Definability of henselian valuations in positive (residue) characteristic

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The road ahead

$$\underbrace{\text{model theory}}_{\text{MATHEMATICAL LOGIC}} \text{ of } \underbrace{\text{valued fields}}_{\text{ALGEBRA}}.$$

Today, we will try to:

- Tell you what valued fields are.
- Give you an idea of what our results look like.
- Tell you about an obstacle in this area and how we turned it into a tool.

Valuations and where to find them

Definition

A valuation on a field K is a surjective map $v\colon K^\times \to \Gamma$, where $(\Gamma, +, \leqslant, 0)$ is an ordered abelian group, such that:

- $v(xy) = v(x) + v(y)$, *multiplying two elements sums their valuations*
- $v(x + y) \geqslant \min\{v(x), v(y)\}$. *all triangles are isosceles*

(Counter)intuition: an element $r \in K^\times$ is *large* if if its valuation $v(r) \in \Gamma$ is *small*, i.e. close to 0. Along this intuition, we usually set $v(0) \coloneqq \infty$.

The ordered abelian group Γ is called the *value group*. We also denote it by vK.

Our favourite example

Fix a prime number p.

- If $a \in \mathbb{Z} \setminus \{0\}$, then
$$v_p(a) \coloneqq \max\{n \in \mathbb{N} \colon p^n \text{ divides } a\}.$$
For example, $v_3(6560) = 0$. According to v_3, then, 6560 is "big". But $v_3(6561) = 8$, which is then "smaller" than 6560. If $a, b \in \mathbb{Z} \setminus \{0\}$ are coprime, then
$$v_p\left(\frac{a}{b}\right) \coloneqq v_p(a) - v_p(b).$$
- This defines a valuation $v_p \colon \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$, called *the p-adic valuation*. With it, we can define a distance on $\mathbb{Q}$ by setting $d_p(a, b) \coloneqq p^{-v_p(a-b)}$.
- If we *complete* the corresponding metric space, we obtain a (new) valued field called $\mathbb{Q}_p$, with its own valuation v_p. These are the *p-adic numbers*.

Why you should like the p-adics

- $(\mathbb{Q}_p, v_p)$ is crucial for algebraic purposes. But we are logicians (allegedly)!
- A valuation is "the same" as its valuation ring, i.e. the subring

$$\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}.$$

  This is the part where we should tell you that ∞ is larger than all elements of Γ, and thus $0 \in \mathcal{O}_v$.

- In the case of $\mathbb{Q}_p$, this subring is called $\mathbb{Z}_p$ (guess why!). Julia Robinson pointed out something remarkable about $\mathbb{Z}_p$ (for $p \neq 2$):

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid \exists Y(Y^2 = 1 + px^2)\}.$$

  There is a similar formula for $p = 2$.

- $\mathbb{Z}_p$ is given, as a subset of $\mathbb{Q}_p$, by a polynomial equation together with some quantifiers. We say that it is a *definable* set in the language of rings.

Logicians, assemble! cont'd

Big question: Is this common? When is some valuation ring definable in the language of rings?

The problem of henselianity

Not all valuations are created equal.

- Take a field K with a valuation v. We give you an algebraic extension L of K, e.g. $L = K(\alpha)$ where α is the root of some polynomial over K. Can you extend v to L?
  Yes, but often in several different ways.
- v is *henselian* if there is a **unique** way to extend v to any algebraic extension of K. A henselian valuation is a bit like a *fill the gaps* exercise in a textbook.
- v_p is henselian. *We will only care about henselian valuations.*

The big question, take 2

Big question: when is
a henselian valuation ring definable in the language of rings?

Two fields in disguise

- To any valued field (K, v) we can associate another "smaller" field, called the *residue field*,

$$Kv := {\{x \in K\colon v(x) \geqslant 0\}}\big/{\{x \in K\colon v(x) > 0\}}.$$

  Indeed, $\mathfrak{m}_v := \{x \in K\colon v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$.

