# Definable Henselian valuations in positive residue characteristic

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## § Motivation

Let $p$ be a prime, $a \in \mathbb{Z} \setminus \{0\}$. Define:

$$v_p(a) := \max\{n : p^n \text{ divides } a \text{ in } \mathbb{Z}\} \in \mathbb{N}$$

Can extend $v_p$ to $\mathbb{Q}$: $a, b \in \mathbb{Z} \setminus \{0\}$

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b) \in \mathbb{Z}, \qquad v_p(0) = \infty$$

$\leadsto$ obtain map $v_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$, the p-adic valuation on $\mathbb{Q}$

$v_p$ induces a metric on $\mathbb{Q}$: $\quad d(x,y) = p^{-v_p(x-y)} \in \mathbb{R}_{>0}$

(Counter-) intuition: $x, y \in \mathbb{Q}$ are close to each other if

$v_p(x-y)$ is big

Completing yields $(\mathbb{Q}_p, v_p)$ - the field of p-adic numbers

$\leadsto$ with the p-adic valuation $v_p : \mathbb{Q}_p \longrightarrow \mathbb{Z} \cup \{\infty\}$

Special subring, the p-adic integers

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : v_p(x) \geq 0\}$$

Fact (J. Robinson) $p \neq 2$

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \exists y \in \mathbb{Q}_p \ (1 + px^2 = y^2)\}$$

i.e. $\mathbb{Z}_p$ is $\underset{=\{0,1,+,-,\cdot\}}{\mathcal{L}_{ring}}$-definable in $\mathbb{Q}_p$

## § Valuations

Def An ordered abelian group $(\Gamma, +, \leq)$ is a group $(\Gamma, +)$ together with a total order $\leq$ on $\Gamma$ such that $+$ and $\leq$ are compatible i.e. $a \leq b \Rightarrow a + c \leq b + c$ for all $a, b, c \in \Gamma$

We introduce a symbol $\infty$ with the usual rules, e.g.

$\forall a \in \Gamma$ : $a \leq \infty$, $\quad \infty + a = a + \infty = \infty$

$\infty \leq \infty$, $\quad \infty + \infty = \infty$

Examples
- $(\mathbb{Z}, +, \leq)$, $(\mathbb{Q}, +, \leq)$, $(\mathbb{R}, +, \leq)$
- in general subgroups of $\mathbb{R}$
- $(\mathbb{Z} \times \mathbb{Z}, +, \leq) =: \mathbb{Z} \oplus_{lex} \mathbb{Z}$, $\mathbb{Q} \oplus_{lex} \mathbb{Z}$, $\mathbb{Q} \oplus_{lex} \mathbb{Q} \oplus_{lex} \mathbb{Q}$, ...

componentwise addition

lexicographic ordering $(a_1, b_1) \leq (a_2, b_2) \iff a_1 < a_2$ OR $(a_1 = a_2$ AND $b_1 \leq b_2)$

**Def** $K$ a field. A *valuation* on $K$ is a surjective map
$$v: K \longrightarrow \Gamma \cup \{\infty\}$$
where $(\Gamma, +, \leq)$ is an ordered abelian group and such that

(1) $v(x) = \infty \iff x = 0$ for all $x \in K$ } i.e. $v|_{K^\times}: K^\times \longrightarrow \Gamma$

(2) $v(x \cdot y) = v(x) + v(y)$ for all $x, y \in K$ } is a group hom.

(3) $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$ "ultrametric triangle inequality"

We say $(K, v)$ is a *valued field*.

