OBERSEMINAR "TOPOLOGICAL HOCHSCHILD HOMOLOGY AND HODGE-DE RHAM DEGENERATION"

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The idea of the seminar is to explain the recent results in topological Hochschild, periodic and cyclic homology. The seminar will not assume any pre-knowledge about these topics but we will use advanced homotopical and algebraic methods (e.g. spectra, spectral sequences, ∞-categories) which we will recall depending on the audience. The main source will be the lecture notes [NK] (which is still subject to slight changes). As an application of the general theory we explain the proof of the degeneration of the non-commutative Hodge-to-de Rham spectral sequence. This result was conjecture by Kontsevich and first proven by Kaledin in a way similar, but considerably more complicated, to the commutative proof given by Deligne-Illusie. We follow the recent very elegant proof given by A. Mathew [M] which fundamentally uses topological periodic homology.

This is a preliminary schedule for the seminar. We might have additional talks explaining preliminaries on request. The time is Tuesday 2-4, please let us know if you want to participate but this time does not work.

(1) Overview of the seminar, goals and discussion of preliminaries. We will also assign talks here. Give some ideas for working in the language of ∞-categories. [**Thomas**]
(2) Section 2 of [NK]: introduce (negative, cyclic, periodic) Hochschild Homology using the cyclic Bar construction and derived functors and explain their ring structures. Mention the relation to the Kähler differentials and the HKR theorem. Explain the computation of the groups in the case $\mathbb{F}_p$ relative to $\mathbb{Z}$ as far as possible. [**Sarah**]
(3) Give an introduction to ∞-categories and the category of spectra. Mention the concept of a symmetric monoidal ∞-category and define an algebra object as a symmetric monoidal functor $\mathrm{NAss}^{\otimes}_{\mathrm{act}} \to \mathcal{C}^{\otimes}$. Section 3 until Example 3.4 in [NK]. A (possible) source is [L] or in principle every introduction to the topic. [**Stefano**]
(4) The rest of Section 3 of [NK]: Give the simplicial definition of Hochschild homology and relate it to the ∞-categorical definition. Use this to introduce topological Hochschild homology as Hochschild homology over the sphere spectrum and explain the result of Bökstedt including the relation to algebraic Hochschild homology of $\mathbb{F}_p$ over $\mathbb{Z}$. Also mention the case of perfect rings and sketch a proof of Bökstedt's result, either using the Thom spectrum idea covered in the notes or the Tor spectral sequence in the $p = 2$ case (skipping the multiplicative extensions). [**Leon**]
(5) Section 5 of [NK]: Explain the group action of the torus $\mathbb{T}$ on $\mathrm{THH}(R)$ and its relationship with Connes operator. More details are given in [NS, Appendix T] and [H]. Introduce the Tate construction and use it to define topological negative

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cyclic homology TC^- and topological periodic homology TP. Finally explain the computations for $\mathbb{F}_p$ and the cyclotomic spectrum $H\mathbb{Z}^{\mathrm{triv}}$. [**Thomas**]

(6) Section 6 of [NK] and Section 2 of [M]: Introduce the Tate diagonal and the Frobenius on abelian groups and spectra. State the universal property of THH with respect to the action of $\mathbb{T}$. Use this to construct the *Frobenius* operator $\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ and mention that it does not exists for ordinary Hochschild homology using the argument of [NS, Theorem III.1.10] or the computation in Example 7.1 of [ABGHLM], but be aware that there is a typo in the right lower corner of the latter: the ring $S_{\mathbb{F}_p}[t, t^{-1}] = \mathbb{F}_p[t^{\pm}]$ is *not* the homotopy ring of $\mathrm{THH}_{H\mathbb{Z}}(\mathbb{F}_p)^{tC_p} = \mathrm{HH}(\mathbb{F}_p)^{tC_p}$ but of $H\mathbb{F}_p^{tC_p}$. [**Stefano**]

(7) Section 7 of [NK]: define the category of cyclotomic spectra and topological cyclic homology TC. Explain that TC given by maps out of the cyclotomic sphere and explain the adjunction with the trivial cyclotomic spectrum. Give the computation of TC for $\mathbb{F}_p$ and $\overline{\mathbb{F}}_p$ and if possible mention the trace and relate it to the K-theory of $\mathbb{F}_p$ (more precisely étale K-theory). [**Jakob**]

(8) Section 8 of [NK]: Define non-commutative Witt vectors following [NK, Appendix B] or the commutative Witt vectors following any of the many other sources and explain the connection with THH or more precisely with TR. It is best to concentrate on the p-typicial case and ignore the integral case. [**Lutz**]

(9) Section 2 of [M]: explain that THH can be defined for dg-categories (or more generally stable ∞-categories). Explain in this generality that TP is a lift of periodic homology over $\mathbb{F}_p$ (i.e. the latter is the mod p reduction of the first) and explain the non-abelian Cartier isomorphism in Proposition 2.7 [M]. For proofs of these results follow the references to [AMN].

(10) Section 3 of [M] until Remark 3.10: Present formality criteria as in Section 3 and work out the proofs (which are very sketchy in the paper). The computation in Proposition 3.6 can be taken as a black box if there is not enough time.

(11) Section 1 of [M]: Introduce (non-)commutative Hodge-to-de Rham degeneration. Explain how to obtain the 2-periodic commutative Hodge-to-de Rham from the non-commutative Hodge-to-de Rham degeneration. Also explain Theorem 1.1 (the characteristic 0 degeneration) and how it is implied by Theorem 1.2 (the mod p degeneration). For this you have to go through Theorem 3.11 and the following proof.

The seminar takes place at the following dates. We might decide to add some more talks, so the numbering of the dates does not correspond to the numbering of the talks.

(1) 17.04
(2) 02.05 Wednesday in Room SRC1
(3) 08.05
(4) 15.05
(5) 22.05
(6) 12.06
(7) 19.06
(8) 26.06
(9) 10.07

(10) 17.07
(11) 24.07

References

[NK] T. Nikolaus and A. Krause, *Lectures on topological Hochschild homology and cyclotomic spectra* `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Lectures.pdf`

[NS] T. Nikolaus and P. Scholze, *On topological cyclic homology* `arXiv:1707.01799`

[M] A. Matthew, *Kaledin's degeneration theorem and topological Hochschild homology* `arXiv:1710.09045`

[H] M. Hoyois, *The fixed points of the circle action on Hochschild homology* `arXiv:1506.07123`

[AMN] B. Antieau, A. Mathew and T. Nikolaus, *On the Blumberg-Mandell Künneth theorem for* TP `arXiv:1709.04828`

[ABGHLM] V. Angeltveit, A. Blumberg, T. Gerhardt, M. Hill, T. Lawson and M. Mandell, *Topological cyclic homology via the norm* `arXiv:1401.5001`

[L] J. Lurie, *Higher Algebra* `http://www.math.harvard.edu/~lurie/papers/HA.pdf`

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