquad H_*^{\mathrm{gp}}(\mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & * = 1, 3, 5, ... \\ 0 & * = 2, 4, 6., ... \end{cases}$$

in particular finitely generated. A general finitely generated abelian group is a product of these groups. Thus it suffices to show if for two groups A and B the homologies $H_*^{\mathrm{gp}}A$ and $H_*^{\mathrm{gp}}B$ are finitely generated, then so is $H_*^{\mathrm{gp}}(A \times B)$. But this follows from the Künneth short exact sequence

$$0 \to H_*^{\mathrm{gp}}(A) \otimes H_*^{\mathrm{gp}}(B) \to H_*^{\mathrm{gp}}(A \times B) \to \mathrm{Tor}(H_*^{\mathrm{gp}}A, H_*^{\mathrm{gp}}B)[-1] \to 0$$

and what we have shown above. □

6.2 Hurewicz' and Whitehead's theorems modulo a Serre class

Now we come to the main theorem of this section.The following result is the first main result of this section and will be proven eventually.

Theorem 6.6. *Let $\mathcal{D}$ be a good Serre class. For a simple*[4] *space X we have that $H_n(X) \in \mathcal{D}$ for all $n > 0$ if and only if $\pi_n(X) \in \mathcal{D}$ for all $n > 0$.*

Corollary 6.7. *All homotopy groups $\pi_*(S^n)$ are finitely generated.*

Corollary 6.8. *The homotopy groups of a finite, simple CW complex (e.g. simply connected) are finitely generated.*

Remark 6.9. Theorem 6.6 is actually true a bit more general than for simple spaces. It is true if X is nilpotent which means that the fundamental group is nilpotent (in the sense of group theory) and the action of the fundamental group on the higher homotopy groups is nilpotent, which means that there is a finite filtration so that the action on the associated graded is trivial. We will not need this extra generality.

In order to proof this theorem we will need some preliminary considerations. Let us begin with those. The next Lemma is the key result:

Lemma 6.10. *Let $\mathcal{D}$ be a good Serre class of abelian groups and let $F \to Y \to X$ be a fibre sequence of path connected spaces with $\pi_1(X)$ acting trivially on F. If two out of three of F, Y, X have $H_n \in \mathcal{D}$ for all $n > 0$ then so does the third.* [5]

Proof. We have to distinguish three cases here.

1. Let us first assume that $H_k(F), H_k(X) \in \mathcal{D}$ for $k > 0$. Then so are all the groups

$$E_{p,q}^2 = H_p(X; H_q(F)) \cong H_p(X) \otimes H_q(F) \oplus \mathrm{Tor}(H_{p-1}(X), H_q(F))$$

for $(p,q) \neq (0,0)$ (the only slightly non-trivial case is $p = 1$ but here $H_{p-1}(X) = H_0(X)$ is free, so the Tor-term vanishes). Then we conclude that $E_{p,q}^\infty \in \mathcal{D}$ and thus also $H_*(Y)$.

[4] Recall that a space is called simple if it is path-connected, the fundamental group is abelian and acts trivially on all higher homtopy groups

[5] We do not really use the full strenghth of being good for the Serre class but only property (1) of Definition 6.4.

2. Now assume that $H_k(F), H_k(Y) \in \mathcal{D}$ for all $k \neq 0$. We first get inductively that $E^\infty_{p,q} \in \mathcal{D}$ for $(p,q) \neq (0,0)$ as it is the associated graded of a filtration on $H_k(Y)$ (Lemma above (3)). We want to induct on the following statement depending on an natural number n:

   All $H_k(X)$ are in $\mathcal{D}$ for $0 < k < n$.

   If we assume this, then we can conclude that $E^r_{p,q} \in \mathcal{D}$ for $p < n$ and $(p,q) \neq (0,0)$ as above. Now we picture the E^r-page as follows:

   picture

   We get a short exact sequence

$$0 \to E^{r+1}_{n,0} \to E^r_{n,0} \to \mathrm{im}(\mathrm{d_r}) \to 0.$$

   We have $\mathrm{im}(\mathrm{d_r}) \in \mathcal{D}$ by what we have said above. Thus we get that for each r we have $E^{r+1}_{n,0} \in \mathcal{D}$ iff $E^r_{n,0} \in \mathcal{D}$. We can now finish the proof by noting that for $r >> 0$ the term $E^r_{n,0} \in \mathcal{D}$ as it stabilizes and $E^\infty_{n,0}$ is.

3. Assume that $H_k(F), H_k(Y) \in \mathcal{D}$ for $k > 0$. We again get that $E^\infty_{p,q} \in \mathcal{D}$ for all $(p,q) \neq (0,0)$. This time we induct on

   All $H_k(F)$ are in $\mathcal{D}$ for $0 < k < n$.

   Again from the r-th page we get an exact sequence

   picture

$$0 \to \ker d_r \to E^r_{r,n-r+1} \to E^r_{0,n} \to E^{r+1}_{0,n} \to 0.$$

   from which we get that $E^r_{0,n} \in \mathcal{D}$ iff $E^{r+1}_{0,n}$ in $\mathcal{D}$ which finishes the proof.

□

Lemma 6.11. *Assume that $\mathcal{D}$ is good. If A belongs to $\mathcal{D}$, then so does $H_k(K(A,n))$ for all $k > 0, n > 0$.*

Proof. By assumption we know this for $n = 1$. Now we consider the fibre sequence

$$K(A,n) \to \mathrm{pt} \to K(A,n+1)$$

and use the last lemma to conclude inductively. □

Now we want to proceed to the proof of 6.15. The key input is going to be the Postnikov tower of a simple space. Let us review that quickly. For every space X there is a tower

$$X \to \qquad ... \to \tau_{\leq 2}X \to \tau_{\leq 1}X \to \tau_{\leq 0}X$$

of maps such that the map $X \to \tau_{\leq n} X$ is an isomorphism on π_k for $k \leq n$ and $\pi_k(\tau_{\leq n} X) = 0$ for $k > n$. Then X is the inverse homotopy limit of the spaces

$$X \simeq \varprojlim(\tau_{\leq n} X).$$

This works for all spaces X but let us for simplicitly assume that X is path connected. Then all the $\tau_{\leq n} X$ are also path connected and there is fibre sequence

$$K(\pi_n X, n) \to \tau_{\leq n} X \to \tau_{\leq n-1} X$$

which follows directly by computing the homotopy groups of the homotopy fibre and observing that it is an Eilenberg MacLane space. Now for X simple the key point is that the action of $\pi_1(\tau_{\leq n-1} X)$ on $K(\pi_n X, n)$ is trivial.

Remark 6.12. The latter condition implies that these fibre sequences can be delooped, that is we have a fibre sequence

$$\tau_{\leq n} X \to \tau_{\leq n-1} X \xrightarrow{k_n} K(\pi_n X, n+1).$$

We think of this as $\tau_{\leq n} X \to \tau_{\leq n-1}$ being a 'principal fibration' (similar to the distinction between fibre bundles and principal bundles where on the second we have an action on the total space by the structure group). In fact it is equivalent that a space X is simple and the fact that the Postnikov tower can be obtained by fibre sequences as in 6.12. The key statement here is the following (which will be an excercise): a fibre sequence $K(A, n) \to Y \to X$ can be obtained as the fibre of a map $Y \to X \to K(A, n+1)$ iff $\pi_1(X)$ acts trivally on $\pi_n(K(A, n)) = A$. This can be proven by showing that the homotopy cofibre of $Y \to X$ is n-connected and has $\pi_{n+1} C(f) = A$.

Our Theorem 6.6 is actually a special case of a more general theorem which we want to prove. We will say that a map $A \to B$ of abelian groups is an isomorphism mod $\mathcal{D}$ if its kernel and cokernel lie in $\mathcal{D}$. We will say that is is a monomorphism (resp. epimorphism) mod $\mathcal{D}$ if the kernel (resp. the corkernel) lie in $\mathcal{D}$.

Theorem 6.13 (Hurewicz theorem mod $\mathcal{D}$)**.** *Let $\mathcal{D}$ be a good Serre class. For an integer $n > 0$ and a simple space X the following are equivalent:*

- *$H_k(X) \in \mathcal{D}$ for all $0 < k < n$;*
- *$\pi_k(X) \in \mathcal{D}$ for all $0 < k < n$.*

In this case the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{D}$.

Proof. Let us first assume that $\pi_k(X) \in \mathcal{D}$ for all $0 < k < n$. We consider the Postnikov tower $X \simeq \varprojlim \tau_{\leq n} X$. Since $H_k(X) = H_k(\tau_{\leq k} X)$ it suffices to show that $H_*(\tau_{\leq k} X) \in \mathcal{D}$ for all $k < n$ and all $* > 0$. This follows inductively from the fibre sequence

$$K(\pi_k X, k) \to \tau_{\leq k} X \to \tau_{\leq k-1} X \tag{2}$$

using Lemma 6.10

Now we want to show by induction on n that $H_k(X) \in \mathcal{D}$ for all $0 < k < n$ implies that the Hurewicz map $\pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{D}$. This then also implies the claim if we apply it for smaller n.

The Hurewicz map $\pi_n(X) \to H_n(X)$ is equivalent to the map

$$\pi_n(X) = \pi_n K(\pi_n X, n) \cong H_n K(\pi_n X, n) \to H_n(\tau_{\leq n} X).$$

By the inductive hypothesis we already know that $\pi_k(X) \in \mathcal{D}$ for all $k < n$. Then by the first part of the proof we know that $H_k(\tau_{\leq n-1} X) \in \mathcal{D}$ for all $k > 0$. We analyze the spectral sequence associated to the fibre sequence

$$K(\pi_n X, n) \to \tau_{\leq n} X \to \tau_{\leq n-1} X \tag{3}$$

it has nothing between the 0th and nth row. The first possible differential is a

$$d^{n+1} : H_{n+1}(\tau_{\leq n-1} X) \to H_n K(\pi_n X, n).$$

The cokernel of this map is $E^\infty_{0,n}$. But $E^\infty_{0,n}$ sits in a short exact sequence

$$0 \to E^\infty_{0,n} \to H_n(\tau_{\leq n} X) \to H_n(\tau_{\leq n-1} X) \to 0$$

coming from the filtration (the diagonal in the picture of the spectral sequence). Thus we get an exact sequence

$$H_{n+1}(\tau_{\leq n-1} X) \to H_n K(\pi_n X, n) \to H_n(\tau_{\leq n} X) \to H_n(\tau_{\leq n-1} X)$$

Since the first and last term are in $\mathcal{D}$ this finishes the proof. $\square$

Now we come to the Whitehead theorem mod $\mathcal{D}$. For this Theorem we need an additional assumption on a good Serre class $\mathcal{D}$, namley that for every abelian group A and every $B \in \mathcal{D}$ thee product $A \otimes B$ lies in $\mathcal{D}$. If a good Serre class satisfies this assumption then we call it a *Serre ideal.* This condition is equivalent to saying that $\mathcal{D}$ is closed under infinite sums.

Examples 6.14. *The good Serre classe of finitely generated abelian groups is not a Serre ideal. The class of torsion abelian groups is a Serre ideal.*

Theorem 6.15 (Whitehead theorem mod $\mathcal{D}$)**.** *Let $\mathcal{D}$ be a Serre ideal. Let $f : X \to Y$ be a map of simply connected spaces and let $n > 0$. Then the following are equivalent*

- $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ *is an isomorphism mod $\mathcal{D}$ for $0 < k < n$ and an epimorphism mod $\mathcal{D}$ for $k = n$.*
- $H_k(f) : H_k(X) \to H_k(Y)$ *is an isomorphism mod $\mathcal{D}$ for $0 < k < n$ and an epimorphism mod $\mathcal{D}$ for $k = n$.*

Proof. Consider the fibre sequence $F \to Y \to X$. The first condition is equivalent to saying that F is $(n-1)$-connected mod $\mathcal{D}$ (i.e. $\pi_i(F) \in \mathcal{D}$ for $i \leq n-1$). But this is by the Hurewicz theorem equivalent to saying that $H_k(F) \in \mathcal{D}$ for $0 < k < n$. Then we study the spectral sequence associated to $F \to X \to Y$. We claim that $H_p(Y, H_q(F))$ is in $\mathcal{D}$ for $0 \leq q \leq n-1$. More generally $H_p(Y, A) \in \mathcal{D}$ for every $A \in \mathcal{D}$. This immediately follows from the fact that we can compute $H_p(Y, A)$ by the cellular chain complex which lies in $\mathcal{D}$ by assumption thus so does its homology. As a result in the Serre spectral sequence the whole region $E^2_{p,q}$ for $0 \leq q \leq n-1$ is in $\mathcal{D}$. Thus so is this region in $E^\infty_{p,q}$. As a result the homology of X in degree k admits a filtration

$$0 \subseteq F_0 \subseteq F_1 \subseteq ... \subseteq F_k = H_k(X)$$

such that for $k < n$ and $i < k$ all quotients F_i/F_{i-1} lie in $\mathcal{D}$ and the last quotient F_k/F_{k-1} is modulo $\mathcal{D}$ isomorphic to $H_k(Y)$ (since all the differentials could only hit elements in $\mathcal{D}$). This implies the result for $k < n$. In the case $k = n$ we get that the first quotient does not need to lie in $\mathcal{D}$ but still the map is surjective.

Now conversely assume that the map $H_k(X) \to H_k(Y)$ is an iso mod $\mathcal{D}$ for $k < n$ and an epi mod $\mathcal{D}$ for $k = n$. We want to show that all $\pi_k(F) \in \mathcal{D}$ for $k < n$. Assume this was not the case, then pick a minimal k for which this fails. As a result we have that $E^2_{0,q} \notin \mathcal{D}$ for all $q < k$. We conclude inductive that $E^l_{0,k} \notin \mathcal{D}$ for all $l \le k$ (by the fact that the source of the differentials is in $\mathcal{D}$). We claim that that this is also true for $E^{k+1}_{0,k} = E^\infty_{0,k}$. To see this we observe that $E^{k+1}_{0,k}$ sits in an exact sequence

$$0 \to \ker d_l \to E^k_{k+1,0} \xrightarrow{d_l} E^k_{0,k} \to E^{k+1}_{0,k} \to 0$$

for which the cokernel of the cokernel of the first two sits in $\mathcal{D}$ by assmuption (edge homo). This shows that $E^{k+1}_{0,k}$ not in $\mathcal{D}$ which is a contradiction because then the edge homorphism can not be a $\mathcal{D}$ iso. $\square$

Note that the theorem fails for good Serre classes that are not Serre ideals. For example consider the map

$$\mathbb{C}P^\infty \times R \to R$$

where R is any simply connected space with $H_2(R)$ not finitely generated (e.g. a big wedge product of S^2's). Then we find that the map

$$\pi_*(\mathbb{C}P^\infty \times R) = \pi_*(\mathbb{C}P^\infty) \oplus \pi_*(R) \xrightarrow{pr} \pi_*(R)$$

is an isomorphism modulo the Serre class of finitely generated abelian groups (since $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$). But in homology we have

$$H_4(\mathbb{C}P^\infty \times R) = \mathbb{Z} \oplus H_2(R) \oplus H_4(R) \xrightarrow{pr} H_4(R)$$

is not an isomorphism modulo this Serre class.

6.3 Some applications

We can now use the results to get information about the homotopy groups of spheres. We start by a direct approach and then phrase the result a little bit more conceptual using Postnikov towers.

Proposition 6.16. *Let n be odd. Then we have $\pi_*(S^n)$ is finite for all $* \neq n$.*

Proof. We consider the Postnikov section $S^n \to K(\mathbb{Z}, n)$. We have seen that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x]/x^2$$

where $|x| = n$. Thus we also have that

$$H_*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0, n \\ 0 & \text{else} \end{cases}$$

The map $S^n \to K(\mathbb{Z}, n)$ by definition induces an isomorphism in degree n rational homology, thus an isomorphism on rational hompology. This means it is an isomorphism modulo the good Serre class of torsion abelian groups (why?). Thus by the Whitehead theorem it is an isomorphism module torsion abelian groups, i.e. on rationalized homotopy groups. The result then follows since we already know that $\pi_*(S^n)$ is degreewise finitely generated. □

Corollary 6.17. *All stable homotopy groups of spheres π_n^s for $n > 0$ are finite abelian groups. In particular, the Hurewicz map $\mathbb{S}_*(X, A) \otimes \mathbb{Q} \to H_*(X, A, \mathbb{Q})$ is an isomorphism for every pair (X, A).*

Proof. We have that $\pi_n^s \cong \pi_{k+n}(S^k)$ for all $k >> 0$. We can choose k odd. □

Proposition 6.18. *Let n be even. Then $\pi_*(S^n)$ is finite for all $* \neq n, 2n-1$. In the latter case we have that $\pi_{2n-1}(S^n) = \mathbb{Z} \oplus F$ where F is a finite abelian group.*

Proof. Recall that $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x]$ with $|x| = n$. In particular the map $S^n \to K(\mathbb{Z}, n)$ is not an isomorphism on rational (co)homology anymore. The reason is that the square of the the cohomology generator of S^n is zero, while it is non-zero in $H^*(K(\mathbb{Z}, n), \mathbb{Q})$. So we simply modify the space $K(\mathbb{Z}, n)$ as follows: we let $q : K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ be the map that represents the square of the cohomology generator. We let F denote the homotopy fibre.

Then the map $S^n \to K(\mathbb{Z}, n)$ admits a lift as in the diagram

$$\begin{tikzcd}
& F \arrow[d] & \\
S^n \arrow[r] \arrow[ur, dashed] & K(\mathbb{Z}, n) \arrow[r, "q"] & K(\mathbb{Z}, 2n)
\end{tikzcd}$$

since the lower composition is nullhomotopic by construction. Moreover from the long exact sequence of homotopy groups we know that

$$\pi_*(F) = \begin{cases} \mathbb{Z} & * = n, 2n-1 \\ 0 & \text{else} \end{cases}$$

and that the map $F \to K(\mathbb{Z}, n)$ is an isomorphism on π_n an therefore the map $S^n \to K(\mathbb{Z}, n)$ is also an isomorphism on π_n and thus also on H_n and H^n.

Now let us try to compute $H^*(F : \mathbb{Q})$ by means of the spectral sequence associated to the fibre sequence

$$K(\mathbb{Z}, 2n-1) \to F \to K(\mathbb{Z}, n).$$

There is a single $d_{2n} : H^{2n-1}(K(\mathbb{Z}, 2n-1), \mathbb{Q}) \to H^{2n}(K(\mathbb{Z}, n))$ that determines everything. We claim that this differential has to be onto, since otherwise we would have a class x^2 in F which is the square of the class in degree n. But this cannot happen, since by construction of F the square of the class x is trivial. Thus we find that

$$H^*(F; \mathbb{Q}) = \mathbb{Q}[x]/x^2$$

where $|x| = n$.

Putting everything together we have a map

$$S^n \to F$$

which induces an isomorphism on rational (co)homology. Thus it is an isomorphism on rational homotopy groups (i.e. modulo the Serre class of torsion groups) thus we get that

$$\pi_*(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & * = n, 2n-1 \\ 0 & \text{else} \end{cases}$$

Together with the finite generation this implies the claim. $\square$

One can express these reults slightly more conceptual using the Postnikov decomposition.

Theorem 6.19. *For n odd, the map*

$$S^n \longrightarrow \tau_{\leq n} S^n$$

is a rational homotopy equivalence and $\pi_n(S^n) = \mathbb{Z}$. For n even, the map

$$S^n \longrightarrow \tau_{\leq n} S^n$$

is a rational $2n-1$-equivalence with $\pi_n(S^n) = \mathbb{Z}$ and

$$S^n \longrightarrow \tau_{\leq 2n-1} S^n$$

a rational homotopy equivalence with $\pi_{2n-1}(S^n) \otimes \mathbb{Q} = \mathbb{Q}$.

Given basic knowledge about cobordism theory and characteristic numbers one can also make short work of the rational cobordism ring:

Theorem 6.20 (Thom)**.**

$$\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^{2i} \mid i \in \mathbb{N}]$$

Proof. We have

$$\Omega_* \otimes \mathbb{Q} = \mathbb{S}_*(MSO) \otimes \mathbb{Q} = H_*(MSO, \mathbb{Q}) = H_*(BSO, \mathbb{Q}) = H^*(BSO, \mathbb{Q})^*$$

and one readily checks that this composite sends a closed, oriented manifold $[M] \in \Omega_n$ to the homomorphism

$$H^n(BSO, \mathbb{Q}) \ni p \longmapsto \langle p(\nu M), [M] \rangle$$

where νM denotes the stable normal bundle of M. The calculation of Pontryagin numbers of $\mathbb{C}P^{2i}$ shows that the composite $\mathbb{Z}[\mathbb{C}P^{2i} \mid i \in \mathbb{N}] \to \Omega_* \to H^*(BSO)^*$ is injective and thus rationally an isomorphism since the ranks of source and target match up in each degree. $\square$

Theorem 6.21 (Serre's rational Hurewicz theorem)**.** *Let X be a simply connected topological space with*

$$\pi_i(X) \otimes \mathbb{Q} = 0$$

for $i \leq n-1$. Then the Hurewicz map

$$\pi_i(X) \otimes \mathbb{Q} \longrightarrow H_i(X; \mathbb{Q})$$

induces an isomorphism for $1 \leq i \leq 2n-2$ and a surjection for $i = 2n-1$.

Proof. Consider the diagram

$$\begin{tikzcd} \pi_i(\tau_{\geq i}X) \arrow[r] \arrow[d] & H_i(\tau_{\geq i}) \arrow[d] \\ \pi_i(X) \arrow[r] & H_i(X) \end{tikzcd}$$

by definition the left hand vertical map is an isomorphism and by Hurewicz' theorem so is the upper horizontal one. Therefore proving that $\pi_i(X) \otimes \mathbb{Q} \to H_i(X, \mathbb{Q})$ is an isomorphism is equivalent to showing the same for $H_i(\tau_{\geq i}X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$.

The claim will now follow by an iterated application of the following lemma. □

Lemma 6.22. *Let X be a simply connected, rationally $n-1$-connected space. Then*

$$H_i(\tau_{\geq n+1}X, \mathbb{Q}) \to H_i(X, \mathbb{Q})$$

is an isomorphism for $n < i < 2n-2$ and a surjection for $i = 2n-1$.

