ASYMPTOTIC ABELIANNESS, WEAK MIXING, AND PROPERTY T

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Abstract. Let G be a second countable locally compact group and H a closed subgroup. We characterize the lack of Kazhdan's property T for the pair (G, H) by the genericity of G-actions on the hyperfinite II_1 factor with a certain asymptotic Abelianness property relative to H, as well as by the genericity of measure-preserving G-actions on a nonatomic standard probability space that are weakly mixing for H. The latter furnishes a definitive generalization of a classical theorem of Halmos for single automorphisms and strengthens a recent result of Glasner, Thouvenot, and Weiss on generic ergodicity. We also establish a weak mixing version of Glasner and Weiss's characterization of property T for discrete G in terms of the invariant state space of a Bernoulli shift and show that on the CAR algebra a type of norm asymptotic Abelianness is generic for G-actions when G is discrete and admits a nontorsion Abelian quotient.

1. Introduction

Among the various asymptotic independence properties in ergodic theory, weak mixing occupies a distinguished intermediate position. It enjoys stability properties that the weaker notion of ergodicity lacks, and it occurs much more commonly than mixing or completely positive entropy. Most importantly, weak mixing captures the appropriate kind of randomness that leads to a structure theorem for ergodic systems, the applications of which include Furstenberg's celebrated dynamical proof of Szemerédi's theorem [12, 25, 31].

A theorem of Halmos asserts that weak mixing is generic for measure-preserving automorphisms of a nonatomic standard probability space with respect to the weak topology [19]. Rokhlin showed on the other hand that mixing fails generically in this context [26]. The proofs of these facts, as presented in Halmos's classic book on ergodic theory [18], rely in one way or another on periodic approximation, which is particular to integer actions or at least suggestive of amenability if understood in some generalized form. One of the goals of the present paper is to extend Halmos's result as far as it will go within the realm of actions of second countable locally compact groups (the second countability assumption ensuring that the set of actions is a Baire space under the weak topology). We prove that, for actions of a second countable locally compact group G on a nonatomic standard probability space, generic weak mixing holds precisely when G lacks Kazhdan's property T [22]. This gives a complete answer to a question of Bergelson and Rosenblatt [5] and yields in particular a new approach to the single automorphism case. The ergodicity version of this dynamical characterization of property T was recently established for discrete G by Glasner, Thouvenot, and Weiss [15] by applying a correspondence principle [14] to transfer an analogous theorem due to Glasner and Weiss [16] from the setting of invariant

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probability measures for Bernoulli shifts. In the case of discrete minimally almost periodic G without property T, generic weak mixing follows from the Glasner-Thouvenot-Weiss result, as minimal almost periodicity is equivalent to the property that every ergodic unitary representation is weakly mixing ([5], Theorem 3.2). We also point out that, as part of his recent solution to the homogeneous spectrum problem, Ageev showed that weak mixing is generic for certain virtually Abelian G [1].

We also establish a weak mixing analogue of Glasner and Weiss's Bernoulli shift result for discrete G. Here the correspondence principle of [14] still applies, but we have developed a separate argument so as to be able to formulate the statement in a general noncommutative framework, to which the correspondence principle does not extend.

In fact one of our original aims was to investigate generic mixing properties for G-actions in the noncommutative domain. For dynamical systems on highly noncommutative operator algebras, mixing properties are closely related to asymptotic Abelianness (see Sections 9 and 10 of [24] and Example 4.3.24 in [6]). We prove that G-actions on the hyperfinite II_1 factor with a certain asymptotic Abelianness property relative to the trace norm are generic precisely when G does not have property T, and as a corollary obtain the parallel statement for weak mixing in accord with the commutative case. These statements also apply to the CAR algebra with respect to the unique tracial state and its associated norm. However, if we replace the trace norm with the operator norm then we can no longer obtain a handle on asymptotic commutation relations using the same measure-theoretic devices. Nevertheless, we are able to show that a type of norm asymptotic Abelianness is generic for G-actions on the CAR algebra when G is discrete and admits a nontorsion Abelian quotient, generalizing a result for single *-automorphisms from [24].

The element that is common to the proofs of all of our dynamical genericity results is an orthogonal-distribution-across-a-product construction which was conceived by Glasner and Weiss in [16]. The crucial difference here is that, while [16] uses joinings to achieve an approximation by invariant states possessing the global property of ergodicity, we employ the product construction in a multifold manner for purely local purposes, with the desired genericity of asymptotic Abelianness or weak mixing resulting from a direct Baire category argument.

For the sake of generality, all results involving property T will actually be formulated and proved for the relative case of a pair (G, H) where H is a closed subgroup of G.

We begin the main body of the paper in Section 2 by characterizing the lack of property T by the genericity of weak mixing for representations on a fixed separable infinite-dimensional Hilbert space. This generalizes a result of Bergelson and Rosenblatt [5], who showed generic weak mixing when G possesses what later became identified as the Haagerup property (see Chapter 2 of [8]). It also illustrates in a more basic linear-geometric framework some of the main ideas involved in the relation between weak mixing and property T and puts into perspective the more delicate local arguments required in the dynamical context, where an additional multiplicative structure must be negotiated. Sections 3 and 4 treat actions on the hyperfinite II_1 factor and on a nonatomic standard probability space, respectively, while Section 5 deals with Bernoulli shifts. Finally in Section 6 we turn to actions on the CAR algebra and norm asymptotic Abelianness.

We now summarize some general facts and terminology used throughout the paper. As above we suppose G to be a second countable locally compact (Hausdorff) group and H a closed subgroup. By a unitary representation of G on a Hilbert space we mean a strongly continuous unitary representation. Given a unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$, a nonempty set $K \subseteq G$ and an $\varepsilon > 0$, we say that a vector $\xi \in \mathcal{H}$ is *(K, ε)-invariant* if $\sup_{s\in K} \|\pi(s)\xi - \xi\| < \varepsilon\|\xi\|$. The representation π *has almost invariant vectors* if it admits a (K, ε)-invariant vector for every nonempty compact set $K \subseteq G$ and $\varepsilon > 0$. The pair (G, H) is said to have *property T* if every unitary representation which has almost invariant vectors admits a nonzero H-invariant vector. Equivalently, there exist a compact set $K \subseteq G$ and an $\varepsilon > 0$ such that every unitary representation of G possessing a (K, ε)-invariant vector admits a nonzero H-invariant vector. The group G itself is said to have property T if the pair (G, G) has property T. If G has property T and is amenable then it must be compact. Prototypical examples of noncompact groups with property T are $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$. The semidirect product $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ for the canonical action along with the subgroup $\mathbb{Z}^2$ yields an example of a pair which has property T although the groups themselves do not. See [21] for information on property T for pairs and [20] for a general reference on property T.

Write $\mathfrak{m}$ for the unique G-invariant mean on the unital C^*-algebra $\mathrm{WAP}(G)$ of weakly almost periodic continuous bounded functions on G. A unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$ is said to be *ergodic* if $\mathfrak{m}(f) = 0$ for every matrix coefficient f, *weakly mixing* if $\mathfrak{m}(|f|) = 0$ for every matrix coefficient f, and *mixing* if every matrix coefficient vanishes at infinity. Ergodicity is equivalent to the nonexistence of nonzero G-invariant vectors. Recall that a subset J of G is said to be *syndetic* if there is a finite set $F \subseteq G$ such that $FJ = G$ and *thickly syndetic* if for every finite set $F \subseteq G$ the set $\bigcap_{s\in F} sJ$ is syndetic, and note that thick syndeticity is closed under taking finite intersections. Weak mixing for π is equivalent to each of the following conditions:

(1) $\pi \otimes \bar{\pi}$ is ergodic,
(2) π has no nonzero finite-dimensional subrepresentations,
(3) for every finite subset Ω of a given dense subset of $\mathcal{H}$ and every $\varepsilon > 0$ there exists an $s \in G$ such that $|\langle \pi(s)\xi, \zeta\rangle| < \varepsilon$ for all $\xi, \zeta \in \Omega$,
(4) for all ξ, ζ in a given dense subset of $\mathcal{H}$ and every $\varepsilon > 0$ the set of all $s \in G$ such that $|\langle \pi(s)\xi, \zeta\rangle| < \varepsilon$ is thickly syndetic.

If π is weakly mixing and $\rho : G \to \mathcal{B}(\mathcal{K})$ is another unitary representation then $\pi \otimes \rho$ is weakly mixing, as is clear from condition (3). We say that π is *H-ergodic*, *weakly H-mixing*, or *H-mixing* if its restriction to H has the corresponding property. For information on weak mixing see [5], where characterizations (3) and (4) are established.

Let α be a continuous σ-preserving action of G on a von Neumann algebra M with normal state σ. On the GNS Hilbert space $L^2(M, \sigma)$ (defined by completing the seminorm $a \mapsto \sigma(a^*a)^{1/2}$ on M) we have, associated to α, the unitary representation π_σ of G given by $\pi_\sigma(s)a\xi = \alpha_s(a)\xi$ for $a \in M$, where ξ is the canonical cyclic vector and M is viewed as acting on $L^2(M, \sigma)$ via left multiplication. Denote by $L^2_0(M, \sigma)$ the orthogonal complement of the scalars in $L^2(M, \sigma)$ and by $\pi_{\sigma,0}$ the restriction of π_σ to $L^2_0(M, \sigma)$. We say that α is *ergodic*, *weakly mixing*, or *mixing* if $\pi_{\sigma,0}$ has the corresponding property, and *H-ergodic*, *weak H-mixing*, or *H-mixing* if the restriction of $\pi_{\sigma,0}$ to H has the corresponding property. We apply the same terminology for a σ-preserving action on a unital C^*-algebra with state

σ by employing the GNS representation for σ. We will also speak of H-ergodicity and weak H-mixing as properties of G-invariant states.