$\Gamma$ is called the *value group* of $(K, v)$, also write $vK = v(K^\times) = \Gamma$

**Recall** intuition: "close" to $0 \iff$ big valuation

**Examples** • $(\mathbb{Q}, v_p)$, $(\mathbb{Q}_p, v_p)$ have value group $\mathbb{Z}$

• $(K, v_{triv})$ $\quad v_{triv}: K \longrightarrow \{0\} \cup \{\infty\}$
$$x \longmapsto \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

**Def** $(K, v)$ valued field.

$\mathcal{O}_v := \{x \in K : v(x) \geq 0\}$, the *valuation ring*

is a local ring with unique maximal $\mathfrak{m}_v := \{x \in K : v(x) > 0\}$
ring with exactly one maximal ideal

$\Rightarrow$ $Kv := \mathcal{O}_v / \mathfrak{m}_v$, the *residue field*

Thus a valued field has two characteristics: $char(K)$ & $char(Kv)$

- $char(K) = char(Kv) = 0$ "equicharacteristic 0"
- $char(K) = 0 < p = char(Kv)$ "mixed characteristic"
- $char(K) = char(Kv) = p > 0$ "positive characteristic"/ "equicharacteristic p"

} positive residue characteristic

**Remark** One can retrieve the valuation $v$ up to an isomorphism of ordered abelian groups. ⇝ equivalent valuation ∼

$$v_{\mathcal{O}_v}: K^\times \longrightarrow K^\times/\mathcal{O}_v^\times \xrightarrow{\cong} \Gamma$$

⇝ $v_{\mathcal{O}_v} \sim v$

↑ ordering: $x\mathcal{O}_v^\times < y\mathcal{O}_v^\times$ iff $\frac{y}{x} \in \mathcal{O}_v$

Have 1:1-correspondence $\{\text{valuations on } K\}/_\sim \overset{1:1}{\longleftrightarrow} \{\text{valuation rings on } K\}$

**Def** We say $v$ is *definable* if $\mathcal{O}_v \subseteq K$ is $\mathcal{L}_{ring}(K)$-definable.

**Examples** • $(\mathbb{Q}, v_p)$: $\quad \mathcal{O}_{v_p} = \mathbb{Z}_{(p)} := \{\frac{a}{b} : a, b \in \mathbb{Z}, p \text{ does not divide } b\} \subseteq \mathbb{Q}$

$\mathfrak{m}_{v_p} = p\mathbb{Z}_{(p)}$

$\mathbb{Q}v_p = \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

• $(\mathbb{Q}_p, v_p)$: $\quad \mathcal{O}_{v_p} = \mathbb{Z}_p$, the *p-adic integers*, $\mathbb{Q}_p v_p = \mathbb{F}_p$.

• $(K, v_{triv})$: $\quad \mathcal{O}_{v_{triv}} = K$, $\mathfrak{m}_{v_{triv}} = \{0\}$, $Kv_{triv} = K/\{0\} = K$.

**Def** $(K, v)$ valued field is called *henselian* if there is a unique (upto equivalence) valuation $\tilde{v}$ on $K^{alg}$ (algebraic closure of $K$) extending $v$.

**Examples** $(\mathbb{Q}, v_p)$ not henselian, $(\mathbb{Q}_p, v_p)$ henselian

$(K, v_{triv})$ always henselian.

A field $K$ is called *henselian* if there is at least one non-trivial henselian valuation on $K$.

**Question (*)** Given a field $K$, when is there a definable non-trivial henselian valuation on $K$?

$\to$ $K$ should be henselian

$\to$ $K$ should not be separably closed

(separably closed fields are stable, and in stable structures no non-trivial valuation is definable)

## § The canonical henselian valuation

Recall the value group can be higher rank ($\not\leq \mathbb{R}$)

$\rightsquigarrow$ can have proper non-trivial convex subgroups

$\Delta \leq \Gamma$ is convex iff $\left( 0 \leq a \leq b \in \Delta \Rightarrow a \in \Delta \quad \forall a, b \in \Gamma \right)$

Example $\Gamma \leq \mathbb{R}$: $\{0\}$ and $\Gamma$ are the only convex subgroups

$\Gamma = \mathbb{Z} \otimes_{lex} \mathbb{Z}$: $\{(0,0)\}$, $\{0\} \times \mathbb{Z}$, $\Gamma$

--- ( $\{-2\}\times\mathbb{Z}$ )( $\{-1\}\times\mathbb{Z}$ )( $\{0\}\times\mathbb{Z}$ )( $\{1\}\times\mathbb{Z}$ )( $\{2\}\times\mathbb{Z}$ ) ---