Proof. We shall need to know the following facts about Eilenberg-Mac Lane spaces

$$H_*(K(V,n), \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } * = 0 \\ 0 & \text{for } 0 < * < n \\ V \otimes \mathbb{Q} & \text{for } * = n \\ 0 & \text{for } n < * < 2n \end{cases}$$

Up to degree $n+1$ this is clear from Hurewicz' theorem and the rest serves as the basic input case. It is readily deduced from the cohomology calculation

$$H^*(K(V,n), \mathbb{Q}) = \mathrm{Sym}^g r(V[n])$$

which you had to perform for $V = \mathbb{Z}$. In particular, we also now that this is the sharpest linear estimate possible, as $H^{4n}(K(\mathbb{Z}, 2n), \mathbb{Q}) = \mathbb{Q}\langle \iota_{2n}^2 \rangle$. Now for the actual proof.

Consider the fibration sequence $\tau_{\geq n+1}X \to X \to \tau_{\leq n}X$ and its shift $\Omega\tau_{\leq n}X \to \tau_{\geq n+1}X \to X$, which fits into a diagram of fibrations

$$\begin{tikzcd} \Omega\tau_{\leq n}X \arrow[r] \arrow[d] & \tau_{\geq n+1}X \arrow[r] \arrow[d] & X \arrow[d] \\ \Omega\tau_{\leq n}X \arrow[r] & * \arrow[r] & \tau_{\leq n}X \\ K(\pi_n(X), n-1) \arrow[r] \arrow[u] & * \arrow[r] \arrow[u] & K(\pi_n(X), n) \end{tikzcd}$$

where the lower vertical maps are all rational equivalences by the Hurewicz theorem mod torsion and assumption. Up to total degree $2n-1$ the rational Serre spectral sequence for the top row fibration sequence therefore has non-trivial entries only in spots $(0,0)$, $(i,0)$ for $n \leq i$, $(0, n-1)$, $(i, n-1)$ again for $n \leq i$ and $(0, 2n-2)$. In this range the first non-trivial differential is the isomorphism

$$d_{n-1} : \mathcal{S}_{n-1}^{(n,0)} = H_n(X, \mathbb{Q}) \to H_{n-1}(K(\pi_n(X), n-1), \mathbb{Q}) = \mathcal{S}_{n-1}^{(0,n-1)}$$

, readily deduced from the comparison to $K(\pi_n(X), n-1) \to * \to K(\pi_n(X), n)$, which is an isomorphism in both these spots, and on the same page we find a surjection

$$d_{n-1} : \mathcal{S}_{n-1}^{(n,n-1)} = H_n(X, \pi_n(X) \otimes \mathbb{Q}) \to H_{2n-2}(K(\pi_n(X), n-1), \mathbb{Q}) = \mathcal{S}_{n-1}^{(0,2n-2)}$$

again because this has to happen in the Serre spectral sequence for $K(\pi_n(X), n-1) \to * \to K(\pi_n(X), n)$. Therefore in the relevant range the third page has entries at most in positions $(0,0), (i,0)$ for $i > n$ and $(n, n-1)$. In the indicated range the spectral sequence then has to collapse for degree reasons and the identification of the edge homomorphism yields the claim. $\square$

7 Mulitplicative structures

In this section we want to study multiplicative structures on and more generally pairings between spectral sequences. Let us start out with the general definitions. We shall assume that we are given three graded abelian categories $\mathcal{A}, \mathcal{A}'$ and $\mathcal{A}''$ (that is categories enriched in the category of graded abelian groups and degree 0 morphisms, which we equip with the monoidal structure given by the tensor product) and a bi-additive functor $-\hat{\otimes}- : \mathcal{A}\hat{\times}\mathcal{A}' \longrightarrow \mathcal{A}''$ which we will soon have to assume right exact. This time around the grading does make a difference, as there seems to be no reasonable way of giving $S \otimes T$ a differential based on given ones for S and T without a grading. With the grading in place $\mathcal{A}\hat{\times}\mathcal{A}'$ denotes the category whose objects are pairs of objects and where

$$hom((A, A'), (B, B')) = hom(A, A') \otimes hom(B, B')$$

with composition picking up a sign, i.e. $(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} fh \otimes gk$. (Therefore the $\hat{}$ in $\hat{\times}$ has mathematical meaning in that it indicates we are using the Koszul braiding as the symmetry operator in the category of graded abelian groups, whereas the $\hat{}$ in $\hat{\otimes}$ serves no purpose other than to remind us of what composition in the source category does. This will hopefully turn out convenient in examples, where $\hat{\otimes}$ will be the tensor product on objects, but only up to a sign on morphisms). Given now differential objects (S, d) of $\mathcal{A}$ and (T, d) of $\mathcal{A}'$ with differentials of the same *odd!* degree the map $id\hat{\otimes}d + d\hat{\otimes}id$ becomes a differential on $S\hat{\otimes}T$. Most of the time the degree of d is either 1 or -1.

The only such structure relevant for us is the usual tensor product of modules, say $R-S$ bimodules, $S-T$-bimodules and $R-T$-bimodules, with the Koszul sign inserted, i.e. $f\hat{\otimes}g$ is the map that sends $x \otimes y$ to $(-1)^{|g||x|} f(x) \otimes g(y)$.

7.1 Multiplicative structures in general

Definition 7.1. If $(E, d), (E', d')$ and (E'', d'') are spectral sequences in $\mathcal{A}, \mathcal{A}'$ and $\mathcal{A}''$, respectively, with differentials of equal odd degree then a *pairing* or *exterior product* is a collection of maps

$$\mu_n : E_n \hat{\otimes} E'_n \longrightarrow E''_n$$

such that (i)

$$d''_n \circ \mu_n = \mu_n \circ (d_n \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'_n),$$

which makes the composition

$$H^*(E_n, d_n) \hat{\otimes} H^*(E'_n, d'_n) \xrightarrow{\times} H^*(E_n \hat{\otimes} E'_n, d_n \hat{\otimes} \mathrm{id} \pm \mathrm{id} \hat{\otimes} d'_n) \xrightarrow{\mu_n} H^*(E''_n, d''_n)$$

well-defined so that we can (ii) require it to correspond to μ_{n+1} under the nameless structure isomorphisms of all spectral sequences involved. If $\mathcal{A} = \mathcal{A}' = \mathcal{A}''$ and $(E, d) = (E', d') = (E'', d'')$ such a pairing is called a *multiplicative structure* on E or E in turn as a *spectral sequence of algebras*. Similarly, one can speak of a spectral sequence being one of modules of a one of algebras.

Remark 7.2. 1. In the concrete case the formula under (i) translates to

$$d_n''(x \cdot y) = d_n(x) \cdot y + (-1)^{|x|} x \cdot d_n(y).$$

2. Let us point out immediately that such pairings are one of the primary ways of computing differentials via property (i). We have already seen examples of this in the beginning.

3. The sequence $(E_n \hat{\otimes} E_n', d_n \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d_n')$ carries all the requisite structure to be a spectral sequence, that is there are canonical maps

$$E_{n+1} \hat{\otimes} E_{n+1}' \xrightarrow{\hat{\times}} H^*(E_n \hat{\otimes} E_n', d_n \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d_n')$$

induced by the structure maps of E and E'. However, this map can easily fail to be an isomorphism. In particular, we have not defined an induced functor $\otimes$ on the categories of spectral sequences. If we call objects as where the structure maps need not be isomorphisms *pre-spectral sequences*, then the above formula does turn pre-spectral sequences into a monoidal category and a spectral sequence of algebras is nothing but a monoid in that category. Does the inclusion of spectral sequences into pre-spectral sequences have the desired adjoint?

To transport this structure to the limit page we need to assume either favourable circumstances or better *right exactness* of $\hat{\otimes}$ in either variable. This will be a standing assumption from this point on.

Proposition 7.3. *μ restricts to maps $Z^\infty(E) \otimes Z^\infty(E') \longrightarrow Z^\infty(E'')$ and $Z^\infty(E) \otimes B^\infty(E') + B^\infty(E) \otimes Z^\infty(E') \longrightarrow B^\infty(E'')$. If $\otimes$ is right exact this induces a pairing*

$$\mu_\infty \colon E_\infty \otimes E_\infty' \longrightarrow E_\infty''$$

Proof. The first two items follow immediately from the Leibniz rule. For the last note that right exactness makes it possible to descend the identity to a map

$$\frac{Z^\infty(E)}{B^\infty(E)} \otimes \frac{Z^\infty(E')}{B^\infty(E')} \longrightarrow \frac{Z^\infty(E) \otimes Z^\infty(E')}{Z^\infty(E) \otimes B^\infty(E') + B^\infty(E) \otimes Z^\infty(E')}$$

and the latter maps to E_∞'' by the first two items. □

To formulate multiplicative convergence, we need similar definitions for filtered objects.

Definition 7.4. Let (T, F) and (S, G) be filtered objects in $\mathcal{A}$ and $\mathcal{A}'$, respectively. Define the filtration $F \otimes G$ of $T \otimes S$ to be

$$(F \otimes G)_i = \bigcup_j \mathrm{im}(F_j \otimes F_{i-j} \to T \otimes S)$$

(HOW DOES THIS WORK CATEGROICALLY?)

Note that this really is a filtration (though good properties of F and G may become lost!). If $\otimes$ is right exact, taking associated graded objects one obtains a map $\mathrm{Gr}_F T \otimes \mathrm{Gr}_G S \to \mathrm{Gr}_{F\otimes G} T \otimes S$:

$$\frac{F_i}{F_{i+1}} \otimes \frac{G_k}{G_{k+1}} \longrightarrow \frac{F_i \otimes G_k}{F_{i+1} \otimes G_k + F_i \otimes G_{k+1}} \longrightarrow \frac{(F \otimes G)_{i+k}}{(F \otimes G)_{i+k+1}}$$

Given a filtration preserving map $m : T \otimes S \longrightarrow U$ (for a filtered object (U, H) in $\mathcal{A}''$) we can define

Definition 7.5. Let E have abutment (T, F) witnessed by $w : \mathrm{Gr}_F T \to E_\infty$ and similarly for E' to (S, G) and E'' to (U, H). We will say *something* if the diagram

$$\begin{tikzcd}
\mathrm{Gr}_F T \otimes \mathrm{Gr}_G S \arrow[r] \arrow[d, "w\otimes w'"'] & \mathrm{Gr}_{F\otimes G}(T \otimes S) \arrow[r, "\mathrm{Gr}(m)"] & \mathrm{Gr}_H U \arrow[d, "w''"] \\
E_\infty \otimes E'_\infty \arrow[rr, "\mu_\infty"'] & & E''_\infty
\end{tikzcd}$$

commutes.

Remark 7.6. Note that for a map $m : T \otimes S \longrightarrow U$ being filtration preserving unfolds to mean $m(F_i \otimes G_j) \subseteq H_{i+j}$. The standard example of such behaviour is an algebra R filtered by the powers of an ideal $I \subseteq R$, i.e. $F_i = I^i$ for $i \geq 0$ and $F_i = R$ for $i \leq 0$.

7.2 Multiplications in the spectral sequences of filtered complexes

As a first example we will now see that there is a pairing

$$\times : \mathcal{L}(C, F)\hat{\otimes}\mathcal{L}(C', F') \longrightarrow \mathcal{L}(C \otimes C', F \otimes F'),$$

where $\otimes$ really is the tensor product over the base ring (that is suppressed from notation),

$$(C \otimes C')^n = \bigoplus_i C^i \otimes C'^{n-i}$$

and similarly

$$(F \otimes F')^n_i = \bigcup_j \bigoplus_k \mathrm{im}(F^k_j \otimes F'^{n-k}_{i-j} \to C^k \otimes C'^{n-k}) \subseteq (C \otimes C')^n.$$

Under the abutments we produced earlier it will turn out to be compatible with the cross product

$$H^*(F_{-\infty}) \otimes H^*(F'_{-\infty}) \xrightarrow{\times} H^*(F_{-\infty} \otimes F'_{-\infty}) = H^*((F \otimes F')_{-\infty}),$$

which is indeed filtration preserving for $HF \otimes HF'$ and $H(F \otimes F')$.

To obtain this pairing recall that $\mathcal{L}_0(C, F) = \mathrm{Gr}_F C$. So we obtain a pairing $\times_0 : \mathcal{L}_0(C, F)\hat{\otimes}\mathcal{L}_0(C', F') \to \mathcal{L}_0(C\hat{\otimes}C', F\hat{\otimes}F')$ by our earlier considerations. Since $d^0 = \mathrm{Gr}(d)$ the Leibniz rule is also inherited. We therefore obtain an induced pairing $\times_1$ on the first pages of the spectral sequence. To see that it satisfies the Leibniz rule, and is well-defined on higher pages recall

$$Z^n = \delta^{-1}(\mathrm{im}(i^n)) \quad B^n = p(\ker(i^n)) \quad d^{n+1}(x) = p(y)$$

where $\delta(x) = i^n(y)$.

Proposition 7.7.

$$Z^n \times_1 Z^n \subseteq Z^n$$

$$Z^n \times_1 B^n + B^n \times_1 Z^n \subseteq B^n$$

These cannot formally be derived from properties of the exact couple, but requires going back to cycles! This is the reason multiplicative exact couples are not so useful.

Proof. Let $x \in F_i$ represent an arbitrary element of $Z^n(C, F)$ and similarly $x' \in F'_j$. This means that $d(x) \in F_{i+1}$, so that $[x] \in F_i/F_{i+1}$ is a cycle, and furthermore that there exists a cycle $y \in F_{i+n+1}$ with $\delta([x]) = i^n([y])$, i.e. there exists a $z \in F_{i+1}$ with

$$d(x) = y + d(z)$$

and similarly for y' and z'. Now $d(x \otimes x') = d(x) \otimes x' + (-1)^{|x|} x \otimes d(x') \in \mathrm{im}(F_{i+1} \otimes F_j + F_i \otimes F_{j+1}) \subseteq (F \otimes F')_{i+j+1}$ so $p_{i+j}(x \otimes x')$ is a cycle in $(F \otimes F')_{i+j}/(F \otimes F')_{i+j+1}$ and we have to show that its boundary (up to a boundary in $(F \otimes F')_{i+j+1}$) admits a representative $a \in (F \otimes F')_{i+j+n+1}$. The obvious candidate for a is $y \otimes x' + x \otimes y'$, however this is not cycle:

$$d(y \otimes x' + (-1)^{|x|} x \otimes y') = (-1)^{|x|+1} y \otimes d(x') + (-1)^{|x|} d(x) \otimes y'$$

Luckily also

$$\begin{aligned} d(y \otimes z' + (-1)^{|x|} z \otimes y') &= (-1)^{|x|+1} y \otimes d(z') + (-1)^{|x|} d(z) \otimes y' \\ &= (-1)^{|x|+1} y \otimes (d(x') - y') + (-1)^{|x|} (d(x) - y) \otimes y' \\ &= (-1)^{|x|+1} y \otimes d(x') + (-1)^{|x|} d(x) \otimes y' \end{aligned}$$

so that $a = y \otimes (x' - z') + (-1)^{|x|} (x - z) \otimes y'$ is a cycle. Also clearly

$$a \in \mathrm{im}(F_{i+n+1} \otimes F'_j + F_i \otimes F'_{j+n+1}) \subseteq (F \otimes F')_{i+j+n+1}$$

so all that remain to be checked is that $d(x \otimes x') = a + d(w)$ for some $w \in (F \otimes F')_{i+j+1}$. But

$$\begin{aligned} d(x \otimes x') - a &= d(x) \otimes x' + (-1)^{|x|} x \otimes d(x') - y \otimes (x' - z') - (-1)^{|x|} (x - z) \otimes y' \\ &= d(z) \otimes x' + (-1)^{|x|} x \otimes d(z') + y \otimes z' + (-1)^{|x|} z \otimes y' \\ &= d(z) \otimes x' + (-1)^{|x|} x \otimes d(z') + (d(x) - d(z)) \otimes z' + (-1)^{|x|} z \otimes (d(x') - d(z')) \\ &= d(z) \otimes x' + (-1)^{|x|} z \otimes d(x') + d(x) \otimes z' + (-1)^{|x|} x \otimes d(z') - d(z) \otimes z' - (-1)^{|x|} z \otimes d(z') \\ &= d(z \otimes x' + x \otimes z' - z \otimes z') \end{aligned}$$

and

$$z \otimes x' + x \otimes z' + z \otimes z' \in \mathrm{im}(F_{i+1} \otimes F'_j + F_i \otimes F'_{j+1} + F_{i+1} \otimes F'_{j+1}) \subseteq (F \otimes F')_{i+j+1},$$

which finishes the proof of the first inclusion.

For the second one let $x \in F_i$ be as before, and $x' \in F'_j$ a cycle representing an arbitrary element in B^n, i.e. there exists an element $z' \in F'_{j-n}$ with $x' = d(z')$. In this case $d(x \otimes x') = d(x) \otimes x' \in \mathrm{im}(F_{i+1} \otimes F'_j \subseteq (F \otimes F')_{i+j+1}$ so $p_{i+j}(x \otimes x')$ is a cycle in $(F \otimes F')_{i+j}/(F \otimes F')_{i+j+1}$ and we have to show that $p_{i+j}(x \otimes x')$ admits a preimage a (under p_{i+j}), that is a cycle in

$(F \otimes F')_{i+j}$ and such that there exists a $w \in F_{i+j-n}$ with $a = d(z)$. The obvious candidate for a, namely $x \otimes x'$ is again not a cycle, but

$$d(x \otimes x') = d(x) \otimes x' = (y + d(z)) \otimes x' = d(z \otimes x' - (-1)^{|x|} y \otimes z')$$

so $a = x \otimes x' + (-1)^{|x|} y \otimes z' - z \otimes x'$ is indeed a cycle and clearly $a \in \operatorname{im}(F_i \otimes F'_j + F_{i+n+1} \otimes F'_{j-n} + F_{i+1} \otimes F_j) \subseteq (F \otimes F')_{i+j}$ and $p_{i+j}(a) = x \otimes x'$. It remains to check that there exists a $w \in (F \otimes F')_{i+j-n}$ with $d(w) = a$. Setting $w = (-1)^{|x|}(x - z) \otimes z'$ we find

$$\begin{aligned} d(w) &= (-1)^{|x|} d(x - z) \otimes z' + (x - z) \otimes d(z') \\ &= (-1)^{|x|} y \otimes z' + (x - z) \otimes x' \\ &= a \end{aligned}$$

and clearly $w \in \operatorname{im}(F_i \otimes F'_{j-n} + F_{i+1} \otimes F'_{j-n}) \subseteq (F \otimes F')_{i+j-n}$, so we are done. □

We therefore obtain well-defined pairing on each page of the spectral sequence by the right-exactness of the tensor product.

Proposition 7.8. *The differentials satisfy the Leibniz-rule.*

Proof. Let $x \in F_i$ and $x' \in F'_j$ be as before. Then $d_{n+1}([p_i x]_{n+1} \times_{n+1} [p'_j x']_{n+1})$ is given by

$$[p''_{i+j+n+1}(y \otimes (x' - z') + (-1)^{|x|}(x - z) \otimes y')]_{n+1}$$

Since

$$-y \otimes z' - (-1)^{|x|} z \otimes y' \in \operatorname{im}(F_{i+n+1} \otimes F'_{j+1} + F_{i+1} \otimes F'_{j+n+1}) \subseteq (F \otimes F')_{i+j+n+2}$$

this reduces to

$$d_{n+1}([x]_{n+1} \times_{n+1} [x']_{n+1}) = [p''_{i+j+n+1}(y \otimes x') + (-1)^{|x|} p''_{i+j+n+1}(x \otimes y')]_{n+1}$$

On the other hand by definition $d^{n+1}([p_i(x)]_{n+1}) \times_{n+1} [p'_j(x')]_{n+1} = [p_{i+n+1}(y)]_{n+1} \times_{n+1} [p'_j(x')]_{n+1}$ is given by $[p''_{i+j+n+1}(y \otimes x')]_{n+1}$ as well and similarly for the other term. □

We thus obtain a pairing of spectral sequences as desired. We can also note immediately that x_1 corresponds exactly to the cross product

$$\times \colon H^*(\mathrm{Gr}_F C) \otimes H^*(\mathrm{Gr}_{F'} C') \longrightarrow H^*(\mathrm{Gr}_{F \otimes F'} C \otimes C')$$

To identify the induced pairing on the limit page we have:

Proposition 7.9. *The diagram*

$$\begin{tikzcd} \mathrm{Gr}_{HF} H^*(F_{-\infty}) \otimes \mathrm{Gr}_{HF'} H^*(F'_{-\infty}) \arrow[r, "\times"] \arrow[d, "p \otimes p'"] & \mathrm{Gr}_{H(F \otimes F')} H^*((F \otimes F')_{-\infty}) \arrow[d, "p''"] \\ \mathcal{L}_\infty(C, F) \otimes \mathcal{L}_\infty(C', F') \arrow[r, "\times_\infty"'] & \mathcal{L}_\infty(C \otimes C', F \otimes F') \end{tikzcd}$$

commutes.

Proof. Every element of HF_i/HF_{i+1} and HF'_j/HF'_{j+1} is represented by a cycle $x \in F_i$ or $y \in F'_j$, respectively. For such elements we find

$$\begin{tikzcd}
{[x] \otimes [x']} \arrow[r, mapsto, "\times"] \arrow[d, mapsto] & {[x \otimes x']} \arrow[d, mapsto] \\
{[p_i(x)]_\infty \otimes [p'_j(x')]_\infty} \arrow[r, mapsto, "\times_\infty"] & {[(p \otimes p')_{i+j}(x \otimes x')],}
\end{tikzcd}$$

which proves the claim. $\square$

In total:

Theorem 7.10. *We have constructed a pairing (natural in maps of filtered chain complexes)*

$$\times \colon \mathcal{L}(C, F) \hat{\times} \mathcal{L}(C', F') \longrightarrow \mathcal{L}(C \otimes C', F \otimes F')$$

such that $\times_2$ *corresponds to*

$$\times \colon H^*(\mathrm{Gr}_F C) \otimes H^*(\mathrm{Gr}_{F'} C') \longrightarrow H^*(\mathrm{Gr}_{F \otimes F'} C \otimes C')$$

under the canonical isomorphisms $H^*(\mathrm{Gr}_-(-)) \cong \mathcal{L}_1(-)$ *and* $\times_\infty$ *corresponds to*

$$\times \colon \mathrm{Gr}_{HF} H^*(F_{-\infty}) \otimes \mathrm{Gr}_{HF'} H^*(F'_{-\infty}) \longrightarrow \mathrm{Gr}_{H(F \otimes F')} H^*((F \otimes F')_{-\infty})$$

under the abutments $p : \mathrm{Gr}_{H-} H^*(-_{-\infty}) \to \mathcal{L}_\infty(-)$.