By characterization (3) above for weak mixing for unitary representations, we see that weak mixing for a σ-preserving action α of G on M is equivalent to the condition that for every finite set $\Omega \subseteq M$ and $\varepsilon > 0$ there is an $s \in G$ such that $|\sigma(b^*\alpha_s(a)) - \sigma(b^*)\sigma(a)| < \varepsilon$ for all $a, b \in \Omega$. Weak mixing for α is also equivalent to the ergodicity of the tensor product action $\alpha \otimes \alpha$ on $M \overline{\otimes} M$ with respect to $\sigma \otimes \sigma$. Ergodicity for α is equivalent to the nonexistence of a nonscalar element of M fixed by α. In the case of a continuous μ-preserving action α of G on a standard probability space (X, μ), weak mixing is furthermore equivalent to the condition that for every finite collection Ω of measurable subsets of X and $\varepsilon > 0$ there is an $s \in G$ such that $|\mu(\alpha_s(A) \cap B) - \mu(A)\mu(B)| < \varepsilon$ for all $A, B \in \Omega$. See [13] for a general reference on mixing properties in ergodic theory.

After this paper was finished we became aware of a preprint of Kechris [23] which contains some overlap with the present work, namely Lemmas 2.1 and 2.3, Theorem 2.5, and the characterization of property T by the closedness of the set of weak mixing actions, which Theorem 4.2 includes and strengthens.

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2. Unitary representations

Fix a separable infinite-dimensional Hilbert space $\mathcal{H}$. Let G be a second countable locally compact group and H a closed subgroup. We denote by $\mathrm{Rep}(G, \mathcal{H})$ the set of unitary representations of G on $\mathcal{H}$ equipped with the topology which has as a basis the sets

$$V(\pi, K, \Omega, \varepsilon) = \{\rho \in \mathrm{Rep}(G, \mathcal{H}) : \|\rho(s)\xi - \pi(s)\xi\| < \varepsilon \text{ for all } s \in K \text{ and } \xi \in \Omega\}$$

where $\pi \in \mathrm{Rep}(G, \mathcal{H})$, K is a compact subset of G, Ω is a finite subset of $\mathcal{H}$, and $\varepsilon > 0$. Taking a dense sequence $\{\xi_k\}_{k=1}^{\infty}$ in the unit ball of $\mathcal{H}$ and an increasing sequence $K_1 \subseteq K_2 \subseteq \dots$ of compact subsets of G whose union is G, we can define on $\mathrm{Rep}(G, \mathcal{H})$ the compatible metric

$$d(\pi, \rho) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} 2^{-j-k} \sup_{s \in K_j} \|\pi(s)\xi_k - \rho(s)\xi_k\|$$

under which $\mathrm{Rep}(G, \mathcal{H})$ is complete, so that it is Polish and hence Baire.

The following is a relativization to pairs of Proposition 2.3 in [5] and follows by essentially the same argument.

Lemma 2.1. *The set of weakly H-mixing representations in* $\mathrm{Rep}(G, \mathcal{H})$ *is a* G_δ *(and hence is itself Polish).*

The next lemma relativizes to pairs a result of Bekka and Valette [4] and appears as Theorem 8 in [3] and part of Theorem 1.2 in [21].

Lemma 2.2. *The pair (G,H) does not have property T if and only if there exists a weakly H-mixing representation of G which has almost invariant vectors.*

The proof of the following Rokhlin-type property for the conjugation action of the unitary group $\mathcal{U}(\mathcal{H})$ on $\mathrm{Rep}(G,\mathcal{H})$ is similar to that of its dynamical analogue in [15].

Lemma 2.3. *The subset of representations in $\mathrm{Rep}(G,\mathcal{H})$ with dense $\mathcal{U}(\mathcal{H})$-orbit is a dense G_δ.*

Proof. Take a dense sequence $\{\pi_n\}_{n=1}^\infty$ in $\mathrm{Rep}(G,\mathcal{H})$ and a unitary isomorphism $V:\mathcal{H}\to\mathcal{H}^{\oplus\mathbb{N}}$ and define the unitary representation $\pi: G\to\mathcal{B}(\mathcal{H})$ by $s\mapsto V^{-1}\big(\bigoplus_{n=1}^\infty \pi_n\big)(s)V$. We will show that π has dense $\mathcal{U}(\mathcal{H})$-orbit in $\mathrm{Rep}(G,\mathcal{H})$. Let ρ be a representation in $\mathrm{Rep}(G,\mathcal{H})$, Ω a finite subset of the unit ball of $\mathcal{H}$, K a compact subset of G, and $\delta>0$. Then there is an n_0 such that $\|\rho(s)\xi-\pi_{n_0}(s)\xi\|<\delta/3$ for all $s\in K$ and $\xi\in\Omega$. The continuity of ρ and π yields a finite set $F\subseteq K$ such that for every $s\in K$ there is an $s'\in F$ for which $\max_{\xi\in\Omega}\|\rho(s)\xi-\rho(s')\xi\|<\delta/3$ and $\max_{\xi\in\Omega}\|\pi(s)V^{-1}\xi^{n_0}-\pi(s')V^{-1}\xi^{n_0}\|<\delta/3$ where ξ^{n_0} denotes the vector in $\mathcal{H}^{\oplus\mathbb{N}}$ which is ξ at the coordinate n_0 and zero elsewhere. Construct a unitary isomorphism $U:\mathcal{H}\to\mathcal{H}^{\oplus\mathbb{N}}$ such that for all ξ in the finite-dimensional subspace $\mathrm{span}(\rho(F\cup\{e\})\Omega)$ we have $(U\xi)(n)=\xi$ if $n=n_0$ and $(U\xi)(n)=0$ otherwise. Then for all $\xi\in\Omega$ and $s\in K$ we have $\|U\rho(s')\xi-V\pi(s')V^{-1}U\xi\|=\|\rho(s')\xi-\pi_{n_0}(s')\xi\|<\delta/3$ and hence

$$\begin{aligned}\|\rho(s)\xi-(V^{-1}U)^{-1}\pi(s)V^{-1}U\xi\| &\leq \|\rho(s)\xi-\rho(s')\xi\|+\|U\rho(s')\xi-V\pi(s')V^{-1}U\xi\| \\ &\qquad+\|\pi(s')V^{-1}U\xi-\pi(s)V^{-1}U\xi\| \\ &< \frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}=\delta.\end{aligned}$$

This shows that π has dense $\mathcal{U}(\mathcal{H})$-orbit.

Now for every $\rho\in\mathrm{Rep}(G,\mathcal{H})$, finite set $\Omega\subseteq\mathcal{H}$, compact set $K\subseteq G$, and $\delta>0$, write $W(\rho,\Omega,K,\delta)$ for the set of all $\gamma\in\mathrm{Rep}(G,\mathcal{H})$ such that there exists a unitary operator U on $\mathcal{H}$ for which $\|\rho(s)\xi-U\gamma(s)U^{-1}\xi\|<\delta$ for all $\xi\in\Omega$ and $s\in K$. Then $W(\rho,\Omega,F,\delta)$ is open, and it is also dense by the above paragraph. Now choose an increasing sequence $\Omega_1\subseteq\Omega_2\subseteq\dots$ of finite subsets of the unit ball of $\mathcal{H}$ whose linear span is dense in $\mathcal{H}$ and an increasing sequence $K_1\subseteq K_2\subseteq\dots$ of compact subsets of G whose union is G. Then $\bigcap_{n=1}^\infty\bigcap_{m=1}^\infty W(\pi_n,\Omega_m,K_m,1/m)$ is a dense G_δ and its elements are precisely the representations in $\mathrm{Rep}(G,\mathcal{H})$ with dense $\mathcal{U}(\mathcal{H})$-orbit. □

Remark 2.4. A variation on the above local absorption argument can be used to show that whenever P is a property of unitary representations of G which is closed under unitary equivalence and taking superrepresentations and holds for some representation on a separable Hilbert space, the set of representations in $\mathrm{Rep}(G,\mathcal{H})$ with property P is dense. In particular, the set of representations in $\mathrm{Rep}(G,\mathcal{H})$ which have almost invariant vectors is always a dense G_δ.

Theorem 2.5. *If the pair (G,H) does not have property T then the set of weakly H-mixing representations in $\mathrm{Rep}(G,\mathcal{H})$ is a dense G_δ, while if (G,H) has property T then* *the set of H-ergodic representations in* $\mathrm{Rep}(G,\mathcal{H})$ *is nowhere dense. Moreover, if G itself has property T then in* $\mathrm{Rep}(G,\mathcal{H})$ *the set of ergodic representations and the set of weakly mixing representations are both closed.*

Proof. Suppose first that the pair (G,H) does not have property T. By Lemma 2.1 it suffices to prove that the set of weakly H-mixing representations in $\mathrm{Rep}(G,\mathcal{H})$ is dense. Let π be an element of $\mathrm{Rep}(G,\mathcal{H})$, K a compact subset of G, Ω a nonempty finite subset of the unit sphere of $\mathcal{H}$, and $\varepsilon > 0$, and let us demonstrate the existence of a weakly H-mixing representation in the open neighbourhood $V(\pi,K,\Omega,\varepsilon)$ of π. By Lemma 2.2 there exists a weakly H-mixing representation ρ of G on a Hilbert space $\mathcal{K}$ and a unit vector $\zeta\in\mathcal{K}$ such that $\|\rho(s)\zeta-\zeta\|<\varepsilon/3$ for every $s\in K$. We may assume $\mathcal{K}$ to be separable by restricting to the closed G-invariant subspace generated by ζ. By the continuity of π, we can find a finite set $F\subseteq K$ such that for every $s\in K$ there is an $s'\in F$ for which $\max_{\xi\in\Omega}\|\pi(s)\xi-\pi(s')\xi\|<\varepsilon/3$. Construct a unitary isomorphism $U:\mathcal{H}\to\mathcal{K}\otimes\mathcal{H}$ such that $U\xi=\zeta\otimes\xi$ for every ξ in the finite-dimensional subspace $\mathrm{span}(\pi(F\cup\{e\})\Omega)$. Then for each $\xi\in\Omega$ and $s\in K$ we have

$$\begin{aligned}\|(\rho\otimes\pi)(s)U\xi-U\pi(s')\xi\| &= \|\rho(s)\zeta\otimes\pi(s)\xi-\zeta\otimes\pi(s')\xi\|\\ &\leq \|\rho(s)\zeta\otimes(\pi(s)\xi-\pi(s')\xi)\|+\|(\rho(s)\zeta-\zeta)\otimes\pi(s')\xi\|\\ &< \frac{2\varepsilon}{3}\end{aligned}$$

and hence

$$\begin{aligned}\|U^{-1}(\rho\otimes\pi)(s)U\xi-\pi(s)\xi\| &\leq \|(\rho\otimes\pi)(s)U\xi-U\pi(s')\xi\|+\|U(\pi(s')\xi-\pi(s)\xi)\|\\ &< \frac{2\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

This shows that the representation γ given by $s\mapsto U^{-1}(\pi\otimes\rho)(s)U$ is contained in $V(\pi,K,\Omega,\varepsilon)$. Now since ρ is weakly H-mixing $\pi\otimes\rho$ is weakly H-mixing and thus γ is weakly H-mixing, yielding the desired density.

