Definable Henselian valuations in positive residue characteristic

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§ Motivation

Let p be a prime. For $a \in \mathbb{Z} \setminus \{0\}$ define

$$v_p(a) := \max\{n \in \mathbb{N} : p^n \mid a\} \in \mathbb{N}$$

extend to $\mathbb{Q}$ by

$a, b \neq 0$

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b) \in \mathbb{Z}, \qquad v_p(0) = \infty$$

$\rightsquigarrow$ $v_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$ is called the p-adic valuation on $\mathbb{Q}$

Induces a metric on $\mathbb{Q}$: $d(x,y) := p^{-v_p(x-y)} \in \mathbb{R}_{>0}$

Completing yields $(\mathbb{Q}_p, v_p)$ - the field of p-adic numbers
with the p-adic valuation
$v_p : \mathbb{Q}_p \longrightarrow \mathbb{Z} \cup \{\infty\}$

There is a special subring

$$\mathbb{Z}_p := \{ x \in \mathbb{Q}_p : v_p(x) \geq 0 \}$$ the p-adic integers

Fact (J. Robinson) $p \neq 2$

$$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : \exists y \in \mathbb{Q}_p \ (1 + px^2 = y^2) \}$$

In other words, $\mathbb{Z}_p$ is $\mathcal{L}_{ring}$ - definable

$= \{0,1,+,-,\cdot\}$ ⇝ polynomial equations over $\mathbb{Z}$
⇝ quatifiers $\exists$ and $\forall$ over the field
⇝ boolean combinations

Def K a field. $M \subseteq K$ is $\mathcal{L}_{ring}(K)$ - definable if

$$M = \{ x \in K : Q_1 Y_1 \dots Q_m Y_m \ \varphi(x, Y_1, \dots, Y_m) \text{ holds} \}$$

where $Q_1, \dots, Q_m \in \{\forall, \exists\}$ are quantifiers (over elements of K)
and φ is a boolean combination (using $\neg, \wedge, \vee$) of
statements of the form $f = 0$ for $f \in K[X, Y_1, \dots, Y_m]$

[M is even $\mathcal{L}_{ring}$-definable if we require the polynomials appearing in φ to come from $\mathbb{Z}[X, Y_1, \dots, Y_m]$]

Intuition: Think of constructible sets, but you are also allowed to take projections and complements

§ Valuations

Recall
- an ordered abelian group is $(\Gamma, +, \leq)$
  ↪ $(\Gamma,+)$ group, $\leq$ total order on Γ
  $a \leq b \implies a+c \leq b+c \quad \forall a,b,c \in \Gamma$
- introduce symbol ∞ with usual rules, e.g.
  $\infty + a = \infty = a + \infty$, $\infty + \infty = \infty$
  $\infty \geq a$, $\infty \geq \infty \quad \forall a \in \Gamma$

Def K a field. A valuation on K is a surjective map

$$v : K \longrightarrow \Gamma \cup \{\infty\}$$

where $(\Gamma, +, \leq)$ is an ordered abelian group, s. th.

(1) $v(x) = \infty \iff x = 0 \quad \forall x \in K$

(2) $v(x \cdot y) = v(x) + v(y)$

(3) $v(x+y) \geq \min\{v(x), v(y)\}$

Then (K, v) is a valued field.

Γ is called the value group, also write $vK = \Gamma$

Intuition x is "close" to $0 \iff v(x) \in \Gamma$ is big

Examples
- $(\mathbb{Q}, v_p)$, $(\mathbb{Q}_p, v_p)$, value group is $\mathbb{Z}$
- (K, v_{triv}), $v_{triv}: K \longrightarrow \{0\} \cup \{\infty\}$
  $x \longmapsto \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

Note: The value group doesn't need to be a subgroup of $\mathbb{R}$

Def: (K,v) valued field.

$\mathcal{O}_v := \{x \in K : v(x) \geq 0\}$, the valuation ring

is a local ring with unique max. ideal

$m_v := \{x \in K : v(x) > 0\}$

$\Rightarrow Kv := \mathcal{O}_v / m_v$, the residue field

A valued field has two characteristics: $\mathrm{char}(K)$ & $\mathrm{char}(Kv)$

- $\mathrm{char}(K) = \mathrm{char}(Kv) = 0$ "equicharacteristic 0" ← "easy mode"
- $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv)$ "mixed characteristic"
- $\mathrm{char}(K) = \mathrm{char}(Kv) = p > 0$ "positive characteristic"/ "equicharacteristic p"

positive residue characteristic

Def We say v is definable ($\emptyset$-definable) if $\mathcal{O}_v$ is $\mathcal{L}_{ring}(K)$-definable ($\mathcal{L}_{ring}$-definable)

Example: $(\mathbb{Q}_p, v_p)$: $\mathcal{O}_{v_p} = \mathbb{Z}_p$

(K, v_{triv}): $\mathcal{O}_{v_{triv}} = K$

Rem For a valued field (K,v) we have $vK \cong K^\times / \mathcal{O}_v^\times$

↖ even as ordered groups with $x\mathcal{O}_v^\times \leq y\mathcal{O}_v^\times$ iff $\frac{y}{x} \in \mathcal{O}_v$

and $v: K^\times \longrightarrow K^\times/\mathcal{O}_v^\times \xrightarrow{\cong} vK$

And we have that valuation rings on K are in one-to-one correspondence with valuations on K up to composing with an order-preserving isomorphism

"equivalent valuation"

⇝ We use $\mathcal{O}_v$ and v interchangeably

Def Let K be a field. A valuation v on K is henselian if there is a unique (upto equivalence) valuation $\tilde{v}$ on K^{alg} with $\tilde{v}|_K = v$.