Corollary 7.11. *Filtering a differential graded algebra by differential ideals yields a multiplicative spectral sequence.*

Examples 7.12.

7.3 Multiplicative double complexes and products in the Serre spectral sequence

Let us now consider double complexes (C, d^h, d^v) and (C', d'^h, d'^v). With the same conventions as before, we obtain a new double complex

$$(C \otimes C', d^h \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'^h, d^v \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'^v).$$

All that remains to be checked is

$$\begin{aligned}
&(d^h \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'^h)(d^v \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'^v) + (d^v \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'^v) \circ (d^h \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} d'^h) \\
&\qquad = d^h d^v \hat{\otimes} \mathrm{id} + d^h \hat{\otimes} d'^v - d^v \hat{\otimes} d'^h + \mathrm{id} \hat{\otimes} d'^h d'^v + d^v d^h \hat{\otimes} \mathrm{id} + d^v \hat{\otimes} d'^h - d^h \hat{\otimes} d'^v + \mathrm{id} \hat{\otimes} d'^v d'^h \\
&\qquad = (d^h d^v + d^v d^h) \hat{\otimes} \mathrm{id} + \mathrm{id} \hat{\otimes} (d'^h d'^v + d'^v d'^h) \\
&\qquad = 0
\end{aligned}$$

Note that

$$(\mathrm{Tot} C \otimes \mathrm{Tot} C')^k = \bigoplus_j \mathrm{Tot} C^j \otimes \mathrm{Tot} C'^{k-j} = \bigoplus_{i,j,l} C^{i,j-i} \otimes C'^{l,k-j-l}$$

whereas

$$\mathrm{Tot}^k C \otimes C' = \bigoplus_j (C \otimes C')^{j,k-j} = \bigoplus_j (\bigoplus_{i,l} C^{i,l} \otimes C' j - i, k - j - l$$

which agree by considering the reindexing given by changing to $j+i$ (since $(i,j) \mapsto (i, j+i)$ is bijective). Now consider the horizontal filtrations, say. They are given by

$$F_i^h C = \oplus_{j \geq i} C^{j,*}$$

and

$$F_i^h C \otimes C' = \oplus_{j \geq i} (C \otimes C')^{j,*} = \bigoplus_{j \geq i} \bigoplus_k C^{k,*} \otimes C^{j-k,*}$$

$$(F^h C \otimes F^h C')_i = \bigcup_j \mathrm{im} F_j^h C \otimes F_{i-j}^h C' = \bigcup_j \oplus_{k \geq j} C^{k,*} \otimes \oplus_{l \geq i-j} C^{l,*}$$

which again agree (both sides consisting of all $C^{k,*} \otimes C'^{l,*}$ with $k + l \geq i$. We obtain a pairing of spectral sequences

$$\mathcal{L}^h C \otimes \mathcal{L}^h C' \to \mathcal{L}^h (C \otimes C')$$

which on the second page is given by the standard exterior product and which is compatible with the abutment constructed earlier.

We shall apply these observations to Dress' construction of the Serre spectral sequence. Recall that this started with the bisimplicial set X with $X_{p,q}$consisting of

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta^p \arrow[r] & B \end{tikzcd}$$

The cohomology in the first direction is constantly that of E and then taking cohomology in the second direction produces a second page consisting of the cohomology of E in a single line, thereby showing that the cohomology of the total complex is that of E. Taking cohomology in the second direction produces a spectral sequence whose first page is quasi-isomorphic to the complex defining $H^*(B, \mathcal{H}^*(F))$ and this was our definition of the Serre spectral sequence. Given now two fibrations $\pi : E \to B$ and $\pi' : E' \to B'$ we claim that there is a map of double complexes

$$EZ : C_*(\pi) \otimes C_*(\pi') \to C_*(\pi \times \pi')$$

which automatically induces a pairing

$$\mathcal{S}(\pi) \otimes \mathcal{S}(\pi') \longrightarrow \mathcal{S}(\pi \times \pi')$$

for the homological Serre spectral sequence. This map of double complexes is induced by the Eilenberg-Zilber map and the identification $X(\pi) \times X(\pi') = X(\pi \times \pi')$: In formulas, a generator

...

However, when switching from homology to cohomology we need a chain map

$$C^*(\pi) \otimes C^*(\pi') \to C^*(\pi \times \pi')$$

and the dual of the Eilenberg-Zilber map goes the opposite way. And while it is a quasi-isomorphism that does not give us sufficient information. The solution is the Alexander-Whitney map

$$AW : C_*(\pi \times \pi') \longrightarrow C_*(\pi) \otimes C_*(\pi'),$$
$$\left[\varphi : \Delta^p \times \Delta^q \to E \times E'\right] \longmapsto \Big[\sum_{\substack{(a,a')+(b,b')\\ =(p,q)}} \delta_f^{(a,b)}\varphi_1 \otimes \delta_b^{(a',b')}\varphi_2\Big]$$

Since products play a bigger role in cohomology we shall analyse this case. First let us check that the abutment is indeed compatible with this pairing.

Lemma 7.13. *The cohomology cross product* $\times : H^*(E)\otimes H^*(E') \to H^*(E\times E')$ *is filtration preserving for the filtrations* $F\pi$, $F\pi'$ *and* $F(\pi\times\pi')$

Proof. We can produce an explicit quasi-isomorphism $i : \mathrm{Sing}_*(E) \to \mathrm{Tot}C_*(\pi)$ by sending a generator $\Delta^p \to E$ to

$$\begin{tikzcd} \Delta^p \times \Delta^0 \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta^p \arrow[r] & B \end{tikzcd} .$$

The verification that this is a quasi-isomorphism boils down to the identification of the edge homomorphism for the spectral sequence associated to a double complex which first takes differentials in the p-direction, as that spectral sequence was concentrated in the $q=0$ row from its second page onwards. By definition of the filtration $x \in H^p(E\times E')$ lies in $F(\pi)_{-k}$ if $i(x)$ can be represented by a chain $\sum n_i\varphi_i$ with $\varphi_i : \Delta^{p_i}\times\Delta^{q_i} \to E\times E'$ ($p_i+q_i=p$) with $p_i \le k$. Then $i(AW(x)) = AW(i(x))$ (by construction of the two Alexander-Whitney maps) so $i(AW(x))$ is represented by $\sum n_i AW(\varphi_i)$ and clearly

$$[\delta_f^{(a,b)}\varphi_1 \otimes \delta_b^{(a',b')}\varphi_2] \in F\pi_{p-a}\otimes F\pi'_{p-a'} \subseteq (F\pi\otimes F\pi')_{2p-a-a'=p}.$$

Dualising this statement shows that the Alexander-Whitney map is filtration preserving on cochains and since it induces the cohomology cross product the claim follows. □

Proposition 7.14. *The diagram*

$$\begin{tikzcd} \mathcal{S}_\infty(\pi)\otimes\mathcal{S}_\infty(\pi') \arrow[r, "AW^*\circ\times"] & \mathcal{S}_\infty(\pi\times\pi') \\ \mathrm{Gr}_{F\pi}H^*(E)\otimes \mathrm{Gr}_{F\pi'}H^*(E') \arrow[u] \arrow[r, "\times"] & \mathrm{Gr}_{F(\pi\times\pi')}H^*(E\times E') \arrow[u] \end{tikzcd}$$

commutes.

Proof. We first enlarge this square as follows:

$$\begin{tikzcd} \mathcal{L}_\infty(C^*\pi)\otimes\mathcal{L}_\infty(C^*\pi') \arrow[r, "\times"] & \mathcal{L}_\infty(C^*(\pi)\otimes C^*(\pi')) \\ & \mathrm{Gr}_{HF_1}H^*(\mathrm{Tot}C^*(\pi)\otimes C^*(\pi')) \arrow[u] \\ \mathrm{Gr}_{HF_1}H^*(\mathrm{Tot}C^*(\pi))\otimes \mathrm{Gr}_{HF_1}H^*(\mathrm{Tot}C^*(\pi')) \arrow[uu] \arrow[r, "\times"] \arrow[d, "\cong"] & \mathrm{Gr}_{HF_1\otimes HF_1}H^*(\mathrm{Tot}C^*(\pi))\otimes H^*(\mathrm{Tot}C^*(\pi')) \arrow[u, "\mathrm{Gr}(\times)"] \arrow[d, "\cong"] \\ \mathrm{Gr}_{F\pi}H^*(E)\otimes\mathrm{Gr}_{F\pi'}H^*(E') \arrow[r, "\times"] & \mathrm{Gr}_{F\pi\otimes F\pi'}H^*(E)\otimes H^*(E') \end{tikzcd}$$

which commutes by our considerations on the spectral sequence of filtered complex and

$$\begin{tikzcd}
\mathcal{L}_\infty(C^*(\pi)\otimes C^*(\pi')) \arrow[r, "AW^*"] & \mathcal{L}_\infty(C^*(\pi\times\pi')) \\
\mathrm{Gr}_{HF_1}H^*(\mathrm{Tot}C^*(\pi)\otimes C^*(\pi')) \arrow[u] \arrow[r, "AW^*"] \arrow[ddr, "\cong"] & \mathrm{Gr}_{HF_1}H^*(\mathrm{Tot}C^*(\pi\times\pi')) \arrow[u] \arrow[d, "\cong"] \\
\mathrm{Gr}_{HF_1\otimes HF_1}H^*(\mathrm{Tot}C^*(\pi))\otimes H^*(\mathrm{Tot}C^*(\pi')) \arrow[u, "\times"] \arrow[d, "\cong"] & \mathrm{Gr}_{F(\pi\times\pi')}H^*(E\times E') \\
\mathrm{Gr}_{F\pi\otimes F\pi'}H^*(E)\otimes H^*(E') \arrow[r, "\times"] & \mathrm{Gr}\,H^*(C^*(E)\otimes C^*(E')) \arrow[u, "AW^*"']
\end{tikzcd}$$

where the upper square clearly commutes, the lower left triangle does so by naturality of the cross product and the upper triangle commutes since $i\circ AW = AW\circ i$ as we established in the last proof already. $\square$

Now we turn to the identification of the induces product on the second page of the spectral sequence. To this end recall the identification $\mathcal{S}_2\cong H^*(B,\mathcal{H}^*(F))$: In homology it is given by the chain map

$$\begin{aligned} j_*: \mathcal{S}^1(\pi) &\longrightarrow C_*(B,\mathcal{H}_*(F)) \\ \big[\sigma:\Delta^p\times\Delta^q\to E\big] &\longmapsto [\sigma_{|e_0\times\Delta^q}]\cdot\sigma_B \end{aligned}$$

which we showed was a quasi-isomorphism way back when (WRITE UP THE IDENTIFICATION WITH WHAT THOMAS DID!). The cohomology version j^* is slightly more complicated: It sends $\varphi\in C^p(B,\mathcal{H}^q(F))$ to the map that sends

$$\big[\sigma:\Delta^p\times\Delta^q\to E\big]\longmapsto\langle\varphi(\sigma_B),\sigma_{e_0\times\Delta^q}\rangle.$$

It remains to check that this map is compatible with the Alexander-Whitney maps. In the source this corresponds to

$$C^p(B,\mathcal{M})\otimes C^{p'}(B',\mathcal{M}')\to C^{p+p'}(B\times B',\mathcal{M}\boxtimes\mathcal{M}')$$

which sends $\varphi\otimes\psi$ to the map that sends $\sigma:\Delta^{p+p'}\to B\times B'$ to

$$\sum(\varphi\otimes\circ\psi)(\partial_f^a\sigma_1\otimes\partial_b^b\sigma_2)$$

The composite $C^*(B,\mathcal{H}^*(F))\otimes C^*(B',\mathcal{H}^*(F'))\to C^*(B\times B',\mathcal{H}^*(F)\boxtimes\mathcal{H}^*(F'))\to C^*(B\times B',\mathcal{H}^*(F\times F'))$ induces the (exterior) exterior product in cohomology with local coefficients.

Proposition 7.15. *The diagram*

$$\begin{tikzcd}
\mathcal{S}_1(\pi)\otimes\mathcal{S}_1(\pi') \arrow[r] & \mathcal{S}_1(\pi\times\pi') \\
C^p(B,\mathcal{H}^q(F))\otimes C^{p'}(B',\mathcal{H}^{q'}(F')) \arrow[u] \arrow[r] & C^{p+p'}(B\times B',\mathcal{H}^{q+q'}(F\times F')) \arrow[u]
\end{tikzcd}$$

commutes up to the sign $(-1)^{pq'+p'q}$*. In particular,*

$$\begin{tikzcd}
\mathcal{S}_2(\pi)\otimes\mathcal{S}_2(\pi') \arrow[r] & \mathcal{S}_2(\pi\times\pi') \\
H^*(B,\mathcal{H}^*(F))\otimes H^*(B',\mathcal{H}^*(F')) \arrow[r] \arrow[u] & H^*(B\times B',\mathcal{H}^*(F\times F')) \arrow[u]
\end{tikzcd}$$

does so as well.

Proof. The upper composition sends $\varphi\otimes\psi\in C^p\otimes C^{p'}$ to $j^*\varphi\otimes j^*\psi$ and then to $AW^*(j^*\otimes j^*(\varphi\otimes\psi))$. This is represented by the homomorphism which sends $\sigma:\Delta^r\times\Delta^s\to E\times E'$ to

$$\sum(j^*\varphi\otimes j^*\psi)(\partial_f^{(a,b)}\sigma_1\otimes\partial_b^{(a',b')}\sigma_2)$$

$$=\sum(-1)^{(p'+q')(r+s-a-b)}\left\langle\varphi((\partial_f^{(a,b)}\sigma_1)_B),(\partial_f^{(a,b)}\sigma_1)_{|e_0\times\Delta^{s-b}}\right\rangle\left\langle\psi((\partial_b^{(a',b')}\sigma_2)_B),(\partial_b^{(a',b')}\sigma_2)_{|e_0\times\Delta^{q-b'}}\right\rangle$$

$$=\sum(-1)^{(p'+q')(r+s-a-b)}\left\langle\varphi(\partial_f^a(\sigma_1)_B),\partial_f^b(\sigma_{|e_0\times\Delta^s})_1\right\rangle\left\langle\psi(\partial_b^{a'}(\sigma_2)_{B'}),\partial_b^{b'}(\sigma_{|e_0\times\Delta^s})_2\right\rangle$$

The lower composition on the other hand sends $\varphi\otimes\psi\in C^p\otimes C^{p'}$ to the homomorphism that sends σ to

$$\sum\left\langle(\varphi\times\psi)(\partial_f^a(\sigma_{B\times B'})_1\otimes\partial_b^{a'}(\sigma_{B\times B'})_2),\sigma_{e_0\times\Delta^s}\right\rangle$$

$$=\sum(-1)^{p'(r-a)}\left\langle\varphi(\partial_f^a(\sigma_1)_B)\otimes\psi(\partial_b^{a'}(\sigma_2)_{B'}),\partial_f^b(\sigma_1)_{e_0\times\Delta^s}\otimes\partial_b^{b'}(\sigma_2)_{e_0\times\Delta^s}\right\rangle$$

$$=\sum(-1)^{p'(r-a)+q'(s-b)}\left\langle\varphi(\partial_f^a(\sigma_1)_B),\partial_f^b(\sigma_1)_{e_0\times\Delta^s}\right\rangle\left\langle\psi(\partial_b^{a'}(\sigma_2)_{B'}),\partial_b^{b'}(\sigma_2)_{e_0\times\Delta^s}\right\rangle$$

Now these numbers can only be non-zero when $p=r-a,q=s-b,p'=r-a'$ and $q'=s-b'$ in which case the two expression differ by the claimed sign. $\square$

8 Steenrod operations

What is a cohomology operation?

Definition 8.1. Let E be a (generalized) cohomology theory on pointed spaces. Then a cohomology operation of degree n is a map $E^*(X)\to E^{*+n}(X)$ of graded abelian groups (sometimes only as graded sets) natural with respect to the structure maps associated to maps $X\to X$. It is called *stable* if it is moreover compatible with respect to the suspension isomorphism

$$\partial:E^*(X)\to E^{*+1}(\Sigma X)$$

in the sense that the diagram

$$\begin{tikzcd}
E^*(X) \arrow[r] \arrow[d] & E^{*+1}(\Sigma X) \arrow[d] \\
E^{*+n}(X) \arrow[r] & E^{*+1+n}(\Sigma X)
\end{tikzcd}$$

commutes for all pointes spaces X.

Recall that cohomology of a pointed space is by definition reduced cohomology (we will supress the tilde)

Example 8.2. *The identity and zero are clearly stable cohomology operations for every E. For the short exact sequence $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$ of abelian groups we get the Bockstein $H^*(-,\mathbb{Z}/2) \to H^{*+1}(-,\mathbb{Z}/2)$ which is a stable cohomology operation.*

For $E = H^*(-,\mathbb{F}_2)$ we can also consider

$$\widetilde{\mathrm{Sq}^n} : H^*(X,\mathbb{F}_2) \to H^{*+n}(X,\mathbb{F}_2)$$

given by sending a class $x \in H^n(X,\mathbb{F}_2)$ to $x^2 \in H^{2n}(X,\mathbb{F}_2)$ and all classes in other degrees to 0. We claim that this is a cohomology operations. It is additive since

$$(x+y)^2 = x^2 + y^2 + 2xy = x^2 + y^2$$

and also natural. This operation is <u>not</u> stable since for $x \in H^n(X,\mathbb{F}_2)$ the element $x^2 \in H^{2n}(X,\mathbb{F}_2) \cong H^{2n+1}(\Sigma X,\mathbb{F}_2)$ is not in the image of a cup product map.

It turns out that one can correct this failure of stability for these operations. We get the following theorem which will be the content of the next two lectures.

Theorem 8.3 (Steenrod)**.** *There are unique cohomology operations $\mathrm{Sq}^i : H^n(X,\mathbb{F}_2) \to H^{n+i}(X,\mathbb{F}_2)$ for every $i \geq 0$ which have the following properties:*

- Sq^0 *is the identity homomorphism.*
- Sq^i *is the cup square on classes of degree i.*
- *If $i > deg(x)$ then $\mathrm{Sq}^i(x) = 0$*
- *They satisfy the Cartan Formula*

$$\mathrm{Sq}^n(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\mathrm{Sq}^j(y).$$

The operations are moreover stable, Sq^1 is the Bockstein of the sequence $\mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2$ and they satisfy the Adem relations

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{n=0}^{\lceil i/2 \rceil} \binom{j-n-1}{i-2n} \mathrm{Sq}^{i+j-n}\mathrm{Sq}^n$$

for all $0 < i < 2j$.

One can work rather axiomatically with these operations. Note that there are similar operations for odd p but we will restrict for simplicity to the case $p = 2$, also since this is the most important prime in homotopy theory.

Example 8.4. *For the space $\mathbb{C}P^2$ we have that $H^*(\mathbb{C}P^2,\mathbb{F}_2) = \mathbb{F}_2[x]/x^3$. Thus we find that $\mathrm{Sq}^2(x) = x^2$. For $\Sigma\mathbb{C}P^2$ we get that*

$$H^*(\Sigma\mathbb{C}P^2,\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{for } * = 0,3,5 \\ 0 & \text{else.} \end{cases}$$

By stability of the Steenrod operations (we suppress the distinction between reduced and unreduced cohomology) we find that $\mathrm{Sq}^2 : H^3(\mathbb{C}P^2, \mathbb{F}_2) \to H^5(\mathbb{C}P^2, \mathbb{F}_2)$ *is an isomorphism. Thus shows that the attaching map of* $\mathbb{C}P^2$ *which is given by the Hopf map* $S^3 \to S^2$ *is non-trivial after suspending once, and by a similar argument suspending as many times as possible. We already knew this, since it is a generator of* $\pi_3(S^2)$ *and the map*

$$\pi_3(S^2) \to \pi_4(S^3) = \mathbb{Z}/2$$

is surjective.

8.1 Construction

We will now prove (at least part of) the Theorem. The idea is to construct Steenrod operations as *power operations.* To this end consider cochain complexes of $\mathbb{F}_2$-vector spaces

$$... \to V^{-1} \to V^0 \to V^1 \to ...$$

an example is given by $V := C^*(X, \mathbb{F}_2)$ for a space X. Then there is a chain level multiplication $m : V \otimes V \to V$ which can be explicitly given by the Alexander-Whitney map. While this can be choosen associative (so that $C^*(X, \mathbb{F}_2)$ is a DGA) it is not symmetric, i.e. the map m is not equal to σm where $\sigma \in \Sigma_2$ acts by permutation. In other words, the map m does not factor through the coinvariants

$$(V \otimes V)_{C_2} \to V.$$

However the map σm is chain homotopic to m (which explains why it is commutative on homotopy). The Steenrod operations will be a consequence of the fact that m is not equal to m'. To see this we have to make precise the fact that m is homotopy coherently symmetric.

For every chain complex we have the n-fold tensor product $V \otimes ... \otimes V$ which is equipped with a natural Σ_n-action by permutation. Using this Σ_n-action we can form the n-th extended power

$$D_n(V) := (V \otimes ... \otimes V)_{h\Sigma_n} := (V^{\otimes n} \otimes E\Sigma_n)_{\Sigma_n}$$

where $E\Sigma_n$ is a resolution of $\mathbb{F}_2$ with trivial Σ_n-action by free $\mathbb{F}_2[\Sigma_n]$-modules. Concretely for $n = 2$ we can take the standard resolution

$$... \to \mathbb{F}_2[\Sigma_2] \xrightarrow{\cdot(1+\sigma)} \mathbb{F}_2[\Sigma_2] \xrightarrow{\cdot(1+\sigma)} \mathbb{F}_2[\Sigma_2] \xrightarrow{\cdot(1+\sigma)} \mathbb{F}_2[\Sigma_2]$$

where $\Sigma_2 = \{1, \sigma\}$. Note that this is the cellular chain complex of S^∞ with respect to the standard cell structure and the antipodal Σ_2-action whose quotient is $\mathbb{R}P^\infty$.