Suppose now that the pair (G,H) has property T. Then there exists a nonempty compact set $K\subseteq G$ and an ε such that every representation $\pi\in\mathrm{Rep}(G,\mathcal{H})$ possessing a (K,ε)-invariant vector has a nonzero H-invariant vector. Thus if ρ is a representation in $\mathrm{Rep}(G,\mathcal{H})$ which has a G-invariant unit vector $\xi\in\mathcal{H}$ then every representation in the neighbourhood $V(\rho,K,\{\xi\},\varepsilon)$ fails to be H-ergodic. Since such ρ evidently exist and by Lemma 2.3 there exist representations in $\mathrm{Rep}(G,\mathcal{H})$ with dense $\mathcal{U}(\mathcal{H})$-orbit, we infer that the set of H-ergodic representations in $\mathrm{Rep}(G,\mathcal{H})$ is nowhere dense. In the case that G itself has property T, we furthermore deduce that the set of ergodic representations in $\mathrm{Rep}(G,\mathcal{H})$ is closed, which implies that the set of weakly mixing representations in $\mathrm{Rep}(G,\mathcal{H})$ is also closed in view of the continuity of the map $\pi\mapsto\pi\otimes\bar{\pi}$ from $\mathrm{Rep}(G,\mathcal{H})$ to $\mathrm{Rep}(G,\mathcal{H}\otimes\overline{\mathcal{H}})$. □

3. Actions on the hyperfinite II_1 factor

We write R for the hyperfinite II_1 factor and τ for the unique normal tracial state on R. The *-automorphism group of R will be denoted by $\mathrm{Aut}(R)$. With the aim of establishing our main result in this section concerning actions on R, we will first examine the relationship of asymptotic Abelianness to weak mixing and construct a certain type of Bogoliubov action on the CAR algebra in the non-property T case.

Associated to a state σ on a unital C^*-algebra A is the seminorm $\|a\|_\sigma = \sigma(a^*a)^{1/2}$, which we will refer to as the *σ-seminorm*, or *σ-norm* if σ is faithful. Let G be a second countable locally compact group and H a closed subgroup. The following is a strong version of the weak type of asymptotic Abelianness that has arisen in the operator-algebraic approach to quantum statistical mechanics (see Section 4 of [6]). It is closely related to the notion of asymptotic Abelianness for a von Neumann algebra [28].

Definition 3.1. Let A be a unital C^*-algebra, σ a state on A, and α a σ-preserving action of G on A which is continuous for the σ-seminorm, i.e., the map $s \mapsto \alpha_s(a)$ is σ-seminorm continuous for every $a \in A$. We say that α is *(H,σ)-Abelian* if for every finite set $\Omega \subseteq M$ and $\varepsilon > 0$ there exists an $s \in H$ such that $\|[\alpha_s(a), b]\|_\sigma < \varepsilon$ for all $a, b \in \Omega$.

If A is separable with respect to the σ-seminorm then the action α is (H,σ)-Abelian if and only if there is a sequence $\{s_n\}_{n=1}^\infty$ in H such that $\lim_{n\to\infty} \|[\alpha_{s_n}(a), b]\|_\sigma = 0$ for all $a, b \in A$. This follows by observing that if α is (H,σ)-Abelian then taking an increasing sequence $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ of finite subsets of A whose union is dense in A for the σ-seminorm we can choose for each $n \in \mathbb{N}$ an $s_n \in H$ such that $\|[\alpha_{s_n}(a), b]\|_\sigma < 1/n$ for all $a, b \in \Omega_n$, in which case $\lim_{n\to\infty} \|[\alpha_{s_n}(a), b]\|_\sigma = 0$ for all $a, b \in M$. Notice also that if σ is faithful and A is not commutative then the sequence $\{s_n\}_{n=1}^\infty$ must tend to infinity, as the existence of a limit point for the sequence implies that A is commutative.

It is shown in Example 4.3.24 in [6] that, for actions on a C^*-algebra, asymptotic Abelianness conditions with respect to the operator norm imply mixing properties for a given factor state σ. In fact the argument there still applies if the operator norm in the asymptotic Abelianness hypothesis is replaced with the σ-norm, and consequently we have:

Proposition 3.2. *Let A be a unital C^*-algebra with factorial state σ and α a σ-preserving action of G on A. Suppose that α is (G,σ)-Abelian. Then α is weakly mixing.*

The converse of the above proposition is false. In fact the examples in Example 9.11 in [24] also work here:

Example 3.3. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. The CAR algebra $A(\mathcal{H})$ is defined as the unique, up to *-isomorphism, unital C^*-algebra generated by the image of an antilinear map $\xi \mapsto a(\xi)$ from $\mathcal{H}$ to $A(\mathcal{H})$ such that the anticommutation relations

$$\begin{aligned} a(\xi)a(\zeta)^* + a(\zeta)^*a(\xi) &= \langle \xi, \zeta \rangle 1_{A(\mathcal{H})}, \\ a(\xi)a(\zeta) + a(\zeta)a(\xi) &= 0, \end{aligned}$$

hold for all $\xi, \zeta \in \mathcal{H}$ (see [7]). Given a unitary operator U on $\mathcal{H}$ we write α_U for the corresponding Bogoliubov automorphism of $A(\mathcal{H})$ determined by $\alpha_U(a(\xi)) = a(U\xi)$ for $\xi \in \mathcal{H}$. The C^*-algebra $A(\mathcal{H})$ has a unique tracial state σ which is given on products of the form $a(\zeta_n)^* \cdots a(\zeta_1)^* a(\xi_1) \cdots a(\xi_m)$ by

$$\sigma(a(\zeta_n)^* \cdots a(\zeta_1)^* a(\xi_1) \cdots a(\xi_m)) = \delta_{nm} \det[\langle \tfrac{1}{2}\xi_i, \zeta_j \rangle].$$

Now if U is a unitary operator on $\mathcal{H}$ with the property that $\lim_{|n|\to\infty}\langle U^n\xi,\zeta\rangle = 0$ for all $\xi,\zeta\in\mathcal{H}$ then α_U is mixing for σ (see Example 5.2.21 in [7]) but for every $\xi\in\mathcal{H}$ we have

$$\begin{aligned}\|[a(U^n\xi),a(\xi)]\|_\sigma = 2\|a(U^n\xi)a(\xi)\|_\sigma &= 2|\sigma(a(\xi)^*a(U^n\xi)^*a(U^n\xi)a(\xi))|^{1/2}\\ &= \sqrt{\|\xi\|^4 - |\langle U^n\xi,\xi\rangle|^2},\end{aligned}$$

which converges to $\|\xi\|^2$ as $|n|\to\infty$, showing that α_U is not $(\mathbb{Z},\sigma)$-Abelian.

Continuing in the framework described in Example 3.3, the even CAR algebra is defined as the unital C^*-subalgebra of the CAR algebra $A(\mathcal{H})$ consisting of those elements which are fixed by the Bogoliubov automorphism associated to scalar multiplication by -1 on $\mathcal{H}$, and it is generated by even products of operators of the form $a(\xi)$ and $a(\xi)^*$ for $\xi\in\mathcal{H}$. Both the CAR algebra and the even CAR algebra are *-isomorphic to the type 2^∞ UHF algebra [7, 30]. A unitary representation $\pi : G\to\mathcal{B}(\mathcal{H})$ gives rise via Bogoliubov automorphisms to continuous actions of G on the CAR algebra and the even CAR algebra (by restriction). Like the Gaussian construction in commutative dynamics as used in [11, 16], Bogoliubov actions provide a means for transporting representation-theoretic phenomena into the dynamical realm (compare for example [2]). In our application of the following lemma in the proof of Theorem 3.7 we will use the fact that every continuous action of G on the CAR algebra extends to a continuous action on the weak operator closure in the tracial GNS representation, which is *-isomorphic to R.

Lemma 3.4. *Suppose the pair (G,H) does not have property T. Let A be the CAR algebra with its unique tracial state σ. Let K be a compact subset of G, $\varepsilon>0$, and $n\in\mathbb{N}$. Then there exist a continuous action β of G on A with the property*

($*$) *for every finite set $\Omega\subseteq A$ and $\delta>0$ the set of all $t\in H$ such that $|\sigma(b^*\beta_t(a)) - \sigma(b^*)\sigma(a)|<\delta$ and $\|[\beta_t(a),b]\|<\delta$ for all $a,b\in\Omega$ is thickly syndetic in H*

and a 2^n-element partition of unity $\mathcal{P}$ in A such that $\sigma(e)=2^{-n}$ and $\|\beta_s(e)-e\|<\varepsilon$ for all $e\in\mathcal{P}$ and $s\in K$.

Proof. By Lemma 2.2 there is a weakly H-mixing representation π of G on a separable infinite-dimensional Hilbert space $\mathcal{H}$ which has almost invariant vectors. Corresponding to the n-fold direct sum representation $\pi^{\oplus\{1,\dots,n\}}$ we have a Bogoliubov action β on the even CAR algebra over $\mathcal{H}^{\oplus\{1,\dots,n\}}$, which we can identify with A according to the comments preceding the lemma. Suppose that we are given a finite set $\Omega\subseteq A$ and a $\delta>0$. Since π is weakly H-mixing so is $\pi^{\oplus\{1,\dots,n\}}$, in which case a straightforward relativization to pairs of Theorem 10.4 of [24] shows that the set of all $t\in H$ such that $|\sigma(b^*\beta_t(a))-\sigma(b^*)\sigma(a)|<\delta$ for all $a,b\in\Omega$ and the set of all $t\in H$ such that $\|[\beta_t(a),b]\|<\delta$ for all $a,b\in\Omega$ are both thickly syndetic in H (note that, as illustrated by Example 3.3 above, Theorem 10.4 of [24] is false for Bogoliubov actions on the CAR algebra itself, necessitating our use of the even CAR algebra here). Since the intersection of two thickly syndetic sets is thickly syndetic, we have verified that β satisfies property ($*$).