K is henselian if there is at least one non-trivial henselian valuation on K.

Question (*) given a field K, when is there a definable non-trivial henselian valuation on K?

non-trivial henselian valuation on K.

→ K should be henselian

→ K should not be separably closed

(separably closed fields are stable, and in stable structures no valuation is definable)

§ The canonical henselian valuation

Recall the value group doesn't need to be $\leq \mathbb{R}$, it can be higher rank ⇝ can have proper convex subgroups

⇝ correspondence: $\left\{\begin{array}{l}\text{convex sub-}\\ \text{groups of } vK\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\mathcal{O}_v \subseteq R \subseteq K\\ \text{rings}\end{array}\right\}$

are always valuation rings again

"coarsenings of v"

Consider the set of all henselian valuations on K.
(= henselian valuation rings)

⇝ Can partially order them by inclusion (of the valuation rings)

Def $H_1(K) := \{\mathcal{O}_v \subseteq K : v \text{ henselian } \& \ Kv \text{ not separably closed}\}$

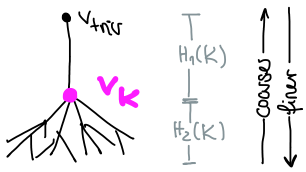
$H_2(K) := \{\mathcal{O}_v \subseteq K : v \text{ henselian } \& \ Kv \text{ separably closed}\}$

The canonical henselian valuation on K, v_K, is

- the finest valuation in H_1, if $H_2 = \emptyset$
- the coarsest valuation in H_2, otherwise

Facts: - v_K exists

- the valuations in H_1 are linearly ordered an coarsenings of the valuations in H_2
- v_K is non-trivial iff K henselian & not separably closed



The answer to question (*) depends on properties of v_K

for $\mathrm{char}(Kv_K) = 0$ ---→ only need 1.-4.

Theorem (Jahnke-Koenigsmann, 2017; K.-Ramello-Szewczyk, 2023+)

K henselian, not separably closed, perfect

If $\mathrm{char}(Kv_K) = p > 0 = \mathrm{char}(K)$, further assume $\mathcal{O}_{v_K}/p\mathcal{O}_{v_K}$ is semi-perfect

Then,

1. Kv_K sep. closed OR
2. Kv_K not t-henselian OR

K admits a non-trivial definable henselian valuation $\iff$

3. $\exists L \geq Kv_K$ with $v_L L$ not divisible OR
4. $v_K K$ not divisible OR
5. (K, v_K) not defectless OR
6. $\exists L \geq Kv_K$ with (L, v_L) not defectless

§ Defect

Fact (fundamental inequality)

(K,v) henselian, $L|K$ finite extension ---> unique extension v to L

$$[L:K] \geq (vL:vK)\cdot[Lv:Kv] \qquad (**)$$

Def $(K,v)\subseteq(L,v)$ is defectless if $(**)$ is an equality

(K,v) is defectless if all finite extensions are.

Otherwise: defect

Remark Usually defect is "bad".

We use it, however, to define a valuation.

$\mathrm{char}(Kv)=0 \implies$ defectless.

From now on assume $\mathrm{char}(Kv)=p>0$

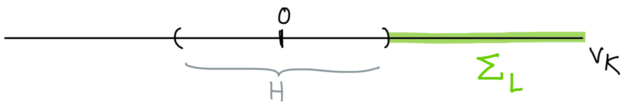
Def Let $(K,v)\subseteq(L,v)$ be a Galois defect extension of degree p

Let $\mathrm{Gal}(L|K)=\langle\sigma\rangle$. Let

$$\Sigma_L := \left\{ v\left(\frac{\sigma(f)-f}{f}\right) : f\in L^\times \right\} \subseteq vL = vK$$

$(K,v)\subseteq(L,v)$ has independent defect if there is $H\leq_{\mathrm{conv}} vK$ s.th.

- $\boxed{\Sigma_L = \{\alpha\in vK : \alpha > H\}}$ AND
- vK/H has no smallest positive element



Proof idea: Find a Galois defect extension of degree p

$K \subseteq K' \subseteq L$

Galois defect degree p

Perfect / semi-perfect $\implies$ has independent defect

Σ_L & the coarsening of v corresponding to H

are "essentially" $\mathcal{L}_{ring}$-definable

Questions: Examples?