Example 8.5. *Let* $V = \mathbb{F}_2[n]$ *the chain complex on a class in cohomological degree* n*. Then*

$$D_2(V) \simeq \mathbb{F}_2[2n] \otimes (E\Sigma_2)_{\Sigma_2} \simeq C^{\mathrm{cell}}_{-*}(\mathbb{R}P^\infty; \mathbb{F}_2)[2n]$$

Thus

$$H^k(D_2V) \cong H_{2n-k}(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & \text{if } k \leq 2n \\ 0 & \text{else} \end{cases}$$

Definition 8.6. We say that a homotopy coherent symmetric multiplication on a complex V is a map $m : D_2(V) \to V$.

A homotopy coherent symmetric multiplication amounts to giving a multiplication $m' : V \otimes V \to V$ together with a homotopy symmetric chain homotopy $m' \simeq (m')^{op}$. The latter fact follows since identity and the non-identity in $(E\Sigma_2)_0$ are cohomologous.

Example 8.7. *Every CDGA V admits a homotopy coherent symmetric multiplication given by the map*

$$D_2(V) = (V \otimes V \otimes E\Sigma_n)_{C_2} \to (V \otimes V)_{C_2} \to V \ .$$

Proposition 8.8. *For every space X there is a homotopy coherent symmetric multiplication on $C^*(X, \mathbb{F}_2)$ which is natural in X and such that it induces on cohomology the cup product.*

Proof. We have to construct a C_2-equivariant map

$$C^*(X, \mathbb{F}_2) \otimes C^*(X, \mathbb{F}_2) \otimes EC_2 \to C^*(X, \mathbb{F}_2)$$

where the target is equipped with the trivial action. We will construct a C_2-equivariant map natural

$$EC_2 \otimes C_*(X, \mathbb{F}_2) \to C_*(X, \mathbb{F}_2) \otimes C_*(X, \mathbb{F}_2)$$

and then get the map in question by dualizing. More precisely as the composition

$$\begin{aligned} C^*(X, \mathbb{F}_2) \otimes C^*(X, \mathbb{F}_2) \otimes EC_2 &\to (C_*(X, \mathbb{F}_2) \otimes C_*(X, \mathbb{F}_2) \otimes EC_2^\vee)^\vee \\ &\to (EC_2 \otimes C_*(X, \mathbb{F}_2) \otimes EC_2^\vee)^\vee \\ &\to C_*(X, \mathbb{F}_2)^\vee = C^*(X, \mathbb{F}_2). \end{aligned}$$

Now for the construction of the map $EC_2 \otimes C_*(X, \mathbb{F}_2) \to C_*(X, \mathbb{F}_2) \otimes C_*(X, \mathbb{F}_2)$ we use the acyclic models theorem for functors from Top to the category of chain complexes over $\mathbb{F}_2[C_2]$. Now the left hand side evaluated on the simplices

$$EC_2 \otimes C_*(\Delta^p, \mathbb{F}_2)$$

is free over $\mathbb{F}_2[C_2]$. Also $C_*(X, \mathbb{F}_2) \otimes C_*(X, \mathbb{F}_2)$ is acylic (over $\mathbb{Z}[F_2]$ for $X = \Delta^p$. It now follows formally (from the acyclic models theorem) that natural transformations $EC_2 \otimes C_*(X, \mathbb{F}_2) \to C_*(X, \mathbb{F}_2) \otimes C_*(X, \mathbb{F}_2)$ up to chain homotopy are in bijection with natural transformations over $\mathbb{F}_2[C_2]$ between

$$H_0(EC_2 \otimes C_*(X, \mathbb{F}_2)) \qquad \text{and} \qquad H_0(C_*(X, \mathbb{F}_2) \otimes C_*(X, \mathbb{F}_2))$$

These groups are given by $H_0(X, \mathbb{F}_2)$ with trivial action of C_2 and $H_0(X, \mathbb{F}_2) \otimes H_0(X, \mathbb{F}_2)$. There is an obvious transformation which is induced by the diagonal of $\pi_0(X) \to \pi_0(X) \times \pi_0(X)$. □

Remark 8.9. There is a notion of having a homotopy coherent and symmetric algebra structure on a chain complex. This is what is called an E_∞-algebra. Every such E_∞-algebra gives rise to a homotopy coherent symmetric multiplication and one can show that $C^*(X, \mathbb{F}_2)$ actually admits an E_∞-structure. This is really the generality in which this should be done.

Definition 8.10. Let V be a chain complex and $v \in H^n V$. For $i \leq n$ define $\overline{\mathrm{Sq}}^i(v) \in H^{n+i} D_2(V)$ as follows:

> Realize v as a map $\mathbb{F}_2[n] \to V$ and consider the element induced from the map $D_2(\mathbb{F}_2[n]) \to D_2 V$ evaluted on the non-trivial element in $H^{n+i}(D_2 \mathbb{F}_2[n])$.

We set $\overline{\mathrm{Sq}}^i(v) = 0$ for $i > n$. If V is equpped with a homotopy coherent symmetric multiplication $m : D_2(V) \to V$ then we set $\mathrm{Sq}^i(v) = m_* \overline{\mathrm{Sq}}^i(v) \in H^{n+i}(V)$.

Note that we do allow $i \leq 0$ here and in fact there are classes in all these degrees of $D_2\mathbb{F}_2[n]$.

Example 8.11. *Let $v \in H^n(V)$ and consider $\overline{\mathrm{Sq}}^n(v)$. This is given by the image of $v \otimes v \in V \otimes V$ under the map $V \otimes V \to D_2(V)$. Thus $\mathrm{Sq}^n(v)$ is given by squaring v.*

Assume that V is equipped with a multiplication which is strictly symmetric, i.e. factors through the quotient

$$D_2(V) \to (V \otimes V)_{\Sigma_2}.$$

Then $\mathrm{Sq}^i(v) = 0$ for $i \neq n$ for v of degree n. To see this we observe that the above quotient map induces for $V = \mathbb{F}_2[n]$ on cohomology the zero map in all degrees except $2n$.

Example 8.12. *Since EC_2 can be chosen to be free on one generator in each nonnegative degree, a map $EC_2 \otimes C_*(X) \to C_*(X) \times C_*(X)$ is the same as a collection of maps μ_i with $d\mu_i + \mu_i d = \mu_{i-1} + \sigma\mu_{i-1}$.*

On $C_*(\Delta^1)$, the map $EC_2 \otimes C_*(\Delta^1, \mathbb{F}_2) \to C_*(\Delta^1, \mathbb{F}_2) \otimes C_*(\Delta^1, \mathbb{F}_2)$ can be chosen as follows: Let σ be the fundamental 1-simplex, and $d\sigma = x_1 - x_0$. Then $\mu_0(x_i) = x_i \otimes x_i$ (by naturality and the values on the 0-simplex), so $d\mu_0(\sigma) = \mu_0(d\sigma) = x_1 \otimes x_1 - x_0 \otimes x_0$.

We can pick $\mu_0(\sigma) = \sigma \otimes x_1 - x_1 \otimes \sigma$, which by naturality determines μ_0 on all 1-chains.

However, $\mu_0(\sigma)$ is not symmetric, which is why $d\mu_1(\sigma) + \mu_1(d\sigma) = \mu_0(\sigma) + \sigma\mu_0(\sigma) = d(\sigma \otimes \sigma)$, so we can pick $\mu_1(\sigma) = \sigma \otimes \sigma$. Note how this is symmetric, so we can choose $\mu_2(\sigma)$ and higher to be trivial.

The dual of μ_{n-i} evaluated on $v \times v$ determines Sq^i on a $v \in C^n(X; \mathbb{F}_2)$. The above computation shows that $\mathrm{Sq}^0(v) = v$ for v a generator in $H^1(S^1; \mathbb{F}_2)$, and similarly $\mathrm{Sq}^{-i}(v) = 0$.

We will now list the basic properties of the Steenrod operations (proofs are mostly sketched or omitted, but fairly standard and well-known).

Proposition 8.13. *The operations $\overline{\mathrm{Sq}}^i : H^n(V) \to H^{n+i}(D_2V)$ are additive. Thus also the Steenrod operations $\mathrm{Sq}^i : H^n(V) \to H^{n+i}(V)$ for V equipped with a homotopy coherent symmetric multiplication.*

Proof. We want to show that for $v, w \in H^n(V)$ we have

$$\overline{\mathrm{Sq}}^i(v + w) = \overline{\mathrm{Sq}}^i(v) + \overline{\mathrm{Sq}}^i(w)$$

If $i > n$ then both sides are zero. For $i = n$ we find that

$$\overline{\mathrm{Sq}}^i(v + w) = (v + w) \otimes (v + w) = \overline{\mathrm{Sq}}^i(v) + \overline{\mathrm{Sq}}^i(w) + v \otimes w + w \otimes v.$$

Since the map $V \otimes V \to D_2V$ is symmetric in cohomology we get that $v \otimes w + w \otimes v = 2v \otimes w = 0$.

Now for $i < n$ we consider the universal case $V = \mathbb{F}_2[n] \oplus \mathbb{F}_2[n]$. One checks by explicit calculation that the canonical map

$$H^m D_2(V) \to H^m D_2(\mathbb{F}_2[n]) \times H^m D_2(\mathbb{F}_2[n])$$

is injective for $m < 2n$. This means that we can assume wlog that either v or w is zero in which case the result is obvious. $\square$

Proposition 8.14 (Cartan Formula)**.** *For V, W, consider the external product $\times : H^n(V) \otimes H^m(W) \to H^{n+m}(V \otimes W)$.*

There is a map $D_2(V \otimes W) \to D_2V \otimes D_2W$, and it maps

$$\overline{\mathrm{Sq}}^k(v \times w) \mapsto \sum_{i+j=k} \overline{\mathrm{Sq}}^i v \times \overline{\mathrm{Sq}}^j w$$

Proof. We can consider the map $D_2(V \otimes W) \to D_2V \otimes D_2W$ as the map

$$(V \otimes V \otimes W \otimes W \otimes EC_2)_{C_2} \to (V \otimes V \otimes W \otimes W \otimes EC_2 \otimes EC_2)_{C_2 \times C_2}$$

which is a map on $EC_2 \to EC_2 \otimes EC_2$ compatible with the action by the diagonal embedding $C_2 \to C_2 \times C_2$.

For the universal example of $\mathbb{F}_2[n] \otimes \mathbb{F}_2[m]$, the left side is a copy of the cellular complex $C^{\text{cell}}_{-*}(\mathbb{R}P^\infty; \mathbb{F}_2)[2n+2m]$, the right side is $C^{\text{cell}}_{-*}(\mathbb{R}P^\infty; \mathbb{F}_2)[2n] \otimes C^{\text{cell}}_{-*}(\mathbb{R}P^\infty; \mathbb{F}_2)[2m]$, and the map is the diagonal on homology, which we know to satisfy

$$x_k \mapsto \sum_{i+j=k} x_i \times x_j$$

where x_i denotes the nontrivial element in $H_i(\mathbb{R}P^\infty; \mathbb{F}_2)$. The result follows. □

Now for spaces, we have

Proposition 8.15. *The following diagram commutes*

$$\begin{tikzcd}
D_2(C^*(X) \otimes C^*(Y)) \arrow[rr] \arrow[d] & & D_2(C^*(X \times Y)) \arrow[d] \\
D_2(C^*(X)) \otimes D_2(C^*(Y)) \arrow[r] & C^*(X) \otimes C^*(Y) \arrow[r] & C^*(X \times Y)
\end{tikzcd}$$

(up to essentially unique chain homotopy)

Proof. By dualizing, we can translate this to a question about a homotopy between two equivariant maps out of $C_*(X \times Y) \otimes EC_2$, natural in X, Y. So it is enough to construct a homotopy on simplices $\Delta^p \times \Delta^q$, and the target will be contractible in higher degrees, so the map is determined by what it does on H_0. There we can check they agree. □

Corollary 8.16. *For Sq^i on spaces, the Cartan formula*

$$\mathrm{Sq}^k(v \times w) = \sum_{i+j=k} \mathrm{Sq}^i v \times \mathrm{Sq}^j w$$

holds for the usual cross product $\times : H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$, and analogously for the cup product and the reduced cross product $\times : \widetilde{H}^*(X) \otimes \widetilde{H}^*(Y) \to \widetilde{H}^*(X \wedge Y)$.

Proof. The observation about the cup product holds by naturality, via the diagonal map $X \to X \times X$. The observation about the reduced cross product holds by the commutative diagram relating the unreduced to the reduced cross product, and naturality. □

Corollary 8.17. *The Sq^i commute with the suspension isomorphism $\widetilde{H}^*(X) \simeq \widetilde{H}^{*+1}(\Sigma X)$.*

Proof. Write the suspension iso as the reduced cross product with the fundamental class $\alpha \in H^1(S^1)$. Then

$$\mathrm{Sq}^n(\alpha \times v) = \sum_{i+j=n} \mathrm{Sq}^i\alpha \times \mathrm{Sq}^j v = \alpha \times \mathrm{Sq}^n v$$

□

Example 8.18. *Let $v \in H^n(S^n)$ be the generator. Then $\mathrm{Sq}^0(v) = v$ and $\mathrm{Sq}^i(v) = 0$ for $i \neq 0$.*

To see this, obtain it from example 8.12 by repeated application of the previous statement.

Proposition 8.19. *Let X be a topological space. Then $\mathrm{Sq}^i : H^*(X; \mathbb{F}_2) \to H^{*+i}(X; \mathbb{F}_2)$ is given by the identity for $i = 0$ and the zero map for $i < 0$.*

Proof. By representability of cohomology and naturality of the Steenrod operations we can assume that $X = K(\mathbb{F}_2, n)$. The canonical map $S^n \to K(\mathbb{F}_2, n)$ induces a map

$$H^{n+k}(K(F_2, n); \mathbb{F}_2) \to H^{n+k}(S^n; \mathbb{F}_2)$$

which is an isomorphism for $k \leq 0$. Thus we may reduce to $X = S^n$ in which case the claim follows from Example 8.18. □

Recall that we have constructed Steenrod operations Sq^i for $i = 0$ and even negative i. These negative Steenrod operations can be non-trivial but are less well known. The reason is that this cannot happen for the cohomology of spaces.

Corollary 8.20. *The cochains on a space X with its homotopy coherent symmetric multiplication can be represented up to quasi-isomorphism by a CDGA only if X only has homology in degree* 0.

Proof. On classes in higher degree the operation Sq^0 is zero for a CGDA but non-zero in $H^*(X, \mathbb{F}_2)$. □

Since we now know that for any $v \in H^n(X)$, only finitely many Sq^i can be nonzero (namely from Sq^0 to Sq^n), we can define the total Steenrod square

$$\mathrm{Sq}\colon H^*(V) \to H^*(V) \qquad v \mapsto \sum \mathrm{Sq}^k(v)$$

as a (nonhomogeneous) operation. The Cartan formula now amounts to this being a multiplicative operation.

Example 8.21. *Consider $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[t]$. Then*

$$\mathrm{Sq}^k t^n = \binom{n}{k} t^{n+k}$$

if $0 \leq k \leq n$ and $\mathrm{Sq}^k t^n = 0$ otherwise. This follows since $\mathrm{Sq}(t) = t + t^2$ using the Cartan formula

$$\mathrm{Sq}(t^n) = \mathrm{Sq}(t)^n = (t + t^2)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n+k}.$$

We won't give a proof of the Adem relations here. In principle, they could be obtained later from our computation of the cohomology of Eilenberg-Maclane spaces, since they only need to be checked there, and we will learn (after the fact) that $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ embeds into $H^*((\mathbb{R}P^\infty)^{\times n}; \mathbb{F}_2)$, where we can compute everything. The combinatorics involved is messy, though. This is done in Mosher-Tangora.

(The computation of $H^*(K(\mathbb{F}_2, n))$ never uses Adem relations, and is in fact historically older).

Alternatively, one can prove them by thinking about a commutative diagram related to $D_2(D_2(V))$, and doing some group cohomology computations. This is the route Adem followed, and it is more enlightening. Turning that computation into the Adem relations as given earlier requires some combinatorics as well, though.

Example 8.22. *We have already seen how to detect the nontriviality of $\Sigma^n\eta$ for every n using Sq^2. The same holds for ν and σ, using Sq^4 and Sq^8. (And technically, Sq^1 detects 2, but we knew better ways to detect that one in Topology 1 already.)*

One can ask for which $n-1$ there are maps $\alpha : S^{n-1+k} \to S^k$ such that the space built by attaching a C^{n+k}-cell to S^k has a nontrivial action by Sq^n.

By the relation of Sq^n to the unstable cup-square operation in degree n, this is a stable version of the question about the existence of maps of odd Hopf invariant.

It turns out that most Sq^n can be decomposed into lower Sq^n by using Adem relations.

By the Adem relations, the coefficient of Sq^{i+j} in $\mathrm{Sq}^i\mathrm{Sq}^j$ is $\binom{j-1}{i}$.

We can compute $\binom{n}{k}$ mod 2 using Lucas' theorem: $\binom{n}{k} = 1$ mod 2 iff each power of 2 that appears in the binary expansion of k also appears in the binary expansion of n.

For $n = i + j$, let i be the smallest power of 2 that doesn't appear in the binary expansion of $n-1$. Then it does appear in the binary expansion of $j - 1 = n - 1 - i$, and so $\binom{j-1}{i} = 1$ mod 2.

This fails, however, precisely if n is a power of 2. Then the binary expansion of $n-1$ consists only of ones, and every choice of i yields $\binom{j-1}{i} = 0$ mod 2.

This means that Sq^n is decomposable in the Steenrod algebra iff n is not a power of 2. It follows that the only dimensions in which elements in $\pi_*(S)$ can be detected by Steenrod operations in the above way are $2^l - 1$.

It is a way harder theorem of Adams, proven by decomposing even those Sq^{2^l} into a different kind of operations, that none of these Hopf invariant 1 maps exist for $l \geq 4$, so $2, \eta, \nu, \sigma$ are the only examples.

Example 8.23. *Similarly to how Steenrod operations detect some stably nontrivial homotopy classes, relations between Steenrod operations detect some stably nontrivial compositions.*

As it turns out, $\Sigma^n\eta \circ \Sigma^{n+1}\eta \neq 0$. To prove this, assume this were nullhomotopic.

Then we could extend the map $\Sigma^n\eta : S^{n+3} \to S^{n+2}$ over the cone of $\Sigma^{n+1}\eta$, which we can identify with $\Sigma^{n+1}\mathbb{C}P^2$.

Taking the cone of that map $\Sigma^{n+1}\mathbb{C}P^2 \to S^{n+2}$, we obtain a CW-complex K with one $n+2$-cell, one $n+4$-cell and one $n+6$-cell (in addition to the basepoint).

Now denote generators of the homology by $x_{n+2}, x_{n+4}, x_{n+6}$. K has $\Sigma^n\mathbb{C}P^2$ as a subcomplex, so naturality tells us that $\mathrm{Sq}^2 x_{n+2} = x_{n+4}$, and $\Sigma^{n+2}\mathbb{C}P^2$ as a quotient complex (by collapsing the $n+2$-skeleton), so $\mathrm{Sq}^2 x_{n+4} = x_{n+6}$.

So $\mathrm{Sq}^2\mathrm{Sq}^2 x_{n+2} = x_{n+6} \neq 0$. However, $\mathrm{Sq}^2\mathrm{Sq}^2 = \mathrm{Sq}^3\mathrm{Sq}^1$ by Adem relations, and $\mathrm{Sq}^1 x_{n+2} = 0$ for degree reasons. Therefore, $\Sigma^n\eta \circ \Sigma^{n+1}\eta$ can't be 0.

The same method works for $\nu^2 \neq 0$, $\sigma^2 \neq 0$, $2\nu \neq 0$, $2\sigma \neq 0$ and $\eta\sigma \neq 0$, by systematically exploiting Adem relations.

(Remark: This is of course the algebraic reason why none of those show up as candidates in the Adams spectral sequence.)

Example 8.24. *It is an easy consequence of Freudenthal's stability theorem that there is a map $S^{2n-1} \to S^n$ of odd Hopf invariant if and only if there is a stable map of degree $n-1$ with stable Hopf invariant 1 (i.e. detected by Sq^n).*

An old question is whether S^n admits an H-space structure, i.e. a unital multiplication map (with no assumptions on associativity or commutativity). From the classical division algebras over $\mathbb{R}$, we have examples of such structures on S^n for $n = 0, 1, 3, 7$.

Obviously we can't get an H-space structure on S^{2k}, as such a map $S^{2k} \times S^{2k}$ already violates cup products.

So assume n odd. Given an H-space map $\mu : S^n \times S^n \to S^n$, let μ^* denote the same map precomposed with a flip, i.e. the opposite H-space structure.

We can build a space

$$B_\mu^{(2)} S^n = * \sqcup (\Delta^1 \times S^n) \sqcup (\Delta^2 \times S^n \times S^n)/\sim$$

where we glue $\Delta^1 \times S^n$ to $*$ at both ends of Δ^1 to give $n+1$-skeleton $\Sigma S^n = S^{n+1}$, and we glue the simplex $\Delta^2 \times (g,h)$ to $\Delta^1 \times g$, $\Delta^1 \times \mu(g,h)$ and $\Delta^1 \times h$. Additionally, identify $\Delta^2 \times (g,e)$ with $\Delta^1 \times g$ along the corresponding projection, and similarly for $\Delta^2 \times (e,h)$.

This space is a complex with one $n+1$-cell and one $2n+2$-cell. It can therefore be built by attaching a cell along a map $S^{2n+1} \to S^{n+1}$, we want to prove it has Hopf invariant 1.