Now take a unit vector $\xi\in\mathcal{H}$ such that $\|\pi(s)\xi-\xi\|<\varepsilon/(2n)$ for all $s\in K$. For each $k=1,\dots,n$ write ξ_k for the vector $(0,\dots,0,\xi,0,\dots,0)$ in $\mathcal{H}^{\oplus\{1,\dots,n\}}$ supported on the kth summand and $e_{k,0}$ and $e_{k,1}$ for the projections $a(\xi_k)^*a(\xi_k)$ and $a(\xi_k)a(\xi_k)^*$, respectively, in A. For each $\kappa\in\{0,1\}^{\{1,\dots,n\}}$ set $e_\kappa = e_{1,\kappa(1)}e_{2,\kappa(2)}\cdots e_{n,\kappa(n)}\in A$. Then

$\{e_\kappa : \kappa \in \{0,1\}^{\{1,\dots,n\}}\}$ is a 2^n-element partition of unity in A, and for each $\kappa \in \{0,1\}^{\{1,\dots,n\}}$ we have $\sigma(e_\kappa) = 2^{-n}$ and, for all $s \in K$,

$$\begin{aligned}\|\beta_s(e_\kappa) - e_\kappa\| &\le \sum_{k=1}^n \|\beta_s(e_{k,\kappa(k)}) - e_{k,\kappa(k)}\| \le \sum_{k=1}^n 2\|a(\pi^{\oplus\{1,\dots,n\}}(s)\xi_k) - a(\xi_k)\| \\ &= 2n\|\pi(s)\xi - \xi\| < \varepsilon,\end{aligned}$$

as desired. □

We denote by $\mathfrak{A}_{R,G}$ the set of continuous actions of G on R by *-automorphisms equipped with the topology which has as a basis the sets

$$V(\alpha, K, \Omega, \varepsilon) = \{\beta \in \mathfrak{A}_{R,G} : \|\beta_s(a) - \alpha_s(a)\|_\tau < \varepsilon \text{ for all } s \in K \text{ and } a \in \Omega\}$$

where $\alpha \in \mathfrak{A}_{R,G}$, K is a compact subset of G, Ω is a finite subset of R, and $\varepsilon > 0$. For $G = \mathbb{Z}$ this topology coincides with the u-topology on $\mathrm{Aut}(R)$ (canonically identified with $\mathfrak{A}_{R,\mathbb{Z}}$) defined via point-norm convergence on the predual [17]. To each action in $\mathfrak{A}_{R,G}$ is canonically associated a representation in the space $\mathrm{Rep}(G, L^2(M,\tau))$ as defined in Section 2, and this map is evidently continuous. It is also readily checked that the image of this map is closed in $\mathrm{Rep}(G, L^2(M,\tau))$, from which we deduce that $\mathfrak{A}_{R,G}$ is a Polish space.

The following type of Rokhlin property for the action of $\mathrm{Aut}(R)$ on $\mathfrak{A}_{R,G}$ by conjugation is a dynamical version of Lemma 2.3. In parallel with the commutative analogue [15], for $G = \mathbb{Z}$ it can be deduced from the fact that every aperiodic automorphism of R has dense conjugacy class, as follows from the work of Connes on outer conjugacy, in which a noncommutative Rokhlin tower theorem plays a key role [10].

Lemma 3.5. *The subset of elements in $\mathfrak{A}_{R,G}$ with dense $\mathrm{Aut}(R)$-orbit is a dense G_δ.*

Proof. Take a dense sequence $\{\beta_n\}_{n=1}^\infty$ in $\mathfrak{A}_{R,G}$ and a *-isomorphism $\Psi : R \to (R,\tau)^{\overline{\otimes}\mathbb{N}}$ and define the action β of G on R by $s \mapsto \Psi^{-1} \circ \left(\bigotimes_{n=1}^\infty \beta_{n,s}\right) \circ \Psi$. To show that β has dense $\mathrm{Aut}(R)$-orbit in $\mathfrak{A}_{R,G}$, suppose we are given an action $\alpha \in \mathfrak{A}_{R,G}$, a finite set Ω in the unit ball of R, a compact set $K \subseteq G$, and a $\delta > 0$. Then we can find an n_0 such that $\|\alpha_s(a) - \beta_{n_0,s}(a)\|_\tau < \delta/5$ for all $a \in \Omega$ and $s \in K$. Take a finite-dimensional subfactor $N \subseteq R$ such that for each $a \in \bigcup_{s\in K\cup\{e\}} \alpha_s(\Omega)$ there is an $E(a)$ in the unit ball of N for which $\|a - E(a)\|_\tau < \delta/5$. Define an injective *-homomorphism $\varphi : N \to (R,\tau)^{\overline{\otimes}\mathbb{N}}$ by $\varphi(a) = \bigotimes_{n=1}^\infty a_n$ where $a_n = a$ if $n = n_0$ and $a_n = 1$ otherwise, and extend φ to a *-isomorphism $\Phi : R \to (R,\tau)^{\overline{\otimes}\mathbb{N}}$. Then, denoting by τ' the product tracial state $\tau^{\otimes\mathbb{N}}$ on $(R,\tau)^{\overline{\otimes}\mathbb{N}}$, for $a \in \Omega$ and $s \in K$ we have

$$\begin{aligned}\|\Phi(E(\alpha_s(a))) - (\Psi \circ \beta_s \circ \Psi^{-1})(\Phi(E(a)))\|_{\tau'} &= \|E(\alpha_s(a)) - \beta_{n_0,s}(E(a))\|_\tau \\ &\le \|E(\alpha_s(a)) - \alpha_s(a)\|_\tau + \|\alpha_s(a) - \beta_{n_0,s}(a)\|_\tau + \|\beta_{n_0,s}(a) - \beta_{n_0,s}(E(a))\|_\tau \\ &< \frac{3\delta}{5}\end{aligned}$$

and so

$$\begin{aligned}&\|\alpha_s(a) - ((\Psi^{-1}\circ\Phi)^{-1} \circ \beta_s \circ (\Psi^{-1}\circ\Phi))(a)\|_\tau \\ &\qquad \le \|\alpha_s(a) - E(\alpha_s(a))\|_\tau + \|\Phi(E(\alpha_s(a))) - (\Psi\circ\beta_s\circ\Psi^{-1})(\Phi(E(a)))\|_{\tau'}\end{aligned}$$

$$+\|(\Psi\circ\beta_s\circ\Psi^{-1})(\Phi(E(a)-a))\|_\tau$$
$$<\frac{\delta}{5}+\frac{3\delta}{5}+\frac{\delta}{5}=\delta.$$

Thus β has dense $\mathrm{Aut}(R)$-orbit in $\mathfrak{A}_{R,G}$.

Now for every $\alpha\in\mathfrak{A}_{R,G}$, finite set $\Omega\subseteq R$, compact set $K\subseteq G$, and $\delta>0$ write $W(\alpha,\Omega,K,\delta)$ for the set of all $\gamma\in\mathfrak{A}_{R,G}$ such that there exists a *-automorphism Φ of R for which $\|\alpha_s(a)-(\Phi\circ\gamma_s\circ\Phi^{-1})(a)\|<\delta$ for all $a\in\Omega$ and $s\in K$. Then $W(\alpha,\Omega,F,\delta)$ is open, and it is dense by the first paragraph. Take an increasing sequence $\Omega_1\subseteq\Omega_2\subseteq\dots$ of finite subsets of the unit ball of R whose linear span is dense in R with respect to the τ-norm, as well as an increasing sequence $K_1\subseteq K_2\subseteq\dots$ of compact subsets of G whose union is G. Then the set of actions in $\mathfrak{A}_{R,G}$ with dense $\mathrm{Aut}(R)$-orbit is equal to $\bigcap_{n=1}^\infty\bigcap_{m=1}^\infty W(\beta_n,\Omega_m,K_m,1/m)$, which is a dense G_δ. □

Lemma 3.6. *Suppose that the pair (G,H) has property T. Then the set of H-ergodic actions in $\mathfrak{A}_{R,G}$ is nowhere dense. Moreover, if G itself has property T then in $\mathfrak{A}_{R,G}$ the set of ergodic actions and the set of weakly mixing actions are both closed.*

Proof. By property T there exist a nonempty compact set $K\subseteq G$ and an $\varepsilon>0$ such that every representation $\pi\in\mathrm{Rep}(G,\mathcal{H})$ possessing (K,ε)-invariant vectors has a nonzero H-invariant vector. Let α be an action in $\mathfrak{A}_{R,G}$ which fails to be G-ergodic. Then there is a nonscalar $a\in R$ which is fixed by α. Then for all β in a small enough neighbourhood of α we will have $\sup_{s\in K}\|\beta_s(a)-a\|_\tau<\varepsilon\|a\|_\tau$ which implies that β is not H-ergodic. It then follows by Lemma 3.5 that the set of H-ergodic actions is nowhere dense. In the case $H=G$, we also deduce that the set of ergodic actions is closed, from which it follows using the continuity of the map $\alpha\mapsto\alpha\otimes\alpha$ from $\mathfrak{A}_{R,G}$ to $\mathfrak{A}_{R\overline{\otimes}R,G}$ that the set of weakly mixing actions is also closed. □

Theorem 3.7. *If the pair (G,H) does not have property T then the set of (H,τ)-Abelian actions in $\mathfrak{A}_{R,G}$ is a dense G_δ, while if (G,H) has property T then the set of (H,τ)-Abelian actions in $\mathfrak{A}_{R,G}$ is nowhere dense.*

Proof. If (G,H) has property T then the nowhere density of the set of (H,τ)-Abelian actions follows by Proposition 3.2, Lemma 3.6, and the fact that weak mixing implies ergodicity. Suppose then that (G,H) does not have property T. Let Ω be a nonempty finite set of unitaries in R and let $\varepsilon>0$. Denote by $W(\Omega,\varepsilon)$ the set of all actions α in $\mathfrak{A}_{R,G}$ such that there exists a $t\in H$ for which $\|[\alpha_t(u),v]\|_\tau^2<\varepsilon$ for all $u,v\in\Omega$. Evidently $W(\Omega,\varepsilon)$ is open in $\mathfrak{A}_{R,G}$. We will show that it is also dense.

