Definable henselian valuation rings in positive residue characteristic

Montag, 4. März 2024 14:26

Joint work with Simone Ramello & Piotr Szewczyk
Young Women in Model Theory and Applications, 13 March 2024, Bonn

§ Intro - valuation rings

Def

K a field. A subring $\mathcal{O} \subseteq K$ is called valuation ring of K if $\forall x \in K$: $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$

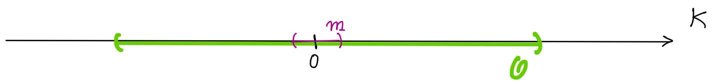
Fact / Def

- $\mathcal{O}$ is local, i.e. it has exactly one maximal ideal $\mathfrak{m} := \mathcal{O} \setminus \mathcal{O}^\times$
- $k_\mathcal{O} := \mathcal{O}/\mathfrak{m}$ is called the residue field
- $\Gamma_\mathcal{O} := K^\times/\mathcal{O}^\times$ is calld the value group

Intuition / Example

If K is an ordered field, not archimedean, then $\mathcal{O} := \langle \mathbb{Z} \rangle_{conv} := \{x \in K : \exists n \in \mathbb{N} : -n < x < n\}$

- $\mathcal{O}$ are the **bounded** elements of K
- $\mathfrak{m}$ are the **infinitesimal** elements of K
- $K \setminus \mathcal{O}$ are elements at **infinity**



Definition

A valuation ring $\mathcal{O}$ is called henselian if for each $f \in \mathcal{O}[X]$, $a \in \mathcal{O}$ such that $f(a) \in \mathfrak{m}$, $f'(a) \notin \mathfrak{m}$ there exists $\alpha \in \mathcal{O}$ with $f(\alpha) = 0$ and $a - \alpha \in \mathfrak{m}$

§ Definable valuation rings

Example for a definable henselian valuation:

Assume $\mathcal{O} \subseteq K$ is a henselian valuation ring s.th. $\mathrm{char}(k_\mathcal{O}) \neq 2$, $\Gamma_\mathcal{O} \cong \mathbb{Z}$, $t \in K$, s.th. $t\mathcal{O}^\times$ is min. pos. in $\Gamma_\mathcal{O}$

Then $\mathcal{O} = \{x \in K : \exists Y(\underbrace{Y^2 - (1 + tx^2)}_{f} = 0)\}$ $\quad f' = 2Y$

is an $\mathcal{L}_{ring}(K)$-definable subset.

not in the talk

Proof: Let $x \in \mathcal{O}$. Then $f(1) = tx^2 \in \mathfrak{m}$, $f'(1) = 2 \notin \mathfrak{m}$ $\xrightarrow{\text{hens.}}$ ex $y \in K$ s.th $f(y) = 0$

Let $y \in K$ s.th. $y^2 = (1 + tx^2)$

even in $K^\times/\mathcal{O}^\times \cong \mathbb{Z}$

also even in $K^\times/\mathcal{O}^\times \cong \mathbb{Z}$

$(1 + tx^2)\mathcal{O}^\times = \mathcal{O}^\times + tx^2\mathcal{O}^\times$

has to dominate

thus has to be infinitesimal $\to x \in \mathcal{O}$

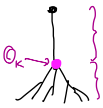
Q: Given a field K, when is there an $\mathcal{L}_{ring}(K)$-def. henselian valuation ring $\mathcal{O} \subsetneq K$?

$\mathcal{L}_{ring} = \{+, -, \cdot, 0, 1\}$

§ The canonical henselian valuation ring

⇝ Need to study ALL henselian valuation rings on K simultaneously.

$H(K) = \{\mathcal{O} \subseteq K : \mathcal{O} \text{ henselian valuation ring of } K\}$



$\mathcal{O}_K \rightarrow$

$\}\ H_1(K) := \{\mathcal{O} \in H(K) : k_{\mathcal{O}} \text{ not sep. closed}\}$

$\}\ H_2(K)$

$\mathcal{O}_K$ is called the canonical henselian valuation ring

↪ value group $K^\times / \mathcal{O}_K^\times = \Gamma_K$

↪ residue field $K / \mathfrak{m}_K = k_K$

↪ for $\mathrm{char}(k_K) = 0$

Thm (Jahnke-Koenigsmann 2017, Ketelsen-Ramello-Szewczyk 2024)

Let K be perfect, $K \neq \mathcal{O}_K$.

If $\mathrm{char}(K) = 0 < p = \mathrm{char}(k_K)$, let $\mathcal{O}_K / p\mathcal{O}_K$ be semiperfect

Then:

[K admits an $\mathcal{L}_{ring}(K)$-definable henselian valuation ring] $\iff$

1. k_K is separably closed
2. k_K is not t-henselian
3. Γ_K is not divisible
4. There is $L \succcurlyeq k_K$ s.th. Γ_L is not divisible
5. $\mathcal{O}_K$ has defect
6. There is $L \succcurlyeq k_K$ s.th. $\mathcal{O}_L$ has defect

NEW conditions needed in positive characteristic