Lets also look at the space obtained from $\Sigma S^n \times \Sigma S^n$ by identifying the two copies of ΣS^n (This is called $J_2(\Sigma S^n)$). This is another complex with one $n+1$-cell and one $2n+2$-cell. Its Hopf invariant is 2. We can also describe it as

$$J_2(\Sigma S^n) = * \sqcup (\Delta^1 \times S^n) \sqcup \Delta^1 \times \Delta^1 \times S^n \times S^n$$

where we identify opposite ends of the square corresponding to (g,h) with $\Delta^1 \times g$ or $\Delta^1 \times h$ respectively.

Now there is a map

$$J_2(\Sigma S^n) \to B_\mu^{(2)} S^n \sqcup_{\Sigma S^n} B_{\mu^*}^{(2)} S^n$$

by subdividing a square $\Delta^1 \times \Delta^1 \times (g,h)$ into two simplices, sending the diagonal to $\Delta^1 \times \mu(g,h)$, and the lower simplex to $\Delta^2 \times (g,h)$ in $B_\mu^{(2)} S^n$, the upper simplex to $\Delta^2 \times (h,g)$ in $B_{\mu^*}^{(2)} S^n$.

This map has degree 1 onto each of the top cells on the right, so an easy computation shows that the Hopf invariants of $B_\mu^{(2)} S^n$ and $B_{\mu^*}^{(2)} S^n$ add up to 2. (This is the same computation as the one that is used when showing that the Hopf invariant is additive)

But there is a map $B_\mu^{(2)} S^n \to B_{\mu^*}^{(2)} S^n$ obtained by flipping all 1-simplices, reflecting the 2-simplices to exchange the first and the last vertex, and swapping $S^n \times S^n$. So this has degree -1 on the $n+1$-cell, but degree $+1$ on the top cell (since both the flip on $S^n \times S^n$ and the reflection on the 2-simplex contribute a sign, because n is odd).

So the Hopf invariants of the two complexes agree and therefore have to be $+1$.

From the above study of the stable Hopf invariant, we immediately observe that S^n can only admit an H-space structure if $n = 2^l - 1$ ($n = 0, 1, 3, 7$ if combined with Adams' hard theorem about Hopf invariant 1 mentioned above.)

The following two statements are easy direct consequences of this:

- $\mathbb{R}^n$ *can only admit an algebra structure without zero divisors if* $n = 2^l$. *(*$n = 1, 2, 4, 8$ *with Adams' theorem).*
- S^n *can only have trivial tangent bundle if* $n = 2^l - 1$ *(*$n = 0, 1, 3, 7$ *with Adams' theorem).*

8.2 Transgression and the cohomology of Eilenberg-Maclane spaces

Let $F \to E \to B$ be a fiber sequence with B connected, with basepoint b_0.

We want to talk about the differentials $E_n^{0,n-1} \to E_n^{n,0}$. This is defined on a certain subgroup of $H^{n-1}(F)$, and takes values in a certain quotient of $H^n(B)$. We want to call that operation the *transgression*, and elements of $H^{n-1}(F)$ on which it is defined (i.e. elements in the kernel of d_i for $i < n$) *transgressive.*

Theorem 8.25. *In the diagram*

$$H^{n-1}(F) \xrightarrow{\delta} H^n(E, F) \xleftarrow{p^*} H^n(B, b_0)$$

the transgressive elements are precisely those that are sent into the image of p^* *by* δ*, and the indeterminacy, i.e. the subgroup of* $H^n(B)$ *modulo which the transgression is defined, is the kernel of* p^**.*

The transgression can be identified with $(p^*)^{-1} \circ \delta$*.*

Proof. The key idea is to compare the Serre spectral sequence for $H^*(E)$ with the relative Serre spectral sequence for $H^*(E, F)$, which just has the first column removed.

First, note that the differentials hitting the entry $H^n(B, b_0)$ in that spectral sequence are precisely those that hit the group $H^n(B)$ in the absolute Serre spectral sequence before the E_n-page. This means that the image of $H^n(B, b_0)$ in $H^n(E, F)$ agrees precisely with the $E_n^{n,0}$-entry in the absolute spectral sequence.

Second, from relating the connecting homomorphism $\delta : H^{n-1}(F) \to H^n(E, F)$ to the differential in the chain complex of $H^n(E)$, we see that the first nonzero differential on $\alpha \in H^{n-1}(F)$ in the absolute spectral sequence coincides with which entry detects $\delta\alpha \in H^n(E, F)$ in the relative spectral sequence.

This means that α is transgressive iff $\delta\alpha$ is in the image of p^*, and that it transgresses to the coset $(p^*)^{-1}(\delta\alpha)$. □

Theorem 8.26 (Transgression theorem)**.** *The subset of transgressive classes in* $H^*(F)$ *is closed under application of the* Sq^i*, and the transgression commutes with the actions of* Sq^i *on* $H^*(F)$ *and* $H^*(B)$*.*

Proof. This is now an easy corollary: The Sq^i commute with the connecting homomorphism as well as induced maps. □

In particular, this means that the square of a transgressive class $\alpha \in H^n(F)$ either lifts to $H^n(E)$ (because the transgression is 0), or Sq^n acts nontrivially on the transgression of α in H^{n+1}.

We now want to apply this to inductively compute the cohomology of $K(\mathbb{F}_2, n)$ via the fibration $K(\mathbb{F}_2, n) \to * \to K(\mathbb{F}_2, n+1)$. Observe that the cohomology of the total space is trivial, so all transgressive elements will transgress to nonzero elements in $H^*(K(\mathbb{F}_2, n+1); \mathbb{F}_2)$.

First, some notation.

Definition 8.27. For $I = (i_n, i_{n-1}, \dots, i_0)$, denote by Sq^I the composite $\mathrm{Sq}^{i_n}\mathrm{Sq}^{i_{n-1}} \dots \mathrm{Sq}^{i_0}$. We will call I *admissible* if $i_k \geq 2i_{k-1}$ for all k and $i_0 \geq 1$. (The empty sequence is admissible, and $\mathrm{Sq}^{()}$ is the identity.)

The total degree $|I|$ is just the sum of the i_k.

Definition 8.28. Define the *excess* $e(I)$ of an admissible sequence I as

$$\begin{aligned} e(I) &= (i_n - 2i_{n-1}) + \dots + (i_1 - 2i_0) + i_0 \\ &= i_n - (i_{n-1} + \dots + i_0) \\ &= 2i_n - |I| \end{aligned}$$

Theorem 8.29 (Cartan-Serre)**.** *For $n \geq 1$, we have $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Sq}^I \iota_n]$, where ι_n is the universal degree n class and I ranges over all admissible sequences of excess $e(I) < n$.*

Additionally, for $n \geq 2$, $H^*(K(\mathbb{Z}, n); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Sq}^I \iota_n]$, where Sq^I ranges over all admissible sequences of excess $e(I) < n$ with $i_0 \geq 2$ (including the empty one).

For example, this tells us that $H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$ is polynomial on the $\mathrm{Sq}^{2^n}\mathrm{Sq}^{2^{n-1}} \dots \mathrm{Sq}^1\iota_2$.

Lemma 8.30. *For x of degree n and $e(I) > n$, $\mathrm{Sq}^I(x) = 0$.*

For $e(I) = n$, $\mathrm{Sq}^I(x) = (\mathrm{Sq}^{I'}(x))^2$ with $e(I') \leq n$.

In particular, the $\mathrm{Sq}^I(x)$ with $e(I) = n$ agree with some $(\mathrm{Sq}^J(x))^{2^k}$ with $e(J) < n$.

Proof. The first observation follows from instability, since $i_n = i_{n-1} + \dots + i_0 + e(I)$ by definition of $e(I)$, and so

$$\mathrm{Sq}^{i_n}(\mathrm{Sq}^{i_{n-1}} \dots \mathrm{Sq}^{i_0} x) = 0$$

for degree reasons. Similarly, if $e(I) = n$, $\mathrm{Sq}^I(x)$ will be $(\mathrm{Sq}^{I'}x)^2$, where $I' = (i_{n-1}, \dots, i_0)$, and $e(I') \leq e(I)$. If equality holds and $e(I') = n$, we can iterate and get higher powers of 2 in the exponent, until we reach a sequence J of lower excess.

We see that this only happens if the original sequence I agreed with

$$I = (2^{k-1}(n + |J|), \dots, n + |J|, i_{n-k}, i_{n-k-1}, \dots, i_0)$$

with $J = (i_{n-k}, i_{n-k-1}, \dots, i_0)$. □

By the transgression theorem, we see that, for $e(J) < n$, the power $(\mathrm{Sq}^J \iota_n)^{2^k}$ transgresses to $\mathrm{Sq}^I \iota_n$ with

$$I = (2^{k-1}(n + |J|), \dots, n + |J|, j_n, j_{n-1}, \dots, j_0)$$

To determine the structure of the Serre spectral sequence, we will use:

Theorem 8.31. *Assume B simply-connected and E contractible. Say $H^*(F; \mathbb{F}_2)$ contains transgressive elements x_i, transgressing to $y_i \in H^*(B; \mathbb{F}_2)$ such that the square-free monomials of the x_i form a basis of $H^*(F; \mathbb{F}_2)$.*

Then $H^*(B; \mathbb{F}_2)$ is polynomial on the y_i.

Proof. We abstractly write down a spectral sequence that looks how we expect our spectral sequence to look like, namely with $E_2^{*,0}$ polynomial in the y_i, $E_2^{0,*}$ freely generated by squarefree products of the x_i, and differentials given by $d_n(x_i) = y_i$ for x_i of degree $n - 1$, and derivation rule.

Now we note that that spectral sequence has a very nice form: The E_r page is given by all squarefree monomials on x_i of degree $\geq r-1$, tensored with a polynomial ring in all y_i of degree $\geq r$.

Therefore, if we compare this to the Serre spectral sequence by mapping to it, we have a map of spectral sequences, since all the differentials are determined by the transgressions.

Now this map is an isomorphism on $E_2^{0,p}$ and on E_∞ (since E_∞ is trivial). We can inductively prove it is an isomorphism on all $E_2^{q,0}$, which will finish the proof.

Assume q is minimal such that the map on $E_2^{q,0}$ is not an isomorphism. First show that it is surjective: Given any α in the actual Serre spectral sequence in bidegree $(q,0)$, it needs to be hit at some stage, say by a d_n from $(q-n, n-1)$. However, we know already that the E_n entry at $(q-n, n-1)$ agrees with the model spectral sequence, in which case it is some multiple of y_i and x_i and the differential of that is definitely some linear combination of y_i. Since the map is already an isomorphism in lower degrees, we know α is in the image.

Now assume it is not injective. This gives some nontrivial combination of y_i in the model spectral sequence that is already 0 in the E_2 page of the Serre spectral sequence.

So the image of the thing that first hits this element in the model spectral sequence is a permanent cycle in the Serre spectral sequence.

Since all the differentials to the left of q are determined by the model spectral sequence though, this cycle can't be killed. Contradiction. $\square$

Now the Cartan-Serre theorem follows:

Proof of theorem 8.29. For $K(\mathbb{F}_2, n)$, we start an induction with $K(\mathbb{F}_2, 1) = \mathbb{R}P^\infty$, which is polynomial in ι_1 (there are no nontrivial sequences of excess < 1).

For $K(\mathbb{Z}, n)$, we start with $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$, which is polynomial in ι_2 (there are no nontrivial sequences of excess < 2 which don't end in 1).

As x_i, we just take all the $(\mathrm{Sq}^I \iota_n)^{2^k}$ with $e(I) < n$, and we've already seen that those transgress to the respective $\mathrm{Sq}^I \iota_{n+1}$ if $k = 0$, and to all the $\mathrm{Sq}^J \iota_{n+1}$ with $e(J) = n$ if $k > 0$. $\square$

Just for reference, we will also quote the Cartan-Serre theorem for $K(\mathbb{Z}/2^k, n)$:

Theorem 8.32. *For $k \geq 2$, $n \geq 2$, we have $H^*(K(\mathbb{Z}/2^k, n); \mathbb{F}_2) = \mathbb{F}_2[\mathrm{Sq}^I \iota_n, \mathrm{Sq}^J \beta_{n+1}]$, where I ranges over all admissible sequences with $e(I) < n$ and $i_0 \geq 2$, and J ranges over all admissible sequences with $e(J) \leq n$ (note $e(J) = n$ is allowed here!) and $j_0 \geq 2$.*

Here, ι_n and β_{n+1} are not connected by a Steenrod operation (they are connected by a higher Bockstein differential).

8.3 Applications

Recall that we call a sequence of natural transformations, $H^n(X; \mathbb{F}_2) \to H^{n+k}(X; \mathbb{F}_2)$, one for each n, a *stable cohomology operation* (with $\mathbb{F}_2$-coefficients) if they commute with suspension isomorphisms.

Definition 8.33. The graded algebra of all stable cohomology operations (over $\mathbb{F}_2$) with product the composition is called the mod 2 *Steenrod Algebra*.

Now note that in degree $< n$, all Sq^I have excess $< n$, since

$$e(I) = 2i_n - |I| \leq 2|I| - |I| = |I|$$

so up to degree $2n$, $H^*(K(\mathbb{F}_2, n); \mathbb{F}_2)$ has a basis consisting of all the admissible $\mathrm{Sq}^I \iota_n$. We obtain:

Corollary 8.34. *The Steenrod algebra has as basis the admissible monomials (i.e.* Sq^I *for* I *admissible). In particular, it is generated by the* Sq^n *modulo the Adem relations.*

Proof. Everything is clear, except for the last part. This just uses that the Adem relations allow us to rewrite a product of admissible monomials as a linear combination of admissible monomials, which can be proven inductively. Since the admissible monomials are linearly independent, the Adem relations generate all relations. □

Now that we understand the cohomology of Eilenberg-Maclane spaces, we can try to leverage this to determine homotopy groups by induction over the Postnikov tower. This is Serre's method of *killing homotopy groups.*

If X_0 is an $n-1$-connected space, the Hurewicz theorem tells us that $\pi_n(X_0) \simeq H_n(X_0; \mathbb{Z})$. We can then build an Eilenberg-Maclane space from X_0 by attaching cells to kill higher homotopy groups, and consider the fiber X_1 of the resulting map $X \to K(\pi_n(X_0), n)$.

X_1 now has the same higher homotopy groups as X_0, but is n-connected, so we can proceed inductively if we maintain enough control over the homology of X_i in each step.

The following little strengthening of the UCT is helpful to reason about $\mathbb{Z}$ homology from the knowledge of the $\mathbb{Z}/2$ cohomology:

Lemma 8.35. *Given a decomposition of* $H_*(X; \mathbb{Z})$ *into cyclic summands* $\mathbb{Z}$ *and* $\mathbb{Z}/p^k$*, we get*

- *For each* $\mathbb{Z}$ *summand in* $H_n(X; \mathbb{Z})$*, an* $\mathbb{F}_2$*-summand in* $H^n(X; \mathbb{F}_2)$*.*
- *For each* $\mathbb{Z}/2$ *summand in* $H_n(X; \mathbb{Z})$*, two* $\mathbb{F}_2$*-summands in* $H^n(X; \mathbb{F}_2)$ *and* $H^{n+1}(X; \mathbb{F}_2)$ *with a nontrivial action of* Sq^1 *between them.*
- *For each* $\mathbb{Z}/2^k$ *summand with* $k > 1$ *in* $H_n(X; \mathbb{Z})$*, two* $\mathbb{F}_2$*-summands in* $H^n(X; \mathbb{F}_2)$ *and* $H^{n+1}(X; \mathbb{F}_2)$ *with a trivial action of* Sq^1 *between them.*
- *Nothing in* $H^n(X; \mathbb{F}_2)$ *for* $\mathbb{Z}/p^k$ *for* p *odd.*

Proof. The splitting of $H_*(X; \mathbb{Z})$ defines a quasi-isomorphism from a direct sum of chain complexes of the form $\mathbb{Z}$ or $\mathbb{Z} \xleftarrow{p^k} \mathbb{Z}$ onto the chain complex of X. So the statement follows from naturality once we've proven it for these simple chain complexes.

The $\mathbb{Z}/2$-dual cochain complexes evidently have the right cohomology.

To establish the connection to Sq^1, look at the long exact sequence in cohomology for $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2$.

It has the form

$$0 \to \mathbb{Z}/2 \xrightarrow{\tilde{\beta}} \mathbb{Z}/2^k \xrightarrow{2} \mathbb{Z}/2^k \to \mathbb{Z}/2 \to 0$$

Now the composite of $\tilde{\beta} : H^n(\bullet; \mathbb{Z}/2) \to H^{n+1}(\bullet; \mathbb{Z})$ with the restriction $H^{n+1}(\bullet; \mathbb{Z}) \to H^{n+1}(\bullet; \mathbb{Z}/2)$ can be seen to be Sq^1.

In our case, that composition $\mathbb{Z}/2 \xrightarrow{\tilde{\beta}} \mathbb{Z}/2^k \to \mathbb{Z}/2$ is 0 if $k > 1$, and the identity if $k = 1$. □

Example 8.36. *Let's compute $\pi_n(S^3)$ for $n \leq 6$. We know that the only odd torsion in this range is a $\mathbb{Z}/3$ in π_6, which is stable (with respect to the stronger stability theorem on odd spheres that shows that $S^{2n-1} \to \Omega^2 S^{2n+1}$ is an isomorphism on odd-primary homotopy groups up to degree $2pn-3$). So we focus on 2^k-torsion.*

First, we compute the fiber $X_1 \to S^3 \to K(Z,3)$. A basis for the mod 2 cohomology of $K(Z,3)$ in a range up to degree 11 is given by

$$\iota_3, \mathrm{Sq}^2\iota_3, \iota_3^2, \iota_3(\mathrm{Sq}^2\iota_3), \mathrm{Sq}^4\mathrm{Sq}^2\iota_3, \iota_3^3, (\mathrm{Sq}^2\iota_3)^2, \iota_3^2\mathrm{Sq}^2\iota_3$$

and everything except for ι_3 needs to be killed.

We get a class a_4 with $d_5(a_4) = \mathrm{Sq}^2\iota_3$, and by transgression, $d_6(\mathrm{Sq}^1 a_4) = \mathrm{Sq}^1\mathrm{Sq}^2\iota_3 = \iota_3^2$.

The 4-row doesn't have any permanent cycles, since $\mathrm{Sq}^2\iota_3$ is not a zero divisor. The first cycle in the 5-row is $(\mathrm{Sq}^1 a_4)(\mathrm{Sq}^2\iota_3)$, with degree 10. The first element that hasn't been killed yet in the bottom row is $\mathrm{Sq}^4\mathrm{Sq}^2\iota_3$, in degree 9.

Thus the next element we get is in degree 8, $d_9(a_4^2) = \mathrm{Sq}^4\mathrm{Sq}^2\iota_3$. After that, $a_4(Sq_1 a_4)$ is also nonzero, because it kills $(\mathrm{Sq}^1 a_4)(\mathrm{Sq}^2\iota_3)$. There's nothing else up to degree 9.

Note that $\mathrm{Sq}^4\mathrm{Sq}^1 a_4$ is zero, since it transgresses to $\mathrm{Sq}^4\iota_2^2 = (\mathrm{Sq}^2\iota_3)^2$, which is killed earlier. We get a basis

$$a_4, \mathrm{Sq}^1 a_4, a_4^2, a_4(\mathrm{Sq}^1 a_4)$$

In the lowest degrees, we see the classes a_4 and $\mathrm{Sq}^1 a_4$ connected by a Sq^1, so $H_4(X_1;\mathbb{Z})$ is $\mathbb{Z}/2$ (mod odd torsion), thus $\pi_4(S^3) = \mathbb{Z}/2$.

Next, we compute the fiber $X_2 \to X_1 \to K(\mathbb{Z}/2,4)$. We get a basis of

$$\iota_4, \mathrm{Sq}^1\iota_4, \mathrm{Sq}^2\iota_4, \mathrm{Sq}^2\mathrm{Sq}^1\iota_4, \mathrm{Sq}^3\iota_4, \mathrm{Sq}^3\mathrm{Sq}^1\iota_4, \iota_4^2, \mathrm{Sq}^4\mathrm{Sq}^1\iota_4, \iota_4(\mathrm{Sq}^1\iota_4)$$

and $\mathrm{Sq}^2\iota_4$, $\mathrm{Sq}^2\mathrm{Sq}^1\iota$, $\mathrm{Sq}^3\iota_4$, $\mathrm{Sq}^3\mathrm{Sq}^1\iota_4$, $\mathrm{Sq}^4\mathrm{Sq}^1\iota_4$ have to be killed.

We get a b_5 with $d_6(b_5) = \mathrm{Sq}^2\iota_4$, $d_6(\mathrm{Sq}^1 b_5) = \mathrm{Sq}^3\iota_4$. Additionally, something has to kill $\mathrm{Sq}^2\mathrm{Sq}^1\iota_4$, this is c_6.

The first cycle is in the 6-row, $(\mathrm{Sq}^1 b_5)(\mathrm{Sq}^2\iota_4)$, of degree 12. This already doesn't affect the range we're interested in. So all differentials in our range are transgressive.

We have $d_8(\mathrm{Sq}^2 b_5) = \mathrm{Sq}^3\mathrm{Sq}^1\iota_4$, and $d_8(\mathrm{Sq}^2\mathrm{Sq}^1 b_5) = \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2\iota_4 = \mathrm{Sq}^2\mathrm{Sq}^3\iota_4 = \mathrm{Sq}^4\mathrm{Sq}^1\iota_4$ since $\mathrm{Sq}^2\mathrm{Sq}^3 = \mathrm{Sq}5 + \mathrm{Sq}^4\mathrm{Sq}^1$. Additionally, $\mathrm{Sq}^1 c_6 = \mathrm{Sq}^2 b_5$ since both transgress to $\mathrm{Sq}^3\mathrm{Sq}^1 a_4$. Also note that $\mathrm{Sq}^3 b_5 = 0$ since it transgresses to 0.

We get a basis of

$$b_5, \mathrm{Sq}^1 b_5, c_6, \mathrm{Sq}^2 b_5, \mathrm{Sq}^2\mathrm{Sq}^1 b_5$$

In the lowest degrees, we again have a class b_5 with nontrivial Sq^1, so modulo odd torsion, $H_5(X_2;\mathbb{Z}) = \mathbb{Z}_2$, and $\pi_5(S^3) = \mathbb{Z}/2$.