Suppose we are given an $\alpha\in\mathfrak{A}_{R,G}$, a nonempty finite set Ω' of unitaries in R, a compact set $K\subseteq G$, and a $\delta>0$, and let us show that there exists an $\tilde{\alpha}\in W(\Omega,\varepsilon)$ such that $\|\tilde{\alpha}_s(u)-\alpha_s(u)\|_\tau<\delta$ for every $u\in\Omega'$ and $s\in K$. Since $W(\Omega,\varepsilon)\supseteq W(\Omega'\cup\Omega,\varepsilon)$ we may assume for simplicity that $\Omega=\Omega'$.

Let n be a positive integer power of 2 such that $n>48/\varepsilon$. Applying Lemma 3.4 and extending the action it yields to the weak operator closure under the tracial representation, we can produce a continuous action β of G on R such that (i) for every finite set $\Theta\subseteq M$ and $\eta>0$ there exists a $t\in H$ for which $|\tau(b^*\beta_t(a))-\tau(b)\tau(a)|<\eta$ and $\|[\beta_t(a),b]\|_\tau<\eta$ for all $a,b\in\Theta$ and (ii) there exists an n-element partition of unity $\mathcal{P}=\{e_1,\dots,e_n\}$ in R $\tau(e_i)=n^{-1}$ and $\|\beta_s(e_i)-e_i\|_\tau<\delta(6n)^{-1}$ for every $i=1,\dots,n$ and $s\in K$.

Take a finite-dimensional subfactor $N \subseteq R$ such that for each $u \in \Omega \cup \Omega^* \cup \bigcup_{s\in K} \alpha_s(\Omega)$ there is an element $E(u)$ in the unit ball of N for which $\|u - E(u)\|_\tau < \min(\delta/(12n), \varepsilon/32)$. For $a \in R$ and $i = 1, \dots, n$ we write $a_{[i]}$ to denote the elementary tensor in $R^{\overline{\otimes}\{1,\dots,n\}}$ with a in the ith factor and $\mathbf{1}$ everywhere else. Define a map $\varphi : N \to R\overline{\otimes}R^{\overline{\otimes}\{1,\dots,n\}}$ by $\varphi(a) = \sum_{i=1}^n e_i \otimes a_{[i]}$ for all $a \in N$. Then φ is an injective *-homomorphism. Since $R\overline{\otimes}R^{\overline{\otimes}\{1,\dots,n\}}$ is *-isomorphic to R and any two type $\mathrm{I_m}$ subfactors of R are unitarily equivalent, we can extend φ to a *-isomorphism $\Phi : R \to R\overline{\otimes}R^{\overline{\otimes}\{1,\dots,n\}}$. Set $\theta = \beta \otimes \alpha^{\otimes\{1,\dots,n\}}$ and define the action $\tilde{\alpha} \in \mathfrak{A}_{R,G}$ by $\tilde{\alpha}_s = \Phi^{-1} \circ \theta_s \circ \Phi$ for all $s \in G$.

Write τ' for the unique normal tracial state $\tau \otimes \tau^{\otimes\{1,\dots,n\}}$ on $R\overline{\otimes}R^{\overline{\otimes}\{1,\dots,n\}}$, and note that $\tau' \circ \Phi = \tau$. For $u \in \Omega$ and $s \in K$ we have

$$\begin{aligned}\|\alpha_s(E(u)) - E(\alpha_s(u))\|_\tau &\leq \|\alpha_s(E(u) - u)\|_\tau + \|\alpha_s(u) - E(\alpha_s(u))\|_\tau \\ &< \frac{\delta}{12n} + \frac{\delta}{12n} = \frac{\delta}{6n}\end{aligned}$$

so that

$$\begin{aligned}\|\tilde{\alpha}_s(E(u)) - E(\alpha_s(u))\|_\tau &= \|\theta_s(\Phi(E(u))) - \Phi(E(\alpha_s(u)))\|_{\tau'} \\ &\leq \sum_{i=1}^n \|\beta_s(e_i) \otimes \alpha_s(E(u))_{[i]} - e_i \otimes E(\alpha_s(u))_{[i]}\|_{\tau'} \\ &\leq \sum_{i=1}^n \big(\|\beta_s(e_i) - e_i\|_\tau + \|\alpha_s(E(u)) - E(\alpha_s(u))\|_\tau\big) \\ &< n\bigg(\frac{\delta}{6n} + \frac{\delta}{6n}\bigg) = \frac{\delta}{3}\end{aligned}$$

and hence

$$\begin{aligned}\|\tilde{\alpha}_s(u) - \alpha_s(u)\|_\tau &\leq \|\tilde{\alpha}_s(u - E(u))\|_\tau + \|\tilde{\alpha}_s(E(u)) - E(\alpha_s(u))\|_\tau + \|E(\alpha_s(u)) - \alpha_s(u)\|_\tau \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.\end{aligned}$$

Let us show now that $\tilde{\alpha} \in W(\Omega, \varepsilon)$. By our choice of β there exists a $t \in H$ such that $\|[\beta_t(e_j), e_k]\|_\tau < \varepsilon/(12n^3)$ for all $j, k = 1, \dots, n$ and $\tau(e_j\beta_t(e_j)) < 2\tau(e_j)^2 = 2/n^2$ for all $j = 1, \dots, n$. Observe that, given any a, b, c, d in the unit ball of N, for $j, k = 1, \dots, n$ with $j \neq k$ we have

$$\begin{aligned}&\big|\tau'\big(\theta_t \circ \Phi(a)\big[\beta_t(e_j) \otimes \alpha_t(b)_{[j]}, e_k \otimes c_{[k]}\big]\Phi(d)\big)\big| \\ &\qquad \leq \|[\beta_t(e_j) \otimes \alpha_t(b)_{[j]}, e_k \otimes c_{[k]}]\|_{\tau'} = \|[\beta_t(e_j), e_k] \otimes \alpha_t(b)_{[j]}c_{[k]}\|_{\tau'} \\ &\qquad \leq \|[\beta_t(e_j), e_k]\|_\tau < \frac{\varepsilon}{12n^2}\end{aligned}$$

while for $j = 1, \dots, n$ we have

$$\begin{aligned}&\big|\tau'\big(\theta_t \circ \Phi(a)\big[\beta_t(e_j) \otimes \alpha_t(b)_{[j]}, e_j \otimes c_{[j]}\big]\Phi(d)\big)\big| \\ &\qquad \leq \sum_{1\leq i,l\leq n} \big|\tau'\big(\beta_t(e_i) \otimes \alpha_t(a)_{[i]}\big[\beta_t(e_j) \otimes \alpha_t(b)_{[j]}, e_j \otimes c_{[j]}\big]e_l \otimes d_{[l]}\big)\big|\end{aligned}$$

$$\begin{aligned}
&\leq |\tau(\beta_t(e_j)e_j)| + \sum_{1\leq i,l\leq n} |\tau(\beta_t(e_i)e_j\beta_t(e_j)e_l)| \\
&\leq 2|\tau(\beta_t(e_j)e_j)| + n^2\|[\beta_t(e_j), e_j]\|_\tau \\
&< \frac{4}{n^2} + \frac{\varepsilon}{12n} < \frac{\varepsilon}{6n}.
\end{aligned}$$

It follows that for $u, v \in \Omega$ we have

$$\begin{aligned}
&\big|\tau\big(\tilde{\alpha}_t(E(u^*))[\tilde{\alpha}_t(E(u)), E(v^*)]E(v)\big)\big| \\
&\quad = \big|\tau'\big(\theta_t\circ\Phi(E(u^*))\big[\theta_t\circ\Phi(E(u)), \Phi(E(v^*))\big]\Phi(E(v))\big)\big| \\
&\quad \leq \sum_{\substack{1\leq j,k\leq n\\ j\neq k}} \big|\tau'\big(\theta_t\circ\Phi(E(u^*))\big[\beta_t(e_j)\otimes\alpha_t(E(u))_{[j]}, e_k\otimes E(v^*)_{[k]}\big]\Phi(E(v))\big)\big| \\
&\qquad\quad + \sum_{1\leq j\leq n} \big|\tau'\big(\theta_t\circ\Phi(E(u^*))\big[\beta_t(e_j)\otimes\alpha_t(E(u))_{[j]}, e_j\otimes E(v^*)_{[j]}\big]\Phi(E(v))\big)\big| \\
&\quad < \frac{\varepsilon}{12} + \frac{\varepsilon}{6} = \frac{\varepsilon}{4}
\end{aligned}$$

and hence

$$\begin{aligned}
\|[\tilde{\alpha}_t(u), v]\|_\tau^2 &= 2\,\mathrm{Re}\,\tau(\tilde{\alpha}_t(u^*)[\tilde{\alpha}_t(u), v^*]v) \\
&\leq 2\big|\tau\big(\tilde{\alpha}_t(E(u^*))[\tilde{\alpha}_t(E(u)), E(v^*)]E(v)\big)\big| \\
&\qquad + 4(\|u - E(u)\|_\tau + \|u^* - E(u^*)\|_\tau + \|u - E(u)\|_\tau + \|v^* - E(v^*)\|_\tau) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore $\tilde{\alpha} \in W(\Omega, \varepsilon)$, and since R is spanned by unitaries we conclude that $W(\Omega, \varepsilon)$ is dense in $\mathfrak{A}_{R,G}$.