$\rightsquigarrow$ correspondence: $\left\{ \begin{matrix} \text{convex sub-} \\ \text{groups of } vK \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \mathcal{O}_v \subseteq R \subseteq K \\ \text{rings} \end{matrix} \right\}$

are always valuation rings again

"coarsenings of $v$"

**Def** $H(K) := \{ \mathcal{O}_v \subseteq K : v \text{ henselian} \}$

$\hookrightarrow$ $H_1(K) := \{ \mathcal{O}_v \in H(K) : Kv \text{ not separably closed} \}$

$\hookrightarrow$ $H_2(K) := \{ \mathcal{O}_v \in H(K) : Kv \text{ separably closed} \}$

$H(K)$ is partially ordered by the coarsening relation / inclusion

Fact: • the valuations in $H_1$ are linearly orded and coarser than (coverings) of the valuations in $H_2$

Picture:


**Def** the canonical henselian valuation on $K$, $v_K$, is

- the finest valuation in $H_1$, if $H_2 = \emptyset$
- the coarsest valuation in $H_2$, otherwise

**The answer to (*) depends on properties of $v_K$**

for $\mathrm{char}(Kv_K) = 0$ $\rightsquigarrow$ only need 1.–4.

Theorem (Jahnke-Koenigsmann, 2017; K.-Ramello-Szewczyk, 2024)

$K$ henselian, not separably closed.

If $\mathrm{char}(K) = p > 0$, assume $K$ is perfect.

If $\mathrm{char}(Kv_K) = p > 0 = \mathrm{char}(K)$, further assume $\mathcal{O}_{v_K}/p\mathcal{O}_{v_K}$ is semiperfect.

Then,

$\boxed{\text{K admits a non-trivial definable henselian valuation}} \iff$

1. $Kv_K$ separably closed OR
2. $Kv_K$ not $t$-henselian OR
3. $\exists L \geq Kv_K$ with $v_L L$ not divisible OR
4. $v_K K$ not divisible OR
5. $(K, v_K)$ not defectless OR
6. $\exists L \geq Kv_K$ with $(L, v_L)$ not defectless

# § Defect

Fact (fundamental inequality)

$(K,v)$ henselian, $L|K$ finite extension $\dashrightarrow$ unique extension $v$ to $L$

$$[L:K] \geq (vL:vK)[Lv:Kv] \qquad (**)$$

Def $(K,v) \subseteq (L,v)$ is defectless if $(**)$ is an equality

$(K,v)$ defectless if all its finite extensions are

Otherwise: defect

Example $\mathrm{char}(Kv)=0 \Longrightarrow (K,v)$ defectless

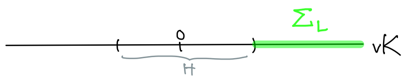
From now on, assume $\mathrm{char}(Kv)=p>0$

Def (Kuhlmann-Rzepka) Let $(K,v) \subseteq (L,v)$ be a Galois defect extension of degree $p$. Let $\mathrm{Gal}(L|K)=\langle\sigma\rangle$.

$$\Sigma_L := \left\{ v\left(\frac{\sigma(f)-f}{f}\right) : f \in L^\times \right\} \subseteq vL = vK$$

We say $(K,v) \subseteq (L,v)$ has independent defect if there is $H \leq_{conv} vK$ s.th.

- $vK/H$ has no smallest positive element
- $\Sigma_L = \{ \alpha \in vK : \alpha > H \}$




Proof idea (of $(S) \Rightarrow \exists$ non-triv. def. hens. val.)

Assume $(K,v_K)$ has defect $(S)$

Find a Galois defect extension of degree $p$:

$$K \subseteq K' \subseteq L$$

finite; Galois defect of deg. $p$

Perfect/semi-perfect $\xRightarrow{\text{Kuhlmann-Rzepka}}$ $K' \subseteq L$ has independent defect

$\Sigma_L$ & the coarsening of $v_K$ corresponding to $H$ are "essentially" $\mathcal{L}_{ring}$-definable (Beth's definability Theorem)

yields a non-trivial definable henselian valuation on $K$.