Finally, let's look at the fibration $X_3 \to X_2 \to K(\mathbb{Z}/2,5)$. The mod 2 cohomology of $K(\mathbb{Z}/2,5)$ has a basis

$$\iota_5, \mathrm{Sq}^1\iota_5, \mathrm{Sq}^2\iota_5, \mathrm{Sq}^3\iota_5, \mathrm{Sq}^2\mathrm{Sq}^1\iota_5$$

and we know c_6 is not in the image. So c_6 restricts to a nontrivial class in $H^6(X_3;\mathbb{Z}/2)$, and there is a class in degree 7 that kills $\mathrm{Sq}^3\iota_5$.

They are not connected by a Sq^1, so $H_6(X_3)$ modulo odd torsion is $\mathbb{Z}/2^k$ for some $k > 1$ (we know it can't be $\mathbb{Z}s$ from our rational computation).

To find k, we look at it with $\mathbb{Z}$-coefficients. Since $H^*(X_2;\mathbb{Z}/2)$ is free over $\Lambda(\mathrm{Sq}^1)$ in our range (up to degree 7), $H^*(X_2;\mathbb{Z})$ is just two $\mathbb{Z}/2$ summands in degrees $6,7$, generated by $\tilde{\beta} b_5$ and $\tilde{\beta} c_6$ (lifts of $\mathrm{Sq}^1 b_5$ and $\mathrm{Sq}^2 b_5$.

The cohomology of $K(\mathbb{Z}/2,5)$ with $\mathbb{Z}$-coefficients also just has $\mathbb{Z}/2$s in our range, generated by $\tilde{\beta}\iota_5$, $\tilde{\beta}\mathrm{Sq}^2\iota_5$ up to degree 8.

Now the $\tilde{\beta}\mathrm{Sq}^2\iota_5$ restricts to 0 in $H^*(X_2)$, so it has to be hit by the group in $E^{0,7}$. Additionally, $\tilde{\beta}c_6$ is not in the image, so has to restrict to a nontrivial element in $E^{0,7}|\infty$. So $H^7(X_3;\mathbb{Z})$ and so $H_6(X_3;\mathbb{Z})$ have order 4, and together with our observation that the group is cyclic, we get $\pi_6(S^3)=\mathbb{Z}/4$ modulo odd torsion (so $\pi_6(S^3)=\mathbb{Z}/12$).

Example 8.37. *The same approach also works for S^5, giving $\pi_6(S^5)=\mathbb{Z}/2$, $\pi_7(S^5)=\mathbb{Z}/2$, $\pi_8(S^5)=\mathbb{Z}/24$. In this case, a $\mathbb{Z}/4$ appears in $H^*(X_2)$ already, which leads to a $\mathbb{Z}/8$ in $H^*(X_3)$.*

Example 8.38. *To understand all the other homotopy groups in this range, let's first understand the suspension map $S^3\to\Omega^2S^5$.*

ΩS^5 has cohomology a divided power algebra over $\mathbb{Z}$ with generator in degree 4, we denote the element in degree $4n$ by $\frac{x^n}{n!}$.

We compute the $\mathbb{Z}$-coefficient Serre spectral sequence for $\Omega^2S^5\to *\to\Omega S^5$. We have an element a_3 killing x via a d_4, so a_3x kills x^2. There remains a $\mathbb{Z}/2$ in the bottom row in degree 8, which needs to be killed by a $\mathbb{Z}/2$ in degree 7 of $H^7(\Omega^2S^5)$.

This leads to a $\mathbb{Z}/2$ in H^6 of the fiber of $S^3\to\Omega^2S^5$. So the fiber is 4-connected, with $\pi_5=\mathbb{Z}/2$.

Looking at the long exact sequence for this fibration, we have

$$?\to\pi_6(S^3)\to\pi_8(S^5)\to\mathbb{Z}/2\to\pi_5(S^3)\to\pi_7(S^5)\to 0\to\pi_4(S^3)\to\pi_6(S^5)\to 0$$

so the suspension $\pi_6(S^3)\to\pi_8(S^5)$ is the injection $\mathbb{Z}/4\to\mathbb{Z}/8$, and the suspensions below that are isomorphisms (which we knew from Freudenthal, since $\pi_5(S^3)\to\pi_6(S^4)$ is already surjective).

Similarly, we can understand the suspension $S^2\to\Omega S^3$. The fiber has homology starting with $\mathbb{Z}$ in degree 3, followed by a 0, from which we see that π_3 and π_4 of this fiber agree with the corresponding groups of S^3. So the long exact sequence goes

$$?\to\pi_5(S^2)\to\pi_6(S^3)\to\mathbb{Z}/2\to\pi_4(S^2)\to\pi_5(S^3)\to\mathbb{Z}\to\pi_3(S^2)\to\pi_4(S^3)$$

so, using that $\pi_n(S^3)\simeq\pi_n(S^2)$, the suspension $\pi_5(S^2)\to\pi_6(S^3)$ is the injection $\mathbb{Z}/2\to\mathbb{Z}/4$.

For S^4, we can use the Hopf fibration $S^3\to S^7\to S^4$. It gives a decomposition $\pi_n(S^4)=\pi_{n-1}(S^3)\oplus\pi_n(S^7)$, so $\pi_7(S^4)$ is $\mathbb{Z}\oplus\mathbb{Z}/12$, with the $\mathbb{Z}$ generated by the Hopf map ν. Since the suspension $\pi_7(S^4)\to\pi_8(S^5)$ is surjective, $\Sigma\nu$ is a generator of it.

Since the isomorphism $\pi_n(S^3)\simeq\pi_n(S^2)$ comes from postcomposition with η, we get explicit descriptions for generators:

[Picture with homotopy groups]

$$\pi_3(S^2)=\mathbb{Z}\eta,\pi_4(S^2)=\mathbb{Z}/2\{\eta\circ\Sigma\eta\},\pi_5(S^2)=\mathbb{Z}/2\{\eta\circ\Sigma\eta\circ\Sigma^2\eta\}$$

and all of these are stable. In particular, $\Sigma^3\eta\circ\Sigma^4\eta\circ\Sigma^5\eta=12\Sigma\nu$, or $\eta^3=12\nu$ stably.

9 The Atiyah-Hirzebruch spectral sequence

9.1 Generalized cohomology theories

So far we have only talked about the Serre spectral sequence which we use the compute ordinary cohomology $H^*(X)$ for several spaces X. But as you might know, there are important cohomology theories besides ordinary cohomology. First we recall the notion of a (generalized) cohomology theory.

Definition 9.1. A cohomology theory E is a functor which contravariantly asssigns to every pointed space X a $\mathbb{Z}$-graded abelian group $\tilde{E}^*(X)$ together with a natural isomorphism $\delta : \tilde{E}^*(X) \to \tilde{E}^{*+1}(\Sigma X)$ ('suspension isomorphism') such that the following axioms are satisfied:

- (Weak homotopy invariance) For a weak homotopy equivalence $X \to Y$ the induced map $\tilde{E}^*(X) \to \tilde{E}^*(X)$ is an isomorphism.
- (Exact sequence) For every cofibration $A \to X$ the sequence
$$\tilde{E}^*(X/A) \to \tilde{E}^*(X) \to \tilde{E}^*(A)$$
is exact in the middle .
- (Additivity) For a family of well pointed spaces X_i the map
$$\tilde{E}^*(\bigvee X_i) \to \prod \tilde{E}^*(X_i)$$
is an isomorphism.

As usual for an unpointed space X we set $E^*(X) := \tilde{E}^*(X_+)$.

Example 9.2. *Ordinary cohomology $H^*(-, M)$ with coefficients in an abelian groups M is a cohomology theory. Another cohomology theory that you might have seen (or maybe only the dual homology theory) is Bordism* $\mathrm{MSO}^*(-)$.

Now we want to recall what a multiplicative structure on such a cohomology theory is, roughly speaking this is a ring structure on $E^*(X)$ for every space X.

Definition 9.3. A multiplicative structure on a cohomology theory E is a given by external products
$$\times : \tilde{E}^*(X) \otimes \tilde{E}^*(Y) \to \tilde{E}^*(X \wedge Y)$$
natural with respect to the functoriality of $\tilde{E}^*$ and the suspension isomorphism such that these products are associative and unital with a unit $1 \in E^0(\mathrm{pt}) = \tilde{E}^0(S^0)$. It is called commutative if the external products satify moreover grade commutativity.

Again we can ignore the reduced versions mostly and only think about $E^*(X)$ which then becomes a ring by means of the product

$$E^*(X) \otimes E^*(X) = \tilde{E}^*(X_+) \otimes \tilde{E}^*(X_+) \xrightarrow{\times} \tilde{E}^*(X_+ \wedge X_+) = E^*(X \times X) \xrightarrow{\Delta^*} E^*(X).$$

For a cohomology theory E we set $E^* := E^*(\mathrm{pt})$ which is a graded ring if E is multiplicative. The group $\tilde{E}^*(X)$ is only a non-unital ring. We will from now on suppress the reduced theory and work with the non-reduced groups for simplicity.

Example 9.4. *Ordinary cohomology $H^*(-;R)$ admits a canonical multiplication if R is a ring and it is commutative if R is commutative.*

If R is a graded abelian group, then we can form a cohomology theory by setting

$$H^*(X,R) := \prod_i H^{*-i}(X,R_i)$$

and we leave it to the reader to verify that this can be extended to cohomology theory. Moreover if R is a multiplicative then the cohomology theory $H^*(-,R)$ is multiplicative as well.

Question 9.5. One could wonder if every cohomology theory E is of the form $H^*(X,R)$, in which case R has to be $E^*(\mathrm{pt})$. Or said differently if $E^*(X)$ is always isomorphic to $H^*(X,E^*)$. This would mean that all cohomlogy theories are variants of ordinary cohomology and the only interesting datum of a cohomology theory E is its group (ring) of coefficients.

We will show that the answer to the question is 'no'. In fact we will construct a very interesting and important cohomology theory soon, namley complex K-theory with $K^* = \mathbb{Z}[\beta^\pm]$ with $|\beta| = -2$ and show that the ablian groups $K^*(X)$ is not isomorphic to $H^*(X,\mathbb{Z}[\beta^\pm])$ for a particular space X (namely $X = \mathbb{R}P^4$). But even so that the answer to the question is no, there is a relation between E^* and $H^*(-,E^*(\mathrm{pt}))$ for every cohomology theory E, namely the Atiyah-Hirzebruch spectral sequence

Theorem 9.6 (Atiyah-Hirzebruch spectral sequence)**.** *For every cohomology theory E and every space X there is a (conditionally convergent) spectral sequence*

$$E_2^{p,q} = H^p(X,E^q) \Rightarrow E^{p+q}(X).$$

If the cohomology theory is multiplicative then so is the spectral sequence.

We will skip the proof of this theorem, we only.... Note that the question above then essentially becomes the question, if there are differentials in this spectral sequence. If this was not the case, then it would always degenerate and we would have $E^*(X) \cong H^*(X,E^*)$ (or more precisely there would be a filtration on $E^*(X)$ such that the associted graded is isomorphic to $H^*(X,E^*)$ which is not the same).

Let us also note that the spectral sequence canonically extends to reduced cohomology groups, is natural in X and the cohomology theory E and compatible with the suspension isomorphism.

The point is that we can make use of the existence of this spectral sequence in a rather formal way. Let us demonstrate this: assume that E^* is a multiplicative cohomology theory with E^* a ring which is concentrated purely in even degrees. We wish to calculate $E^*(\mathbb{C}P^n)$. The AHSS takes the form

$$H^*(\mathbb{C}P^n,E^*) = E^*[x]/x^{n+1} \Rightarrow E^*(\mathbb{C}P^n)$$

Everything is concentrated in even degrees, thus there can not be any differential, since a differential from a spot $(2p,2q)$ lands in $(2p+r, 2q-r+1)$ and one of the two is definitely odd depending on r. Thus the Atiyah-Hirzebruch spectral sequence degenerates at E_2 and this implies (by the convergence properties that we will discuss later) that there is an isomorphism of graded rings

$$E^*[x]/x^{n+1} \cong \mathrm{Gr}E^*(\mathbb{C}P^n).$$

But this formally implies that $E^*(\mathbb{C}P^n) \cong E^*[\bar{x}]/\bar{x}^{n+1}$ since for every lift $\bar{x}$ of x to $E^2(\mathbb{C}P^n)$ we get a map which is an isomorphism on associated graded pieces.

9.2 K-theory

Now we want to recall the notion of vector bundles.

Definition 9.7. A complex vector bundle over a space X is given by a continuous map $V \to X$ together with the structure of a complex vector space on each fibre V_x for $x \in X$ such that for every point $x \in X$ there exists an open neighboorhood $U \subseteq X$ of X and a homeomorphism

$$\begin{tikzcd} V\mid_U \arrow[rr, "\cong"] \arrow[dr] & & \mathbb{C}^n \times U \arrow[dl] \\ & U & \end{tikzcd}$$

over U that is fibrewise $\mathbb{C}$-linear. A morphism of complex vector bundles is a continuous map $V \to V'$ over U that is fibrewise $\mathbb{C}$-linear.

Note that the dimension of a the vector spaces V_x as x varies is constant on connected components. Since usually X is connected we will speak of the dimension (or the rank) of a vector bundle.

Example 9.8. *Hopf bundle $H \to \mathbb{C}P^1$ as tautological bundle.*

Now we will need some constructions of vector bundles. First there is a natural pullback operation of vector bundles:

Now let $T : \mathrm{Vect}_{\mathbb{C}} \to \mathrm{Vect}_{\mathbb{C}}$ be a continuous functor, by which we mean a functor such that for every pair of vector spaces V and W the induced map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(TV, TW)$ is continuous.

Proposition 9.9. *For every vector bundle $V \to X$ there is a natural topology on the set*

$$T(V) = \coprod_{x \in X} T(V_x) \to X$$

such that it becomes a vector bundle over X (with the evident fibrewise vector space structure) and which is natural in X (i.e. for every continuous homomorphism $f : X \to Y$ the evident bijection $f^*T(V) \to T(f^*V)$ is a vector bundle isomorphism.

Proof. TODO □

The case when T has several variables both covariant and contravariant, proceeds similarly. Therefore we can define for vector bundles V, W corresponding bundles:

- $V \oplus W$, the direct sum
- $V \otimes W$, the tensor product
- $\underline{\mathrm{Hom}}(V, W)$
- $V^\vee$, the dual bundle

We also obtain natural isomorphisms

- $V \oplus W \cong W \oplus V$

- $V \otimes W \cong W \otimes V$
- $V \otimes (W \oplus W') \cong (V \otimes W) \oplus (V \otimes W')$
- $\underline{\mathrm{Hom}}(V, W) \cong V^\vee \otimes W$

We denote by $\mathrm{Vect}(X)$ the set of isomorphism classes of vector bundles over X [6]. Then the addition $\oplus$ equips $\mathrm{Vect}(X)$ with the structure of a commutative monoid. Moreover the tensor product makes $\mathrm{Vect}(X)$ into a semiring (i.e. a ring without additive inverses). For every map $f : X \to Y$ the pullback $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is a map of semirings. Similarly there are the submonoids $\mathrm{Vect}^n(X)$ of isomorphism classes of n-dimensional vector bundles over X which are closed under sums and pullbacks, but not under tensor products.

Example 9.10. *For $X = \mathrm{pt}$ we have that $\mathrm{Vect}(X) = \mathbb{N}$.*

Definition 9.11. Let M be an abelian monoid. A group completion GM is an abelian group together with a homomorphism $i : M \to GM$ of abelian monoids such that for every homomorphism $M \xrightarrow{f} A$ where A is an abelian group there exists a unique factorization

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & GM \arrow[d, dashed, "\bar{f}"] \\ & A. \end{tikzcd}$$

Example 9.12.
- *The inclusion $\mathbb{N} \to \mathbb{Z}$ of additive monoids is a group completion as one can easily check the universal property.*
- *The inclusion $\mathbb{Z} \setminus 0 \to \mathbb{Q} \setminus 0$ of multiplicative monoids is a group completion.*
- *The map $\mathbb{Z} \to \{1\}$ is a group completion when $\mathbb{Z}$ is equipped with the multiplicative monoid struture. To see this it suffices to see that every map of abelian monoids*

$$f : (\mathbb{Z}, \cdot) \to (A, \cdot)$$

where A is an abelian group is necessarily the constant map which sends each $n \in \mathbb{Z}$ to $1 \in A$. This is true since $f(x) \cdot f(0) = f(x \cdot 0) = f(0)$ which implies that $f(x) = 1$.

Proposition 9.13. *For every abelian monoid M there exists a group completion GM. If M was a semiring, then GM has a unique ring structure compatible with the one on M (i.e. such that the map $M \to GM$ is a map of semirings).*

Proof. Consider the abelian monoid $M \times M$ with pointwise addition. Now we let GM be the quotient by the diagonal submonoid $M \subseteq M \times M$ in the world of monoids, i.e. (m, n) is identified with (m', n') if there exists an element $x \in M$ such that $m + n' + x = m' + n + x$. Then the map $M \to M \times M \to GM$ given by sending m to $[m, 0]$ is the map $i : M \to GM$. We leave it to the reader to verify the universal property and establish the ring structure in the case that M was a semiring. $\square$

Our goal is to understand vector bundles and we want to use the tools of generalized cohomology theories for this. Ultimately we will prove the following result.

[6] We will see that this is actually a set soon

Theorem 9.14. *There is a commutative multiplicative cohomology theory K^* (called complex K-theory) together with a natural homomorphism of commutative semirings*

$$\mathrm{Vect}(X) \to K^0(X)$$

which is a group completion if X is a finite CW complex. This cohomology theory has the coefficient ring $K^* = \mathbb{Z}[\beta^{\pm}]$ with $|\beta| = -2$.

Example 9.15. *From the first part of the theorem we can already deduce that $K^0 = K^0(\mathrm{pt})$ is the group completion of $\mathbb{N}$, i.e. isomorphic to $\mathbb{Z}$.*

Also note that the homomorphism $\mathrm{Vect}(X) \to K^0(X)$ even for finite CW complexes does not need to be injective. In fact, two bundles V and V' over X will be identified in $K^0(X)$ if there exists a third bundle W over X such that $V \oplus W \cong V' \oplus W$. One can show that W can always be choosen to be the trivial bundle $\mathbb{C}^n \times X \to X$ for some large enough n. We will call two bundles V, W over X stably equivalent if there is an isomorhism $V \oplus \mathbb{C}^n \cong V \oplus \mathbb{C}^m$ for some n, m.

Let us start by proving this last fact first:

Proposition 9.16. *For a compact Hausdorff space X the group completion of* $\mathrm{Vect}(X)$ *is isomorphic to*

$$\mathbb{Z} \oplus (\mathrm{Vect}(X)/\sim)$$

where $\sim$ is the equivalence relation stable equivalence.

Proof. The key fact to note is the following claim: on a compact Hausdorff space X for every vector bundle $V \to X$ there is another vector bundle $W \to X$ such that $V \oplus W$ is the trivial bundle. To see this we proceed as follows: we first note that it is enough to embedd the bundle V into a trivial bundle $V \to \mathbb{R}^n \times X$ since every subbundle has a complement.

Now this last fact implies that $\mathbb{Z}^{\pi_0(X)} \oplus (\mathrm{Vect}(X)/\sim)$ is a group, since there are inverses. Now there is an obvious map $\mathrm{Vect}(X) \to \mathbb{Z}^{\pi_0(X)} \oplus (\mathrm{Vect}(X)/\sim)$ given by... We claim that it is a group completion. Thus we have to verify the universal property. For simplicity we assume that ....

□

For example the tangent bundle TS^2 of S^2 is a non-trivial real line bundle by the hairy ball theorem. But since S^2 embeds into $\mathbb{R}^3$ with an obvious normal vector field we can deduce that $TS^2 \oplus \mathbb{R} \cong \mathbb{R}^3 \times S^2$. Thus TS^2 and $\mathbb{R}^2 \times S^2$ are stably isomorphic.

We will start our attempt to construct complex K-theory by trying to homotopytheoretically classifying vector bundles. Note that Theorem 9.14 at least implies that the group completion of $\mathrm{Vect}(-)$ is invariant under homotopy equivalences between finite CW complexes. The first step in proving the result/constructing K-theory is to show this independently. This will be a consequence of the following more general claim:

Proposition 9.17. *Let $f, g : X \to Y$ be two homotopic maps, X a paracompact space and $V \to Y$ be a vector bundle. Then f^*V and g^*V are isomorphic vector bundles.*

Proof. Take a homotopy $h : X \times I \to Y$, then the bundles f^*V and g^*V are the restrictions of the bundle $h^*V \to X \times I$ along $X \times \{0\}$ and $X \times \{1\}$. Thus it will suffices to show that for every vector bundle W over $X \times I$ the restrictions along $X \times \{0\}$ and $X \times \{1\}$ are isomorphic as vector bundles over X.

Now in order to prove this claim we first note the following two facts

Hatcher Proposition 1.7 □

Corollary 9.18. *For a homotopy equivalence $f : X \to Y$ between paracompact spaces the induced map $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is an isomorphism. In particular every bundle over a contractible space is trivial.*

Proof. Choose a homotopy inverse $g : Y \to X$. Then both compositions gf and gf are homotopic to the identity, thus induce the identity on Vect. □

Now we want to define a universal n-dimensional vector bundle $V_n \to G_n$ where G_n is a certain space G_n. First we consider the Grasmanian

$$G_n(\mathbb{C}^k) := \{V \subseteq \mathbb{C}^k \mid V \text{complex linear subspace}\}.$$

We want to make this into a topological spaces. To this end we observe that there is a transtive action of the unitary group $U(k)$ on $G_n(\mathbb{C}^k)$. The stabilizer of this action for the subspace $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^k$ is given by

$$U(n) \times U(k-n) \subseteq U(n)$$

and we get a bijection

$$\frac{U(k)}{U(n) \times U(n-k)} \cong G_n(\mathbb{C}^k).$$

We declare the topology on $G_n(\mathbb{R}^k)$ so that this bijection is a homeomorphism. Then we define $G_n(\mathbb{C}^\infty)$ to be the colimit of the topological spaces $G_n(\mathbb{C}^k)$ along the canonical inclusions $G_n(\mathbb{C}^n) \to G_n(\mathbb{C}^{n+1})$ induced from the inlcusion $\mathbb{C}^n \to \mathbb{C}^{n+1}$ into the first n-coordinates.