Now take an increasing sequence $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ of nonempty finite sets of unitaries in R whose union has dense linear span in R with respect to the τ-norm. Then the set of (H, τ)-Abelian actions in $\mathfrak{A}_{R,G}$ is equal to $\bigcap_{j=1}^\infty W(\Omega_j, 1/j)$, which is a dense G_δ. □

Theorem 3.8. *If the pair (G, H) does not have property T then the set of weakly H-mixing actions in $\mathfrak{A}_{R,G}$ is a dense G_δ, while if (G, H) has property T then the set of H-ergodic actions in $\mathfrak{A}_{R,G}$ is nowhere dense. Moreover, if G itself has property T then in $\mathfrak{A}_{R,G}$ the set of ergodic actions and the set of weakly mixing actions are both closed.*

Proof. In the case that (G, H) does not have property T, the desired conclusion follows by combining Theorem 3.7 and Proposition 3.2. The rest of the theorem is Lemma 3.6. □

The statement of Theorem 3.8 remains valid if we replace $\mathfrak{A}_{R,G}$ by the set $\mathfrak{A}_{A,G}$ of G-actions on the CAR algebra A with the topology described at the beginning of Section 6 and understand weak mixing and ergodicity to be relative to the unique tracial state. We leave the details to the reader (compare the proof of Theorem 4.2).

4. Actions on a nonatomic standard probability space

Fix a nonatomic standard probability space (X, μ). As before G will be a second countable locally compact group and H a closed subgroup. We write $\mathfrak{A}_{X,G}$ for the set of continuous measure-preserving actions of G on (X,μ) equipped with the topology which has as a basis the sets

$$V(\alpha,K,\Omega,\varepsilon)=\{\beta\in\mathfrak{A}_{X,G}:|\mu(\beta_s(A)\Delta\alpha_s(A))|<\varepsilon \text{ for all } s\in K \text{ and } A\in\Omega\}$$

where $\alpha\in\mathfrak{A}_{X,G}$, K is a compact subset of G, Ω is a finite collection of measurable subsets of X, and $\varepsilon>0$. As is well known, the space $\mathfrak{A}_{X,G}$ is Polish, since the canonical embedding into the unitary representation space $\mathrm{Rep}(G,L^2(X,\mu))$ has closed image.

The following is a multiset variation on the Connes-Weiss-type result which appears in [16].

Lemma 4.1. *Suppose that the pair (G,H) does not have property T. Let K be a compact subset of G, $\varepsilon>0$, and $n\in\mathbb{N}$. Then there is a weakly H-mixing action $\beta\in\mathfrak{A}_{X,G}$ for which there exists a measurable partition $\mathcal{P}$ of X into 2^n sets such that $\mu(A)=2^{-n}$ and $\mu(\beta_s(A)\Delta A)<\varepsilon$ for all $A\in\mathcal{P}$ and $s\in K$.*

Proof. Choose a $\delta>0$ such that $(\frac{1}{2}+\delta)^n<(\frac{1}{2}-\delta)^n+\varepsilon$. A straightforward relativization to pairs of the Gaussian construction in [16] using Lemma 2.2 produces a weakly H-mixing action $\alpha\in\mathfrak{A}_{X,G}$ and a set $A\subseteq X$ of μ-measure 1/2 such that $\mu(\alpha_s(A)\Delta A)<\delta$ for all $s\in K$. Set $A_0=A$ and $A_1=X\setminus A$.

Let β be the action of G on (X^n,μ^n) given by the product of n copies of α. Since α is weakly H-mixing so is β. For each $\kappa\in\{0,1\}^{\{1,\dots,n\}}$ set $A_\kappa=A_{\kappa(1)}\times A_{\kappa(2)}\times\cdots\times A_{\kappa(n)}\in X^n$. Then $\{A_\kappa:\kappa\in\{0,1\}^{\{1,\dots,n\}}\}$ is a measurable partition of X^n into 2^n sets of equal μ^n-measure. Now suppose we are given $s\in F$ and $\kappa\in\{0,1\}^{\{1,\dots,n\}}$. For each $i=1,\dots,n$ we have $\mu(\alpha_s(A_{\kappa(i)})\cap A_{\kappa(i)})>\frac{1}{2}-\delta$ and $\mu(\alpha_s(A_{\kappa(i)})\cup A_{\kappa(i)})<\frac{1}{2}+\delta$ and so the complement C of $(\alpha_s(A_{\kappa(1)})\cap A_{\kappa(1)})\times\cdots\times(\alpha_s(A_{\kappa(n)})\cap A_{\kappa(n)})$ in $(\alpha_s(A_{\kappa(1)})\cup A_{\kappa(1)})\times\cdots\times(\alpha_s(A_{\kappa(n)})\cup A_{\kappa(n)})$ has μ^n-measure less than $(\frac{1}{2}+\delta)^n-(\frac{1}{2}-\delta)^n$, which in turn is less than ε. Since $\beta_s(A_\kappa)\Delta A_\kappa$ is contained in C it follows that $\mu^n(\beta_s(A_\kappa)\Delta A_\kappa)<\varepsilon$. Since (X^n,μ^n) is isomorphic to (X,μ) this completes the proof. □

Theorem 4.2. *If the pair (G,H) does not have property T then the set of weakly H-mixing actions in $\mathfrak{A}_{X,G}$ is a dense G_δ, while if (G,H) has property T then the set of H-ergodic actions in $\mathfrak{A}_{X,G}$ is nowhere dense. Moreover, if G itself has property T then in $\mathfrak{A}_{X,G}$ the set of ergodic actions and the set of weakly mixing actions are both closed.*

Proof. The nowhere density of H-ergodic actions when (G,H) has property T and the closedness of the set of ergodic actions and the set of weakly mixing actions when G has property T follow by an argument parallel to that of Lemma 3.6 using the weak Rokhlin property established in [15] (the proof given there also works in the nondiscrete case).

Suppose then that (G,H) does not have property T. Let $\mathcal{P}$ be a finite measurable partition of X and let $\varepsilon>0$. Denote by $W(\mathcal{P},\varepsilon)$ the set of all actions $\alpha\in\mathfrak{A}_{X,G}$ such that there exists a $t\in G$ for which $|\mu(\alpha_t(A)\cap B)-\mu(A)\mu(B)|<\varepsilon$ for all $A,B\in\mathcal{P}$. This is clearly an open subset of $\mathfrak{A}_{X,G}$. We will argue that it is also dense.

Suppose we are given an $\alpha\in\mathfrak{A}_{X,G}$, a finite measurable partition $\mathcal{P}'$ of X, a compact set $K\subseteq G$, and a $\delta>0$, and let us show that there exists an $\tilde{\alpha}\in W(\mathcal{P},\varepsilon)$ such that $\mu(\tilde{\alpha}_s(A)\Delta\alpha_s(A))<\delta$ for every $A\in\mathcal{P}'$ and $s\in K$. We may assume by refining the partitions if necessary that $\mathcal{P}=\mathcal{P}'$. Let $n\in\mathbb{N}$ be a positive integer power of 2 such that $n>6/\varepsilon$. By Lemma 4.1 there is a weakly H-mixing action $\beta\in\mathfrak{A}_{X,G}$ for which there exists a measurable partition $\{D_1, \dots, D_n\}$ of X with $\mu(D_i) = 1/n$ and $\mu(\beta_s(D_i)\Delta D_i) < \delta/(4n)$ for every $i = 1, \dots, n$ and $s \in K$. By the continuity of α we can find a finite set $F \subseteq G$ such that for every set B of the form $\alpha_s(A)$ for $s \in K$ and $A \in \mathcal{P}$ there is a set $E(B) \in \bigvee_{s\in F} \alpha_s(\mathcal{P})$ for which $\mu(E(B)\Delta B) < \delta/(4n)$.

For each $A \in \mathcal{P}$ define the subset $A^\natural$ of $X \times X^n$ by

$$A^\natural = (D_1 \times A \times X \times \cdots \times X) \cup (D_2 \times X \times A \times X \times \cdots \times X) \cup \\ \cdots \cup (D_n \times X \times \cdots \times X \times A)$$

where in the kth component of this disjoint union the jth factor in X^n is A if $j = k$ and X otherwise. Defining ω as the product measure $\mu \times \mu^n$ on $X \times X^n$ we have $\omega(A^\natural) = \mu(A)$ for all measurable $A \subseteq X$. We can then construct a measure space isomorphism $\varphi : (X, \mu) \to (X \times X^n, \omega)$ which at the algebra level induces the bijective map $\bigvee_{s\in F\cup\{e\}} \alpha_s(\mathcal{P}) \to \bigvee_{s\in F\cup\{e\}}(\mathrm{id} \times \alpha^{(n)})_s(\mathcal{P}^\natural)$ given by $\alpha_s(A) \mapsto (\mathrm{id} \times \alpha^{(n)})_s(A^\natural) = \alpha_s(A)^\natural$, where $\mathcal{P}^\natural$ is the partition $\{A^\natural : A \in \mathcal{P}\}$ of $X \times X^n$ and $\alpha^{(n)}$ denotes the n-fold product $\alpha \times \alpha \times \cdots \times \alpha$. Set $\theta = \beta \times \alpha^{(n)}$ and define the action $\tilde{\alpha} \in \mathfrak{A}_{X,G}$ by $\tilde{\alpha}_s = \varphi^{-1} \circ \theta_s \circ \varphi$ for all $s \in G$.

For $A \in \mathcal{P}$ and $s \in K$ we have

$$\begin{aligned} \mu(\tilde{\alpha}_s(A)\Delta E(\alpha_s(A))) &= \omega(\theta_s(A^\natural)\Delta E(\alpha_s(A))^\natural) \\ &\leq \sum_{i=1}^{n} \big(\mu(\beta_s(D_i)\Delta D_i) + \mu(\alpha_s(A)\Delta E(\alpha_s(A)))\big) \\ &< n\left(\frac{\delta}{4n} + \frac{\delta}{4n}\right) = \frac{\delta}{2} \end{aligned}$$

and hence

$$\mu(\tilde{\alpha}_s(A)\Delta\alpha_s(A)) \leq \mu(\tilde{\alpha}_s(A)\Delta E(\alpha_s(A))) + \mu(E(\alpha_s(A))\Delta\alpha_s(A)) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

so that $\tilde{\alpha} \in V(\alpha, K, \Omega, \delta)$.