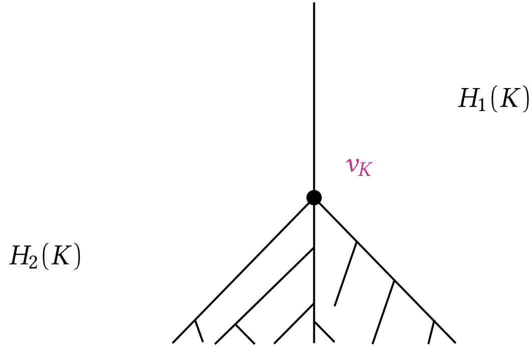
- Example: $(\mathbb{Q}, v_p)$ and $(\mathbb{Q}_p, v_p)$ both have residue field $\mathbb{F}_p$, the finite field with p elements
  In fact, $\mathbb{Q} \subseteq \mathbb{Q}_p$ is an *immediate extension*: They have the same value groups and residue fields.
- So a valued field consists of *two fields*: the "big" valued field and the "smaller" residue field. If we talk about the characteristic of a valued field, we talk about the characteristics of the two fields
  - equicharacteristic zero: $\mathrm{char}(K) = \mathrm{char}(Kv) = 0$
  - mixed characteristic: $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv)$, where p is prime
  - positive characteristic: $\mathrm{char}(K) = \mathrm{char}(Kv) = p$, where p is prime

A canonical friend

- Henselian valuations on a given field K arrange themselves nicely according to whether their residue field is separably closed or not,

$$H_1(K) := \{v \colon Kv \text{ is not separably closed}\} \text{ vs. } H_2(K) := \{v \colon Kv \text{ is separably closed}\}.$$

- $H_1(K)$ is linearly ordered by inclusion.
  The "middle point" between $H_1(K)$ and $H_2(K)$ is the *canonical henselian valuation* v_K.



The gist of it

$$\underbrace{\exists \text{ definable (non-trivial) henselian valuation}}_{\text{Logic question}} \iff \underbrace{\text{Conditions on the canonical henselian valuation}}_{\text{(Almost) algebra answer}}$$

What we proved

Theorem (Jahnke, Koenigsmann, 2017; Ketelsen, Ramello, Szewczyk, 2023)

Let K be a non-separably closed henselian field.
If $\mathrm{char}(K) = p > 0$, *then assume that* K *is perfect.*
If $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv_K)$, *then assume that* $\mathcal{O}_{v_K}/p$ *is semi-perfect.*
Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not t-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ (K, v_K) \text{ is not defectless}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless}. & \end{cases}$$

What we had before

Theorem (Jahnke, Koenigsmann, 2017; ~~Ketelsen, Ramello, Szewczyk, 2023~~)

Let K be a non-separably closed henselian field, $\mathrm{char}(Kv) = 0$.
~~*If* $\mathrm{char}(K) = p > 0$*, then assume that* K *is perfect.*~~
~~*If, further,* $\mathrm{char}(K) = 0 < p = \mathrm{char}(Kv_K)$*, then further assume that* $\mathcal{O}_{v_K}/p$ *is semi-perfect.*~~
Then,

$$K \text{ admits a definable non-trivial henselian valuation} \iff \begin{cases} Kv_K = Kv_K^{\mathrm{sep}}, & \text{or} \\ Kv_K \text{ is not t-henselian}, & \text{or} \\ \exists L \succeq Kv_K \text{ with } v_L L \text{ divisible}, & \text{or} \\ v_K K \text{ is not divisible}, & \text{or} \\ \cancel{(K, v_K) \text{ is not defectless}}, & \text{or} \\ \cancel{\exists L \succeq Kv_K \text{ with } (L, v_L) \text{ not defectless.}} & \end{cases}$$

We don't talk about defect

Actually, we do now.

- Given a henselian valuation v and a finite field extension $K \subseteq L$, then there is a unique extension of v to L, which we denote by v again. Then, we have

  $$[L : K] \geqslant [Lv : Kv](vL : vK).$$

  More precisely,

  $$[L : K] = p^d [Lv : Kv](vL : vK),$$

  where $p = \mathrm{char}(Kv)$, if the latter is positive, and $p = 1$ if $\mathrm{char}(Kv) = 0$.
- We say that $(K, v) \subseteq (L, v)$ is *defectless* if

  $$[L : K] = [Lv : Kv](vL : vK).$$

  In particular, then, if $\mathrm{char}(Kv) = 0$, then $p = 1$ and so equality holds. Otherwise, not being defectless (= *having defect*) is a problem.
- For us, however, defect is a **source of information**! (At least when it is "of independent type").