Now we define vector bundles

$$E_n^k \to G_n(\mathbb{C}^k)$$

by setting $E_n^k = \{(V, v) \mid V \in G_n(\mathbb{C}^k), v \in V\}$ and topologize this as a subbundle of the trivial bundle $\mathbb{C}^k \times G_n(\mathbb{C}^k)$. There are canonical maps

$$E_n^k \to E_n^{k+1}$$

and we define $E_n \to G_n(\mathbb{C}^\infty)$ as the colimit of the $E_n^k \to G_n(\mathbb{C}^k)$ as k tends to ∞.

Lemma 9.19. *All the V_n^k and V_n are vector bundles over the respective $G_n(\mathbb{C}^k)$.*

Proof. Excercise. □

Now the importance of these vector bundles is that they are the universal examples of vector bundles. For simplicity we will write G_n for $G_n(\mathbb{C}^\infty)$.

Theorem 9.20. *For a paracompact space X, the map*

$$[X, G_n] \to \mathrm{Vect}^n(X)$$

which sends f to f^*E_n is a bijection.[7].

In other words, every n-dimensional vector bundle $V \to X$ over a paracompact space is (up to isomorphism) the pullback of the bundle $E_n \to G_n$.

[7]Note that this map is well-defined by Proposition 9.17

Proof. For a given vector bundle $V \to X$ a pair consisting of a map $f : X \to G_n$ together with an isomorphism $V \xrightarrow{\cong} f^*E_n$ is the same as a continuous map $g : V \to \mathbb{C}^\infty$ that is injective and linear on fibres. This is because such map g determines a map $f : X \to G_n$ sending $x \in X$ to the subspace $g(V) \subseteq \mathbb{C}^\infty$ and then the pullback of the universal bundle comes with an isomorphism to V.

Thus it will suffice to construct for every vectorbundle $V \xrightarrow{p} X$ such a map $V \to \mathbb{C}^\infty$. To this end we choose a trivializing cover U_α for V and trivializations $V|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C}^n$. Then the composite

$$g_\alpha : V|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C}^n \to \mathbb{C}^n \hookrightarrow \mathbb{C}^\infty$$

defines a map that is continuous, linear and injective on fibres for the restricted bundle. Now we choose a partition of unity $e_\alpha : X \to \mathbb{R}$ subordinate to the cover U_α and consider the map

$$g := \sum (p^*e_\alpha) \cdot g_\alpha \colon V \to \mathbb{C}^\infty$$

which has by construction the required property.

Thus we have shown that the map $[X, G_n] \to \mathrm{Vect}^n(X)$ is surjective. For injectivity assume that for two maps $f_0, f_1 : X \to G_n$ the bundles $f_0^*E_n$ and $f_1^*E_n$ are isomorphic. The two maps f_0 and f_1 give rise to maps $g_0, g_1 : V \to \mathbb{C}^\infty$ where $V := f_0^*E_n \cong f_1^*E_n$. We want to show that g_0 and g_1 are homotopic through homotopies that are linear injections on fibres. Then we get an associated homotopy between f_0 and f_1 as in the first part of the proof.

To obtain a homotopy between g_0 and g_1 we first note that there is a linear, injective homotopy $h_t : \mathbb{C}^\infty \to \mathbb{C}^\infty$ from the identity to the map

$$(x_0, x_1, ...) \mapsto (x_0, 0, x_1, 0, ...)$$

for example pick $h_t(x_0, x_1, ..) = t(x_0, x_1, ...) + (1-t)(x_0, 0, x_1, 0, ...)$. Thus after postcomposing with this homotopy we can assume that g_0 is purely on even degrees and by a similar argument we assume that g_1 is purely in odd degrees. Now there is a homotopy

$$g_t := tg_0 + (1-t)g_1$$

which has the desired properties. □

Corollary 9.21. *For every paracompact space X there is a bijection*

$$[X, \coprod_n G_n] \to \mathrm{Vect}(X)$$

obtained by pullback of the universal bundle $\coprod E_n \to \coprod G_n$.

Remark 9.22. Since $[X, \coprod_n G_n]$ is a semiring, the Yoneda lemma implies that the object G_n has the strucutre of a semiring object in the homotopy category of topological spaces, that is there are maps $+ : G_n \times G_m \to G_{n+m}$ and $\cdot : G_n \times G_m \to G_{nm}$ that are associative, unital and distributive up to homotopy and that classify the direct sum and the tensor product of vector bundles.

Unfortunately the computation of $[X, G_n]$ is in general very hard and untractable. This is why we pass to K-theory and the group completion since then it magically becomes more tractable.

Note that there are canonical maps $G_n \to G_{n+1}$ classifying the bundle $V_n \oplus \mathbb{C}$ (or concretely the subspace $V \oplus \mathbb{C}$).

Definition 9.23. We define the space BU as the colimit of

$$G_1 \to G_2 \to G_3 \to \dots$$

Now there is a canonical map

$$\coprod_{n\in\mathbb{N}} G_n \xrightarrow{i} \coprod_{n\in\mathbb{Z}} BU = BU \times \mathbb{Z}$$

given by the inclusions $G_n \to BU$ on the respective factors. Moreover the addition and multiplication on $\coprod G_n$ make $BU \times \mathbb{Z}$ into a homotopy ring in such that the canonical map is a map of homotopy rings (and such that the projection $BU \times \mathbb{Z} \to \mathbb{Z}$ is a map of rings in Ho(Top). As a result the set $[X, BU \times \mathbb{Z}]$ inherits the structure of a ring for every space X.

Proposition 9.24. *Let X be a compact Hausdorff space. Then the map i induces a map*

$$\mathrm{Vect}(X) = [X, \coprod G_n] \to [X, BU \times \mathbb{Z}]$$

which is a group completion.

Proof. We assume without loss of generality that X is connected. The group $[X, BU \times \mathbb{Z}]$ splits into a sum $[X, BU] \oplus [X, \mathbb{Z}]$ the latter factor is isomorphic to $\mathbb{Z}$. Thus by Proposition 9.16 and the agreement of the maps it suffices to prove that $[X, BU]$ classifies stable equivalence classes of bundles. But this is true by construction: for a compact space X every map $X \to BU$ factors through a finite stage $X \to G_n$, i.e. classifies a vector bundle over X. Two such maps are then identified if the are stable equivalent. □

Now we almost want to define $K^0(X)$ to be $[X, BU \times \mathbb{Z}]$ except that this is not weakly invariant in X since we have taken homotopy classes. It is only invariant under homotopy equivalence. But there is an easy way to correct this, namely by replacing X with a CW approximation.

Definition 9.25. For every pair of topological space X, Y we define the set of weak homotopy classes denoted $[X, Y]^w$ by $[\hat{X}, Y]$ where $\hat{X} \to X$ is a CW approximation. It comes with a canonical map $[X, Y] \to [X, Y]^w$.

There are some things to check here, namely that this is well-defined, that this defines the set of morphisms in a category (called the weak homotopy category) and so on. But these are all straightforward and left to the reader.

Definition 9.26. For every topological space X we define

$$K^0(X) := [X, BU \times \mathbb{Z}]^w.$$

Note that this is a ring for every X as $BU \times \mathbb{Z}$ is a ring object as noted before. For every space X we get a canonical map

$$\mathrm{Vect}(X) \to [X, BU \times \mathbb{Z}] \to [X, BU \times \mathbb{Z}]^w$$

and since for X a CW complex the second map is an isomorphism we know that for X a finite CW complex this map is a group completion.For a pointed space X we define $\tilde{K}^0(X)$ as the group of pointed maps $[X, BU \times \mathbb{Z}]_*$ which is a subset of $[X, BU \times \mathbb{Z}]$. For X connected it is also isomorphic to the set of maps $[X, BU]$. We leave it to the reader to verify that there are external products

$$\tilde{K}^0(X) \otimes \tilde{K}^0(Y) \to \tilde{K}^0(X \wedge Y)$$

induced from the external products $K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$ which are themselves induced from the ring structure on $K^0(X \times Y)$.

Definition 9.27. For a pointed space X we define define $\tilde{K}^{-n}(X) := \tilde{K}^0(\Sigma^n X)$ and for a non-pointed space $K^{-n}(X) = \tilde{K}^{-n}(X_+)$.

Example 9.28. *By definition we have $K^2(\mathrm{pt}) = \tilde{K}^2(S^0) = \tilde{K}^0(S^2)$. We define the Bott element $\beta \in K^{-2}(pt)$ as the element $[H] - 1 \in \tilde{K}^0(S^2)$.*

Then based on the previous ring structures it is easy to see that $K^*(X)$ is a $-\mathbb{N}$-graded ring (i.e. indices are negative) for every space X and similarly there are external products

$$\tilde{K}^m(X) \otimes \tilde{K}^n(Y) \to \tilde{K}^{m+n}(X \wedge Y)$$

for $m, n \leq 0$. By definition there are suspension isomorphisms

$$\delta : \tilde{K}^n(X) \to \tilde{K}^{n+1}(\Sigma X)$$

for $n \leq -1$. It is not hard to check that, where defined, these groups satisfy the axioms of a (multiplicative) cohomology theory, except that the positive K-groups are missing.[8] To obtain positive groups we will make the groups periodic by brute force. First note that there is a $-\mathbb{N}$-graded ring map $K^*(\mathrm{pt}) \to K^*(X)$ for every space X which is split (every choice of basepoint in X gives a splitting). In particular every element in $K^*(\mathrm{pt})$ gives rise to an element in $K^*(X)$ for every space X. For a pointed space X the $-\mathbb{N}$-graded group $\tilde{K}^*(X)$ is still a module over $K^*(\mathrm{pt})$.

Construction 9.29. Let A^* be a $-\mathbb{N}$-graded ring with an element $x \in A^{-n}$ for some $n > 0$. Then we define a ring $A^*[x^{-1}]$ as the localization of A^* at x. This is then a $\mathbb{Z}$-graded ring where x^{-1} has degree n.

Concretely we consider A^* first as $\mathbb{Z}$-graded ring with zero positive components and then take the colimit

$$A^* \xrightarrow{\cdot x} A^* \xrightarrow{\cdot x} ...$$

in the category of $\mathbb{Z}$-graded modules. This colimit can be equipped with a ring structure as usual such that it satisfies the universal property of the localization. In fact this construct also show that more generally if M^* is a $-\mathbb{N}$-graded module over A^* then we get an induced $\mathbb{Z}$-graded module $M^*[x^{-1}]$ as the same colimit

$$M^* \xrightarrow{\cdot x} M^* \xrightarrow{\cdot x} ...$$

Example 9.30. *We note that the construction $A^*[x^{-1}]$ can heavily effect the groups in negative degrees. It can either kill them or enlarge them enormously as the following two examples show:*

[8] The hardest thing to check is exactness, but this is more generally true for the functor $[-, Y]^w$ where Y is a pointed space.

- *Consider the ring $A^* = \mathbb{Z}[x]/x^n$ with $|x| = -2$. Then $A^*[x^{-1}] = 0$ since x is nilpotent and thus in the defining colimit everything is equivalent to zero.*

- *Consider the ring $A^* = \mathbb{Z}[x,y]$ with $|x| = -2$ and $|y| = -4$. In particular $A^{-2} = \mathbb{Z}$ generated by x. Then $A^*[x^{-1}] = \mathbb{Z}[x^{\pm}, y]$. In particular in degree -2 we have the elements x and yx^{-1} that generate a $\mathbb{Z}^2$.*

However if the element x that we invert has the property that multiplication by x induces an isomorphism $\cdot x : A^{-n} \to A^{-n+|x|}$ for $n \geq 0$ then the localized object $A^*[x^{-1}]$ will have the same negative groups as A^* but new positive groups. The positive groups are then isomorphic to the respective negative ones by periodicity.

Definition 9.31. For a space X we define the $\mathbb{Z}$-graded ring $K^*(X)$ as $K^*(X)[\beta^{-1}]$ where the latter denotes the $-\mathbb{N}$-graded ring. For a pointed space X we define $\tilde{K}^*(X) := \tilde{K}^*(X)[\beta^{-1}]$.

In particular we have that $K^*(X)$ and $\tilde{K}^*(X)$ are 2-periodic. A priori there is clash in notation now, because we do not know that in negative degrees the K-theory groups agree. However, it turns out that they do by the following important theorem.

Theorem 9.32 (Bott peridocity)**.** *For every pointed space X the map*

$$\tilde{K}^0(X) \to \tilde{K}^0(\Sigma^2 X)$$

induced by external tensor product with $\beta = [H] - 1 \in \tilde{K}^0(S^2)$ is an isomorphism. In particular multiplication with the element $\beta \in K^{-2}(pt)$ acts invertibly on negative K-groups $\tilde{K}^*(X)$ for every pointed space X.

We will not give a prove of this theorem but refer to [] or [] for proofs. The theorem in particular implies that our old K^0 agrees with the new one and the same for negative groups. Thus these can be expressed in terms of vector bundles and group completion for finite CW complexes. A more abstract way of stating the result would be that there is a homotopy equivalence

$$\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$$

induced by multiplication with the Bott element.

Now finally we need to show that K-theory forms a multiplicative cohomology theory. Thus let us first collect all the necessary data:

- The groups $\tilde{K}^*(X)$ are functorial in X. This is because for a map $X \to Y$ we first get an induced map on negative groups which is a module map over the negative part of K^* and then inverting β is functorial as well.

- There are external product

  $$\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \to K^*(X \wedge Y)$$

  which are induced from the respective external product for the negative part by inverting β as well. The key is to note that inverting β is compatible with the tensor product since the module structures over K^* are.

- The map
$$\delta : \tilde{K}^n(X) \to \tilde{K}^{n+1}(\Sigma X)$$
that exists for $n \leq 1$ is K^*-linear and thus induces a map on localizations. Since it is an isomorphism in positive degrees this induced map is an isomorphism that is compatible with the respective structures.

Since we also have long exact sequence for K^* (which exist for the positive part and thus by periodicity also for the negative part this proves most of Theorem 9.14 above, in particular that complex K-theory is a multiplicative, commutative cohomology theory.

We see from the proof that this proposition is rather formal and would also be true without Bott peridocity (but then K-theory would be way more complicated).

Proposition 9.33. *$K^{-1}(\mathrm{pt}) = 0$, in particular $K^* = K^*(\mathrm{pt}) = \mathbb{Z}[\beta^{\pm}]$ with $|\beta| = -2$.*

Proof. To see this we claim that we have $\Omega(BU \times \mathbb{Z}) \simeq U$ where
$$U = \varinjlim U(n).$$
First it is clear that $\Omega(BU \times \mathbb{Z}) \simeq \Omega BU$. Thus we have to compute ΩBU. By definition BU is the colimit of G_n. Since S^1 is compact and the maps in the colimit are closed embeddings we get that
$$\Omega BU = \Omega \varinjlim G_n = \varinjlim \Omega G_n.$$
Now we see that G_n is the quotient $U/U(n) \times U$ similar to $G_n(\mathbb{C}^k) \cong U(k)/U(k) \times U(n-k)$. But since $U \to U(n) \times U$ is a fibre bundle we get a fibre sequence
$$U(n) \times U \to U \to G_n$$
which implies that ΩG_n is the fibre of the map $U(n) \times U \to U$ given by the canonical embedding $U(n) \to U$ and the inclusion $U \to U$ given by setting the first n-coordinates trivial. We claim that this map is homotopic to the projection map $pr_2 : U(n) \times U \to U$. But this is clear since....

Now we find that $\tilde{K}^1(\Sigma X) = [\Sigma X, BU \times \mathbb{Z}]_* = [X, \Omega BU] = [X, U]$. Thus vector bundles over ΣX are determined by a 'clutching function' $X \to U$. For $X = \mathrm{pt}$ we have to compute $\pi_0(U)$. Since all $U(n)'s$ are connected this is trivial which shows the claim. □

Note that the fact that $\Omega BU \simeq U$ is really the justification for the name BU that was given ad hoc before.

9.3 Properties and computations

Now that we have established, that K-theory is a generalized cohomology theory, we can in particular apply Theorem 9.6 and get a spectral sequence
$$E_2^{p,q} = H^p(X, K^q) = H^*(X)[\beta^{\pm}] \Rightarrow K^{p+q}(X) \ .$$
which converges conditionally. There is also a relative version of course:
$$E_2^{p,q} = \tilde{H}^p(X, K^q) = \tilde{H}^*(X)[\beta^{\pm}] \Rightarrow \tilde{K}^{p+q}(X) \ .$$

Pictorially this looks as follows. We see that for degree reasonst there can not be a d_2-differential and the first possible differential is a map

$$d_3 : H^*(X) \to H^{*+3}(X) \ .$$

We will come back to the d_3 soon.

We first want to explain, what conditional convergence means. Since we do not enough time for that, we will just list some of its consequences. For all this the paper "Conditionally Convergent Spectral Sequences" of Boardman is the standard reference.

Definition 9.34. We say that a weakly convergent spectral sequence $E_r^{p,q} \Rightarrow H$ convergence strongly to a filtred graded group

$$... \subseteq F^3 \subseteq F^2 \subseteq F^1 \subseteq F^0 \subseteq F^{-1} \subseteq ... \qquad \subseteq H.$$

if (F, H) is Hausdorff, exhaustive and complete. Here complete means that Cauchy sequences converge with respect to the topology induced from the filtration (i.e. the F_i form a neighbourhood basis of 0).

Lemma 9.35. *The following are equivalent for a filtred (graded) abelian group with Hausdorff filtration:*

- *The filtration is complete.*
- *The canonical map*
$$H \to \varprojlim H/F^i = H^{\wedge}_F$$
is an isomorphism.
- *The term $\varprojlim^1 F^i$ is zero.*

Proof. For the equivalence between the first two terms one shows that $H^{\wedge}_F$ is the completion. This is easy as one can just identify this with equivalence classes of Cauchy sequence in a straightforward way. For the second part we consider the short exact sequence

$$0 \to F^s \to H \to H/F^s \to 0$$

and taking the limit as s tends to ∞ we obtain a long exact sequence

$$0 \to \varprojlim F^s \to H \to \lim H/F_s \to \varprojlim^1 F^s \to \varprojlim^1 H = 0.$$

The first term is zero by assumption, thus it follows that the map is an isomorphism iff the term $\varprojlim^1 F^s$ is zero. □

Remark 9.36. Note that without the Hausdorff assumption the last lemma would still be true if we replace isomorphism in the second condition by surjection. Clearly being Hausdorff makes limits of sequences unique.

Clearly we want to know that this is the case, as it guarantees that one can hope to reconstruct H from the E_∞-page of the spectral sequence.

Proposition 9.37 (Boardman). *Let $E_r \Rightarrow H$ be a conditionally convergent right half plane spectral sequence. Then E is strongly convergent iff*

$$\varprojlim{}^1 E_r = 0.$$

Clearly we have that $E_{r+1}^{p,q} \subseteq E_r^{p,q}$ as soon as we are outside the range where boundaries can happen, i.e. $r >> 0$. [9] This statement is really useful as it allows us to deduce information about the filtration from the inner structure of the spectral sequence.

Example 9.38. *Assume that the spectral sequence is conditionally convergent first and fourth quadrant spectral sequence. If for fixed p, q there are only finitely non trivial differentials out of $E_*^{p,q}$ then clearly we have that $\varprojlim^1 E_r = 0$ and thus it converges strongly.*

As an example of this example for every finite dimensional space (meaning that the cohomology vanishes above some degree) we get that the AHSS converges strongly. In this case also the filtrations will be finite and there are no issues coming up.

Another instance is if all the groups on the E_2-page (or any later pager) are finite groups. Then there can only be finitely many morphisms out of every spot, thus it also strongly converges.

Proposition 9.39. *We have*

$$K^*(\mathbb{C}P^n) = K^*[x]/x^{n+1} \qquad |x| = 0$$
$$K^*(\mathbb{C}P^\infty) = K^*[[x]] \qquad |x| = 0$$

Proof. We have already explained the case of $\mathbb{C}P^n$. The case of $\mathbb{C}P^\infty$-follows similar: the E_2-page takes the following form:

There can not be any differentials. As a result we get that the E_∞-page is given by $K^*[x]$ and the filtration is complete and Hausdorff. It follows that it is the associated graded of the power series ring (the polynomial ring is not complete).

□

Remark 9.40. One could also try to approach the case of $\mathbb{C}P^\infty$ by writing it as the direct limit of the $\mathbb{C}P^n$'s. For the computation of the $\mathbb{C}P^n$'s we use that the AHSS converges always for finite dimesnional spaces. Whenever we have a space X as the (homotopy) colimit of spaces X_i (e.g. its skeleta) then there is always a Milnor short exact sequence

$$0 \to \varprojlim{}^1 K^{*-1}(X_i) \to K^*(X) \to \varprojlim K^*(X_i) \to 0$$

in fact this exists for every cohomology theory. In our case one can see directly that the $\lim^1$-term vanishes, since all the groups are zero.

The notion of conditional convergence does really only take care of complications with this additional $\lim^1$-term. So in practice one can just use that the AHSS converges stronly in the finite dimensional case and then treat the $\lim^1$-problem by hand. The point is that in ordinary cohomology this $\lim^1$-term for the skeletal filtration automatically vanishes for degree reasons, thus the AHSS can never see the difference.

[9] In the more general situation on has to quotient by the eventual boundaries as explain in Section, but we will not need this here.

Now we want to come to a bit more elaborate case, namely the space $\mathbb{R}P^\infty$. The reduced AHSS takes the following form.

picture

There are no differentials possible. As a result we know that there is a descending, complete, Hausdorff filtration on $\tilde{K}^0(\mathbb{R}P^\infty)$ whose quotients are $\mathbb{Z}/2$ and $\tilde{K}^1(\mathbb{R}P^\infty)$ is zero. This leaves a lot of possibilities. It could be that $\tilde{K}^0(\mathbb{R}P^\infty) = \prod_{i=0}^\infty \mathbb{Z}/2$ with the product filtration, i.e.

$$F_n = \prod_{i=n}^\infty \mathbb{Z}/2 \subseteq \prod_{i=0}^\infty \mathbb{Z}/2.$$

(It can not be the direct sum, since this is not complete....). Another possibility is that $\tilde{K}^0(\mathbb{R}P^\infty) = \mathbb{Z}_2$ are the 2-adic integers. In this case we have a filtration

$$... \subseteq 4\mathbb{Z}_2 \subseteq \ 2\mathbb{Z}_2 \subseteq \mathbb{Z}_2$$

and by definition the 2-adics are the completion of this filtration. This makes a bit difference, as for example $\mathbb{Z}_2$ is torsion free. The question is, how we can solve this extension problem. There are several ways of doing that, either with characteristic classes, with a comparison to complex projective space. We will do it in a rather conceptual way that will give us a bit more information.