Let us show now that $\tilde{\alpha} \in W(\Omega, \varepsilon)$. Since β is weakly H-mixing there exists a $t \in H$ such that

$$\big|\mu(\beta_t(D_i) \cap D_j) - n^{-2}\big| \leq \frac{\varepsilon}{3n^2}$$

for all $i, j = 1, \dots, n$. For $A, B \in \mathcal{P}$ we then have

$$\begin{aligned} |\mu(\tilde{\alpha}_t(A) \cap B) - \mu(A)\mu(B)| &= |\omega(\theta_t(A^\natural) \cap B^\natural) - \mu(A)\mu(B)| \\ &\leq \sum_{\substack{1\leq i,j\leq n \\ i\neq j}} \big|\mu(\beta_t(D_i) \cap D_j)\mu(A)\mu(B) - n^{-2}\mu(A)\mu(B)\big| \\ &\quad + \sum_{1\leq i\leq n} \big|\mu(\beta_t(D_i) \cap D_i)\mu(\alpha_t(A) \cap B) - n^{-2}\mu(A)\mu(B)\big| \\ &\leq (n^2 - n)\frac{\varepsilon}{3n^2} + n\left(\frac{\varepsilon}{3n^2} + \frac{1}{n^2} + \frac{1}{n^2}\right) < \varepsilon. \end{aligned}$$

Therefore $\tilde{\alpha} \in W(\mathcal{P}, \varepsilon)$, and so we conclude that $W(\mathcal{P}, \varepsilon)$ is dense in $\mathfrak{A}_{X,G}$.

Taking now an increasing sequence $\mathcal{P}_1 \leq \mathcal{P}_2 \leq \dots$ of finite measurable partitions of X whose union generates a dense subalgebra of the measure algebra, we can express the set of weakly H-mixing actions in $\mathfrak{A}_{X,G}$ as $\bigcap_{j=1}^{\infty} W(\mathcal{P}_j, 1/j)$, which is a dense G_δ. □

Remark 4.3. In [9] Chou gave examples of nondiscrete noncompact solvable G which have the property that every weakly mixing unitary representation is mixing (see the remark on page 77 of [5]). In this case a generic action in $\mathfrak{A}_{X,G}$ is mixing by Theorem 4.2. Therefore Rokhlin's theorem that mixing fails generically in $\mathfrak{A}_{X,\mathbb{Z}}$ [26] does not extend to amenable G. It seems to be unknown however whether there exist infinite discrete G for which every weakly mixing unitary representation is mixing (see [29]).

5. Bernoulli shifts

In [16] Glasner and Weiss showed for countable discrete G that the space of G-invariant states (i.e., Borel probability measures) for the Bernoulli shift on $\{0,1\}^G$ is a Bauer simplex or the Poulsen simplex depending on whether or not G has property T (they also established the same dichotomy for a suitable Bernoulli-type action in the nondiscrete case). In the Poulsen simplex the extreme points (which are the ergodic states in this case) form a dense G_δ set. In a Bauer simplex the set of extreme points is closed. So the set of ergodic G-invariant states on $\{0,1\}^G$ is closed or a dense G_δ depending in whether or not G has property T. We establish here a weak mixing version of this fact, relativized to pairs as usual. We will also work in a general noncommutative framework.

Given an action of a group G on a unital C^*-algebra A, we write $S_G(A)$ for the set of G-invariant states on A equipped with the relative weak* topology. Ergodicity implies extremality in $S_G(A)$, but the converse need not hold if A is not commutative. However, if the action is asymptotically Abelian in a certain weak sense then extremality is equivalent to ergodicity (see Section 4 of [6]). This is the case for the Bernoulli shifts considered in the theorem below.

The symbol $\otimes$ can be interpreted below as either the minimal or maximal C^*-tensor product, and so we will fix its meaning to be one or the other for the remainder of the section. For a unital C^*-algebra A we write $A^{\otimes I}$ for the tensor product of copies of A indexed over the set I, which for I infinite is defined as the direct limit of the C^*-algebras $A^{\otimes J}$ over finite sets $J \subseteq I$ with respect to the canonical unital embeddings $A^{\otimes J} \hookrightarrow A^{\otimes J'}$ for $J \subseteq J'$ under which $A^{\otimes J}$ is identified with $A^{\otimes J} \otimes \mathbf{1} \subseteq A^{\otimes J} \otimes A^{\otimes J' \setminus J} = A^{\otimes J'}$.

Theorem 5.1. *Let G be a countable discrete group and H a subgroup. Let A be a separable unital C^*-algebra not equal to the complex numbers and let α be the shift action of G on $A^{\otimes G}$. If the pair (G,H) does not have property T then the set of weakly H-mixing states in $S_G(A^{\otimes G})$ is a dense G_δ, while if (G,H) has property T then the set of H-ergodic states in $S_G(A^{\otimes G})$ is nowhere dense. Moreover, if G itself has property T then in $S_G(A^{\otimes G})$ the set of G-ergodic states and the set of weakly G-mixing states are both closed.*

Proof. Suppose first that (G,H) does not have property T. Let $E = \{s_1, \dots, s_k\}$ be a nonempty finite subset of G and let Ω be a nonempty finite set of elementary tensors in the unit ball of $A^{\otimes E}$. Let $\varepsilon > 0$. Denote by $W(\Omega, \varepsilon)$ the set of all $\omega \in S_G(A^{\otimes G})$ such that there exists a $t \in H$ for which $|\omega(b^*\alpha_t(a)) - \omega(b^*)\omega(a)| < \varepsilon$ for all $a, b \in \Omega$. Then $W(\Omega, \varepsilon)$ is evidently open in $S_G(A^{\otimes G})$, and we will show that it is also dense.

Let σ be a state in $S_G(A^{\otimes G})$, Ω' a finite set of elementary tensors in the unit ball of $A^{\otimes E'}$ for some finite set $E' \subseteq G$, and $\varepsilon' > 0$. Since the set of such Ω' is total in $A^{\otimes G}$, the density of $W(\Omega, \varepsilon)$ will follow once we show the existence of a $\sigma' \in W(\Omega, \varepsilon)$ such that $|\sigma'(a) - \sigma(a)| < \varepsilon'$ for all $a \in \Omega'$. Since $W(\Omega, \varepsilon) \supseteq W(\Omega' \cup \Omega, \min(\varepsilon, \varepsilon'))$ we may assume for simplicity that $\Omega' = \Omega$ and $\varepsilon' = \varepsilon$.

Let $n \in \mathbb{N}$ be a power of 2 such that $n > 16/\varepsilon$. By Lemma 4.1 there is a nonatomic standard probability space (X, μ) and a weakly H-mixing measure-preserving action β of G on $L^\infty(X, \mu)$ for which there exists an n-element partition of unity $\mathcal{P} = \{e_1, \dots, e_n\}$ in $L^\infty(X, \mu)$ with $\mu(e_i) = 1/n$ and $\|\beta_{s_j}(e_i) - e_i\|_\mu < \varepsilon(32kn^k)^{-1}$ for every $i = 1, \dots, n$ and $j = 1, \dots, k$.

We construct a *-homomorphism $\Phi : A^{\otimes G} \to L^\infty(X, \mu) \otimes (A^{\otimes G})^{\otimes\{1,\dots,n\}}$ as follows. Let $F = \{t_1, \dots, t_r\}$ be a nonempty finite subset of G and let $a = a_1 \otimes \cdots \otimes a_r$ be an elementary tensor in $A^{\otimes F}$. For a set $I \subseteq \{1, \dots, r\}$ we write a_I to denote the elementary tensor $\bar{a}_1 \otimes \cdots \otimes \bar{a}_r \in A^{\otimes F}$ where $\bar{a}_i = a_i$ if $i \in I$ and $\bar{a}_i = \mathbf{1}$ if $i \notin I$. We then define $\Phi_F(a)$ to be

$$\sum_{\kappa \in \{1,\dots,n\}^{\{1,\dots,r\}}} \beta_{t_1}(e_{\kappa(1)})\beta_{t_2}(e_{\kappa(2)}) \cdots \beta_{t_r}(e_{\kappa(r)}) \otimes a_{\kappa^{-1}(1)} \otimes a_{\kappa^{-1}(2)} \otimes \cdots \otimes a_{\kappa^{-1}(n)}.$$

Since in each of the above pairwise orthogonal summands we have simply redistributed the factors of the elementary tensor a according to a fixed scheme, we obtain an isometric map on the algebraic tensor product (whether we are using the minimal or maximal C^*-tensor norm) and hence a *-homomorphism $\Phi_F : A^{\otimes F} \to L^\infty(X, \mu) \otimes (A^{\otimes G})^{\otimes\{1,\dots,n\}}$. The *-homomorphisms so defined are compatible in the sense that for any finite sets $F_1, F_2 \subseteq G$ the restrictions of Φ_{F_1} and Φ_{F_2} to $A^{\otimes(F_1 \cap F_2)}$ agree. We then define Φ to be the resulting direct limit *-homomorphism. Setting $\theta = \beta \otimes \alpha^{\otimes\{1,\dots,n\}}$, it is readily seen that $\Phi \circ \alpha = \theta \circ \Phi$.

Write ω for the state $\mu \otimes \sigma^{\otimes\{1,\dots,n\}}$ on $L^\infty(X, \mu) \otimes (A^{\otimes G})^{\otimes\{1,\dots,n\}}$. Define σ' to be $\omega \circ \Phi$ and let us show that this element of $S_G(A^{\otimes G})$ has the desired properties.