To this end, we recall that for every compact Lie group G (e.g. a finite group) we can study finite dimensional, complex representations of G. A representation V of G is a finite dimensional, complex vector space with a continuous, $\mathbb{C}$-linear G-action. We denote the set of isomorphism classes of complex G-representations by $\mathrm{Rep}(G)$. This set, much like the set $\mathrm{Vect}(X)$ of vector bundles over a space X, inherits extra structure, namely one can take the direct sum and the tensor product of two represrentation. This equips the set $\mathrm{Rep}(G)$ with the structure of a semiring. We will denote the ring, obtained by group completion from $\mathrm{Rep}(G)$ as $R(G)$ and refer to it as the (complex) representations ring. There is a canonical map

$$\dim : R(G) \to \mathbb{Z}$$

induced by the dimension of representations. We refer to the kernel of this map as the augmentation ideal $I \subseteq R(G)$. Since dim has a canonical section by assigning the trivial representation. Thus we have a canonical isomorphism of abelian groups $R(G) \cong \mathbb{Z} \oplus I$.

Example 9.41. *Consider an abelian group A. Then every irreducible representation is 1-dimensional, thus the irreducible representations are given by the Pontriyagin dual group $A^\vee = \mathrm{Hom}(A, \mathbb{C}^*)$. Thus we find that $\mathrm{Rep}(A) \cong \mathbb{N}[A^\vee]$ as semirings. Thus we find that $R(A) = \mathbb{Z}[A^\vee]$. For example we get that*

$$R(C_2) = \mathbb{Z}[C_2] = \mathbb{Z}[\sigma]/(\sigma^2 - 1) = \mathbb{Z} \oplus \mathbb{Z}\{\sigma\}$$

where σ is the sign representation. This ring can also be written as

$$R(C_2) = \mathbb{Z}[x]/(2x + x^2) = \mathbb{Z} \oplus \mathbb{Z}\{x\}$$

where $x = \sigma - 1$ is in the kernel of the augmentation ideal, so this is the decomposition $R(G) = \mathbb{Z} \oplus I$.

The filtration induced by I is then given by

$$.. \subseteq (x^2) \subseteq (x) \subseteq R(C_2)$$

and thus the completion at I is given by

$$R(C_2)^\wedge_I = \mathbb{Z}[[x]]/(2x + x^2) = \mathbb{Z} \oplus \mathbb{Z}_2^\wedge$$

Now we can consider the universal G-bundle $EG \to BG$. For example in the case $G = C_2$ we have $S^\infty \to \mathbb{R}P^\infty$. For a G-representation V we can thus assign the associated vector bundle $EG \times_G V \to BG$, which is a vector bundle over BG. Clearly for a sum or tensor product of representations this results in a sum (resp. tensor products) of vector bundles. This construction gives a homomorphism

$$\mathrm{Rep}(G) \to \mathrm{Vect}(BG)$$

and therefore a morphism of rings

$$R(G) \to K^0(BG). \tag{4}$$

Proposition 9.42. *The map* (4) *is compatible with filtrations in that it takes the n-th power of the augmentation ideal to the n-filtration of $K^0(BG)$ with respect to the AHSS filtration.*

Proof. The first step in the filtration of $K^0(BC_p)$ is the reduced K-group $\tilde{K}^0(BC_p)$ (those classes that vanish on the point). Clearly the map sends the augmentation ideal to this subgroup. Then the square gets mapped to the second filtration and so on. □

In particular the map is continuous for the I-adic topology on the left hand side and the adic topology induced from the AHSS filtration (i.e. the skeletal filtration) on the right. We claim that $K^0(BG)$ is always complete with respect to the skeletal filtration, i.e the one arising in the Atiyah Hirzebruch spectral sequence. We have shown this for $G = C_2$ and the other cases will be skipped (they are considerably harder). Since the target is complete with respect to this topology we get a map

$$R(G)^\wedge_I \to K^0(BG).$$

Proposition 9.43. *The map* $\mathrm{Rep}(C_2) \to \mathrm{Vect}(BC_2)$ *is a completion, i.e.* $\mathrm{Vect}(BC_2)$ *is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2^\wedge$ generated by* 1 *and $x = \sigma - 1$.*

Proof. We consider the generator $x = 1 - x \in I$. It lands in the first filtration and thus gives rise to an element on the associated graded, which is a $\mathbb{Z}/2$ on the E_∞-page. We claim that this element is non-trivial. To see this it suffices to pull it back to $\mathbb{R}P^2$ and observe the following two facts:

- The first Stiefel-Whitney class $w_1(V)$ of a vector bundle is a stable invariant. This is because for a sum we have

$$c_1(V \oplus \mathbb{R}^n) = c_1(\mathbb{R}^n) + c_1(V) = c_1(V).$$

- The first chern-class of the complexification of the tautological bundle over $\mathbb{R}P^2$ (which is the one associated to σ is a generator of $H^2(\mathbb{R}P^\infty) = \mathbb{Z}/2$. This can be seen by noting that the Stiefel-Whitney class of the non real tautological bundle is the generator of $H^1(\mathbb{R}P^2, \mathbb{F}_2) = \mathbb{F}_2$ and the complexification is given by the Bockstein of this class which generates $H^2(\mathbb{R}P^2)$.

Now we find that the element $1-\sigma$ maps to a generator of F^1/F^2 in the filtration of $K^0(BC_2)$. The square of $1-\sigma$ is given by 2 times $1-\sigma$ and generates the second filtration. On the associated graded of $K^0(BC_2)$ the square generates the associated graded. Thus we find that the map is an isomrophism on associated gradedes and thus an isomorphism since both filtrations are exhaustive, Hausdorff and complete. □

A similar proof works in the case BC_p (of course one can not use the Stiefel-Whitney classes). Moreover it turns out that the result is way more generally true.

Theorem 9.44 (Atiyah-Segal completion)**.** *The map* $\mathrm{Rep}(G) \to \mathrm{Vect}(BG)$ *exhibits* $\mathrm{Vect}(BG)$ *as the completion* $\mathrm{Rep}(G)^\wedge_I$ *at the augmentation ideal for every compact Lie group.*

Corollary 9.45. *For even n we have that*

$$K^*(\mathbb{R}P^n) = K^*[x]/(x^2+2x, 2^n x) = (\mathbb{Z} \oplus \mathbb{Z}/2^n)[\beta^\pm].$$

Recall the question about whether every cohomology theory is a form of ordinary cohomology

Corollary 9.46. *K-theory is not a form of ordinary cohomology*

Proof. If K-theory was a form of ordinary cohomology in the sense that $K^*(X) \cong H^*(X.K^*)$ for all spaces X then for $\mathbb{R}P^\infty$ we would obtain in degree zero the infinite product $\prod \mathbb{Z}/2$. □

In some sense this means, that K-theory is more powerful of an invariant than ordinary cohomology. One can show way more precise theorems along these lines, but we will not pursue this here...

Theorem 9.47. *The first differential of the AHSS* $d_3 : H^*(X) \to H^{*+3}(X)$ *is given by* $\beta \circ \mathrm{Sq}^2 \circ \mathrm{red}$*, i.e. the composition*

$$H^*(X,\mathbb{Z}) \xrightarrow{\mathrm{red}} H^*(X,\mathbb{Z}/2) \xrightarrow{Sq^2} H^{*+2}(X,\mathbb{Z}/2) \xrightarrow{\beta} H^{*+3}(X,\mathbb{Z})$$

Proof. FIrst we observe that the d_3-differential of the AHSS defines a stable cohomology operation $d_3 : H^*(X) \to H^{*+3}(X)$. We claim that there are exactly two such operations: the zero operation and the operation in question. The fact that the two are not identitical, i.e. that the operation $\beta \circ \mathrm{Sq}^2 \circ \mathrm{red}$ is non-trivial, can be easily verified by looking at concrete examples. For example for the class $\iota_n \in H^3(K(\mathbb{Z},3))$ we find that it is non-trivial.

Now we have to show that there are no more than two such operations. This can be shown by computing the cohomology opertions

$$\varprojlim H^{n+3}(K(\mathbb{Z},n)) \cong \mathbb{Z}/2.$$

After these preliminary considerations we have to find a single space X sucht that the d_3 in the AHSS is non-trivial.

We claim, that such a space is given by $\mathbb{R}P^\infty \wedge \mathbb{R}P^2$. First we note that the cell structure of $\mathbb{R}P^2$ gives us a cofibre sequence $S^1 \to S^1 \to \mathbb{R}P^2$. We suspend this to get $\mathbb{R}P^2 \to S^2 \xrightarrow{\cdot 2} S^2$. As a result we get a long exact sequence

$$... \to \tilde{K}^{i+2}(\mathbb{R}P^\infty) \xrightarrow{\cdot 2} \tilde{K}^{i+2}(\mathbb{R}P^\infty) \to \tilde{K}^{i+1}(\mathbb{R}P^\infty \times \mathbb{R}P^2) \to \tilde{K}^i(\mathbb{R}P^\infty) \to ...$$

To see this we compute the K-theory of $\mathbb{R}P^\infty \wedge \mathbb{R}P^2$ by means of the long exact sequence induced from the cofibre sequence $S^1 \xrightarrow{\cdot 2} S^1 \to \mathbb{R}P^2$:

Recall that we had $\tilde{K}^{i+1}(\mathbb{R}P^\infty) = \mathbb{Z}_2$ for i even and 0 for i odd. Thus we get an exact sequence

$$0 \to \tilde{K}^0(\mathbb{R}P^\infty \wedge \mathbb{R}P^2) \to \mathbb{Z}_2 \xrightarrow{\cdot 2} \mathbb{Z}_2 \to \tilde{K}^1(\mathbb{R}P^\infty \wedge \mathbb{R}P^2) \to 0$$

which implies that $\tilde{K}^0(\mathbb{R}P^\infty \wedge \mathbb{R}P^2) = 0$ and $\tilde{K}^1(\mathbb{R}P^\infty \wedge \mathbb{R}P^2) = \mathbb{F}_2$.

Now we study the AHSS for the space $\mathbb{R}P^\infty \wedge \mathbb{R}P^2$. First of all, what is the ordinary cohomology of this space? We can use a similar long exact sequence to see that

$$\tilde{H}^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^2) = \mathbb{F}_2$$

for $* \geq 1$. Thus the reduced AHSS takes the following form: □

Remark 9.48. We see that for each cohomology theory the first non-trivial differential in the AHSS is a cohomology operation for ordinary cohomology. In fact, this operation for a given cohomology theory is called the first k-invariant of the spectrum representing the cohomology theory. Thus we have shown that the first k-invariant for the k-theory spectrum (more precisely of the spectrum bu).

Now we want to understand rational K-theory, i.e. the rationalization

$$K^*(X) \otimes \mathbb{Q}.$$

Note that this does not quite define a cohomology theory, as it does not satify Additivity, i.e. the wedge axiom. For finite CW complex it is does so. Therefor we just alter the definition on infinite spaces.

Definition 9.49. We define rational K-theory as

$$K^*_{\mathbb{Q}}(X) := \varinjlim \big(K^*(X_i) \otimes \mathbb{Q}\big)$$

where the colimit runs over all finite CW complexes X_i with a map to X.

This is natural in X, has external and internal products and comes with a natural transformation $\delta : \tilde{K}^*(X) \to \tilde{K}^{*+1}(\Sigma X)$.

Proposition 9.50. *Rational K-theory is a multiplicative cohomology theory.*

The coefficient ring of $K_{\mathbb{Q}}$ is given by $\mathbb{Q}[\beta^{\pm}] = K^* \otimes \mathbb{Q}$.

Remark 9.51. One can define the rationalization for every cohomology theory E in this way.

Definition 9.52. We say that a cohomology theory E is rational if for each space X the groups $E^*(X)$ are rational vector spaces.

Note that it is a property of an abelian group to be a rational vector space, i.e. that every element is uniquely divisible. The last remark implies that for every cohomology theory E there is a rationalization $E_{\mathbb{Q}}$ which is rational and which comes with a canonical map $E \to E_{\mathbb{Q}}$ of cohomology theories. It is not hard to see that $E_{\mathbb{Q}}$ is universal with respect to these properties:

Lemma 9.53. *Every map $E \to E'$ of cohomology theories for E' rational factors uniquely through $E_{\mathbb{Q}}$.*

The main point about rational cohomology theories is, that in contrast to integral ones, they are uniquely determined by their coefficients:

Theorem 9.54. *Every rational cohomology theory E is canonically isomorphic to ordinary cohomology with coefficients in E^*. In particular we have a canonical isomorphism*

$$K^*_{\mathbb{Q}}(X) \cong H^*(X, \mathbb{Q}[\beta^{\pm}]) \cong H^*(X, \mathbb{Q})((\beta)) \ .$$

Note that the ring $H^*(X, \mathbb{Q})((\beta))$ is by definition $\prod_i H^{*+2i}(X, \mathbb{Q})\beta^i$ as a group. For finite dimensional spaces X it is isomorphic to the ring $H^*(X, \mathbb{Q})[\beta^{\pm}]$ but for general spaces they might differ.

Proof. The first non-trivial differential d_r in the Atiyah-Hirzebruch spectral sequence gives a cohomology operation $H^*(X, E^i) \to H^{i+r}(X, E^{i-r+1})$. We claim that all such cohomology operations $H^*(-, E^i) \to H^{i+r}(-, E^{i-r+1})$ are trivial since E^i and E^{i-r+1} are rational vector spaces. To see that there are no cohomology operations in rational cohomology theories we first observe that

$$H^*(K(\mathbb{Q}, n), V)$$

for V a rational vector space is given for even n by

$$H^*(K(\mathbb{Q}, n), V) = H^*(K(\mathbb{Q}, n), \mathbb{Q}) \otimes_{\mathbb{Q}} V = \begin{cases} V & \text{for } * = kn \\ 0 & \text{else} \end{cases}$$

and for n odd by

$$H^*(K(\mathbb{Q}, n), V) = H^*(K(\mathbb{Q}, n), \mathbb{Q}) \otimes_{\mathbb{Q}} V = \begin{cases} V & \text{for } * = n \\ 0 & \text{else} \end{cases}$$

In the limit we get that all stable operations from rational cohomology to cohomology with values in V are concentrated in degree 0 and given by V. Then an arbitrary rational vector space W is a direct sum of $\mathbb{Q}$"s so that we get the same result (since all $\lim^1$-terms here vanish which can be seen by using the fact that cohomology is the dual of homology).

As a result we have deduced that the AHSS spectral sequence for each rational cohomology theory collapses at E_2. Since it is also conditionally convergent we deduce that $E^*(X)$ admits an exhaustive, complete and Hausdorff filtration all of whose quotients are given by ordinary cohomology of X with values in E^*. Every such group non-canonically splits (since we are working over $\mathbb{Q}$). With some more work one can show that there is even a canonical splitting.

□

10 Outlook: Adams spectral sequence

In this last section we want to outline the Adams spectral sequence. Recall which part of the stable stems we have computed in this lecture:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$				
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$			
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/12$		
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Thus we have computed the first three stable stems:

$$\pi_0 = \mathbb{Z} \quad \pi_1 = \mathbb{Z}/2 \quad \pi_2 = \mathbb{Z}/2 \quad \pi_3 = \mathbb{Z}/24 \ .$$

With some more work one can push the methods slightly further, but nut much. The Adams spectral sequence gives a way of computing way further and also much easier. It was introduced by Adams in 1958 and until today remains the most important tool to compute stable homotopy groups of spheres (maybe with its cousins, the Adams Novikov spectral sequence and the motivic Adams spectral sequence). We will first explain the statement and then say a couple of words about the construction of the Adams spectral sequence.

First we want to introduce some terminology: we work at the fixed prime 2 throughout this section. There are Adams spectral sequences for each prime p but we will restrict to $p = 2$. We denote the Steenrod algebra by $\mathcal{A}$. It is a graded commutative algebra. Recall that it has a basis consisting of Sq^I for admissible I. As an algebra is is generated by Sq^i for $i \geq 1$ modulo the Adem relations:

$$\mathcal{A}_* = \frac{\mathbb{F}_2\big[\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3, ...\big]}{\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{n=0}^{\lceil i/2 \rceil} \binom{j-n-1}{i-2n} \mathrm{Sq}^{i+j-n}\mathrm{Sq}^n} \qquad |\mathrm{Sq}^i| = i$$

Since it is a graded algebra, we can consider Ext-terms over it. That is Ext-groups in the abelian category of graded modules over $\mathcal{A}$. These Ext-groups are bigraded: one grading degree comes from the grading of $\mathcal{A}$ and the other one comes from the Ext-functor (i.e.

since we are resolving through chain complexes of graded modules). More precisely these Ext groups are the derived functor of the functor

$$\mathrm{Hom}^*(-,\mathbb{F}_2) : \mathrm{Mod}^{op}_{\mathcal{A}_*} \to \mathrm{grAb}.$$

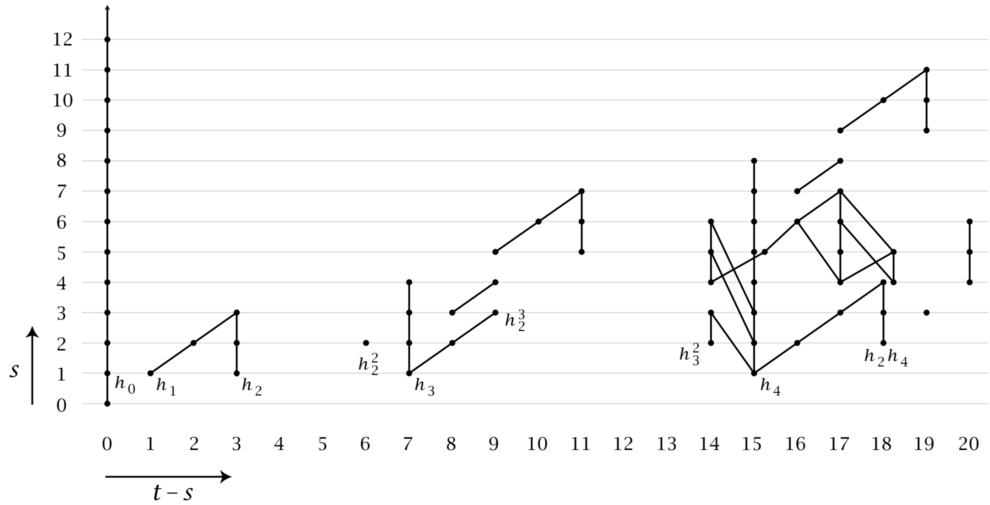
Theorem 10.1 (Adams)**.** *There is a multiplicative, strongly convergent spectral sequence*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2.\mathbb{F}_2) \Rightarrow (\pi_{t-s})^{\wedge}_2.$$

where t is the degree coming from the grading of $\mathcal{A}$, s is the 'derivation degree' and we use Adams grading, i.e. the differential d_r has bidegree $(r, r-1)$ i.e. $d_r : E^{s.t} \to E^{s+r,t+r-1}$.

Draw a picture where the x-axis is $t-s$ and y is s. In this drawing the differentials go up and left and the filtrations go up.

Note that since all stable homotopy groups of spheres are finite, the 2-completion is given in positive degrees by removing all the torsion which is not 2-power torsion. For the graphical picturing we use Adams grading. This is the picture of the first lines of the E_2-term of the Adams spectral sequence:



Here all dot's are $\mathbb{F}_2$. Horizontal lines are given by multiplication with h_0 and diagonal lines are multiplication with h_1. The differentials are given by lines that go to the left. We see that there is no differential until the 15'th line at all. This means we can just read off the first thirteen stable, 2 complete, homotopy groups:

$$\mathbb{Z}^{\wedge}_2, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/8, 0, 0, \mathbb{Z}/2, \mathbb{Z}/16, \mathbb{Z}/2\oplus\mathbb{Z}/2, \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/8, 0, 0$$

The first differentials can also be worked out quite formally: the element h_3^2 has to be off order 2 in π_* since it is the square of an odd element. Therefore $h_0 \cdot h_3^2$ has be killed by a differential and the only thing that can kill it is h_4 etc.

Now let us explain how to compute these Ext-groups. This is not easy, but at least it is pure algebra. By definition we have to choos a projective resolution of $\mathbb{F}_2$ as an $\mathcal{A}$-module:

$$\to P_2 \to P_1 \to P_0 \to \mathbb{F}_2.$$

We want to choose all P_i to be free and minimal, meaning that in each step the number of free generators is minimal (think about how resolutions can be bulit). Then we have to calculat the Ext-groups as the homology of the complex

$$\mathrm{Hom}(P_0,\mathbb{F}_2) \to \mathrm{Hom}(P_1,\mathbb{F}_2) \to \mathrm{Hom}(P_2,\mathbb{F}_2) \to \dots.$$

A very useful observation is the following:

Lemma 10.2. *If the resolution $P_\bullet$ is free and minimal then all differentials in the complex* $\mathrm{Hom}(P_\bullet, \mathbb{F}_2)$ *are zero. Thus the Ext-groups over $\mathcal{A}$ are given by*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Hom}^t_{\mathcal{A}}(P_s, \mathbb{F}_2)$$

where t is the graded degree.

Proof. $\square$

Now we can start resolving. With computer help this can be done roughly until $s = 100$.

Let us finish this lecture by just giving an idea from where the Adams spectra sequence comes. There is the notion of spectra.... Outline: spectra. Cool construction:

$$\mathrm{Map}_{\mathrm{End}(H\mathbb{F}_2)}(H\mathbb{F}_2, H\mathbb{F}_2) \simeq \mathbb{S}_2^\wedge$$

References