Set $\Sigma = \{1, \dots, n\}^{\{1,\dots,k\}}$ and denote by Σ_{c} the set of constant functions in Σ. Let $a \in \Omega$. For $\kappa \in \Sigma$ set

$$\hat{a}_\kappa = \beta_{s_1}(e_{\kappa(1)})\beta_{s_2}(e_{\kappa(2)}) \cdots \beta_{s_k}(e_{\kappa(k)}) \otimes a_{\kappa^{-1}(1)} \otimes a_{\kappa^{-1}(2)} \otimes \cdots \otimes a_{\kappa^{-1}(n)}.$$

For $i = 1, \dots, n$ set

$$\hat{a}_i = e_i \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

where a appears in the ith tensor product factor of $(A^{\otimes G})^{\otimes\{1,\dots,n\}}$. Put $\hat{a} = \sum_{i=1}^n \hat{a}_i$. We will show that $\|\Phi(a) - \hat{a}\|_\omega < \varepsilon/8$. Given a $\kappa \in \Sigma \setminus \Sigma_{\mathrm{c}}$ we have $\kappa(i) \neq \kappa(j)$ for some $i, j \in \{1, \dots, k\}$ in which case

$$\begin{aligned}
\|\hat{a}_\kappa\|_\omega &\leq \|\beta_{s_i}(e_{\kappa(i)})\beta_{s_j}(e_{\kappa(j)})\|_\mu \\
&= |\mu(\beta_{s_i}(e_{\kappa(i)})\beta_{s_j}(e_{\kappa(j)}))|^{1/2} \\
&\leq |\mu((\beta_{s_i}(e_{\kappa(i)}) - e_{\kappa(i)})\beta_{s_j}(e_{\kappa(j)}))|^{1/2} + |\mu(e_{\kappa(i)}(\beta_{s_j}(e_{\kappa(j)}) - e_{\kappa(j)}))|^{1/2} \\
&\leq \|\beta_{s_i}(e_{\kappa(i)}) - e_{\kappa(i)}\|_\mu \|e_{\kappa(j)}\| + \|e_{\kappa(i)}\| \|\beta_{s_j}(e_{\kappa(j)}) - e_{\kappa(j)}\|_\mu \\
&< \frac{\varepsilon}{16n^k}.
\end{aligned}$$

For $\kappa \in \Sigma_{\rm c}$ we have

$$\begin{aligned}\|\hat{a}_\kappa - \hat{a}_{\kappa(1)}\|_\omega &= \|\beta_{s_1}(e_{\kappa(1)})\beta_{s_2}(e_{\kappa(1)})\cdots\beta_{s_k}(e_{\kappa(1)}) - e_{\kappa(1)}\|_\mu \|a\|_\sigma \\ &\leq \sum_{i=1}^{k} \|\beta_{s_i}(e_{\kappa(1)}) - e_{\kappa(1)}\|_\mu < \frac{\varepsilon}{16n}.\end{aligned}$$

Since $\Phi(a) = \sum_{\kappa\in\Sigma} \hat{a}_\kappa$ it follows that

$$\|\Phi(a) - \hat{a}\|_\omega \leq \sum_{\kappa\in\Sigma_{\rm c}} \|\hat{a}_\kappa - \hat{a}_{\kappa(1)}\|_\omega + \sum_{\kappa\in\Sigma\setminus\Sigma_{\rm c}} \|\hat{a}_\kappa\|_\omega < n\frac{\varepsilon}{16n} + (n^k - n)\frac{\varepsilon}{16n^k} < \frac{\varepsilon}{8}.$$

Because $|\omega(\Phi(a) - \hat{a})| \leq \|\Phi(a) - \hat{a}\|_\omega$ and $\omega(\hat{a}) = \sigma(a)$ this shows in particular that $|\sigma'(a) - \sigma(a)| < \varepsilon/8 < \varepsilon$.

Now since β is weakly H-mixing there exists a $t \in H$ such that $|\mu(e_j\beta_t(e_i)) - n^{-2}| \leq \varepsilon(4n)^{-2}$ for all $i, j = 1, \ldots, n$. Let $a, b \in \Omega$. Then for $i = 1, \ldots, n$ we have

$$\omega(\hat{b}_i^*\theta_t(\hat{a}_i)) = \mu(e_i\beta_t(e_i))\sigma(b^*\alpha_t(a)) \approx_{\varepsilon(4n)^{-2}} \frac{1}{n^2}\sigma(b^*\alpha_t(a))$$

while for $i, j = 1, \ldots, n$ with $i \neq j$ we have

$$\omega(\hat{b}_j^*\theta_t(\hat{a}_i)) = \mu(e_j\beta_t(e_i))\sigma(b^*)\sigma(a) \approx_{\varepsilon(4n)^{-2}} \frac{1}{n^2}\sigma(b^*)\sigma(a).$$

Therefore

$$\begin{aligned}\omega(\hat{b}^*\theta_t(\hat{a})) &= \sum_{1\leq i\leq n} \omega(\hat{b}_i^*\theta_t(\hat{a}_i)) + \sum_{\substack{1\leq i,j\leq n \\ i\neq j}} \omega(\hat{b}_j^*\theta_t(\hat{a}_i)) \\ &\approx_{\varepsilon/8}\ n\frac{1}{n^2}\sigma(b^*\alpha_t(a)) + (n^2 - n)\frac{1}{n^2}\sigma(b^*)\sigma(a) \\ &= \frac{1}{n}\sigma(b^*\alpha_t(a)) + \frac{n-1}{n}\sigma(b^*)\sigma(a)\end{aligned}$$

so that $|\omega(\hat{b}^*\theta_t(\hat{a})) - \sigma(b^*)\sigma(a)| \leq \varepsilon/8 + 2/n < \varepsilon/4$. Thus since

$$\begin{aligned}|\omega(\Phi(b)^*\theta_t(\Phi(a))) - \omega(\hat{b}^*\theta_t(\hat{a}))| &\leq |\omega(\Phi(b)^*\theta_t(\Phi(a) - \hat{a}))| + |\omega((\Phi(b) - \hat{b})^*\theta_t(\hat{a}))| \\ &\leq \|b\|\|\Phi(a) - \hat{a}\|_\omega + \|\Phi(b) - \hat{b}\|_\omega\|\hat{a}\| \\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}\end{aligned}$$

and $\sigma'(b^*\alpha_t(a)) = \omega(\Phi(b)^*\Phi(\alpha_t(a))) = \omega(\Phi(b)^*\theta_t(\Phi(a)))$ we obtain

$$\begin{aligned}|\sigma'(b^*\alpha_t(a)) - \sigma'(b^*)\sigma'(a)| &\leq |\omega(\Phi(b)^*\theta_t(\Phi(a))) - \omega(\hat{b}^*\theta_t(\hat{a}))| + |\omega(\hat{b}^*\theta_t(\hat{a})) - \sigma(b^*)\sigma(a)| \\ &\quad + |(\sigma(b^*) - \sigma'(b^*))\sigma(a)| + |\sigma'(b^*)(\sigma(a) - \sigma'(a))| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Consequently $\sigma' \in W(\Omega, \varepsilon)$, and we conclude that $W(\Omega, \varepsilon)$ is dense in $S_G(A^{\otimes G})$.

Let $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$ be an increasing sequence of nonempty finite subsets of $A^{\otimes G}$ such that for each $j \geq 1$ the elements of Ω_j are elementary tensors in the unit ball of $A^{\otimes E}$ for some finite set $E \subseteq G$ and $\bigcup_{j=1}^{\infty} \Omega_j$ is total in $A^{\otimes G}$. Then the set of a weakly mixing states in $S_G(A^{\otimes G})$ is equal to $\bigcap_{j=1}^{\infty} W(\Omega_j, 1/j)$, which is a dense G_δ in view of what we proved above.

Now suppose that (G, H) has property T. Then there exists a nonempty finite set $F \subseteq G$ and an $\varepsilon > 0$ such that every unitary representation of G possessing an (F, ε)-invariant vector has a nonzero H-invariant vector. Let σ be an element of $S_G(A^{\otimes G})$ which is not G-ergodic. Then the representation $\pi_{\sigma,0}$ admits a nonzero invariant vector, and so given a $\delta > 0$ we can find an $a \in A$ such that $|\sigma(a)| < \delta$, $\sigma(a^*a) > 1 - \delta$, and $\sigma((\alpha_s(a) - a)^*(\alpha_s(a) - a)) < \delta$ for all $s \in F$. These inequalities also clearly hold with σ replaced by any state in some neighbourhood N of σ in $S_G(A^{\otimes G})$. It follows that if δ is small enough then for each $\omega \in N$ the orthogonal projection of the vector $\pi_\omega(a)\xi_\omega \in \mathcal{H}_\omega$ onto $\mathcal{H}_{\omega,0}$ is (F, ε)-invariant (where $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ is the GNS triple for ω and $\mathcal{H}_{\omega,0}$ is the orthogonal complement of $\mathbb{C}\xi_\omega$ in $\mathcal{H}_\omega$), which yields the existence of a nonzero H-invariant vector in $\mathcal{H}_{\omega,0}$, so that ω is not H-ergodic. Since $A \neq \mathbb{C}$ the set $S_G(A^{\otimes G})$ contains more than one product state and thus has cardinality greater than one, and so the states in $S_G(A^{\otimes G})$ that fail to be G-ergodic, being precisely those that fail to be extremal ([6], Theorem 4.3.17), form a dense subset, from which we conclude that the set of H-ergodic states is nowhere dense. We also see that if G itself has property T then the set of G-ergodic states in $S_G(A^{\otimes G})$ is closed. In this case the weak* continuity of the map $\sigma \mapsto \sigma \otimes \sigma$ from $S_G(A^{\otimes G})$ to $S_G(A^{\otimes G} \otimes A^{\otimes G})$ (with G acting as $\alpha \otimes \alpha$ on $A^{\otimes G} \otimes A^{\otimes G} \cong (A \otimes A)^{\otimes G}$) shows that the set of weakly G-mixing states in $S_G(A^{\otimes G})$ is also closed. □

6. Actions on the CAR algebra

Let A be the CAR algebra, i.e., the UHF algebra $M_2^{\otimes \mathbb{N}}$. Let G be a second countable locally compact group. We write $\mathfrak{A}_{A,G}$ for the Polish space of continuous actions of G on A whose topology has as a basis the sets

$$V(\alpha, K, \Omega, \varepsilon) = \{\beta \in \mathfrak{A}_{A,G} : \|\beta_s(a) - \alpha_s(a)\| < \varepsilon \text{ for all } s \in K \text{ and } a \in \Omega\}$$

where $\alpha \in \mathfrak{A}_{A,G}$, K is a compact subset of G, Ω is a finite subset of A, and $\varepsilon > 0$.

The following is the topological analogue of Definition 3.1.

Definition 6.1. Let B be a C^*-algebra and α an action of G on B. We say that α is G-*Abelian* if for every finite set $\Omega \subseteq A$ and $\varepsilon > 0$ there is an $s \in G$ such that $\|[\alpha_s(a), b]\| < \varepsilon$ for all $a, b \in \Omega$.

By Proposition 9.2 of [24], α is G-Abelian if and only if there is a net $\{s_\gamma\}_\gamma$ in G (which can be taken to be a sequence if B is separable) such that $\lim_\gamma \|[\alpha_{s_\gamma}(a), b]\| = 0$ for all $a, b \in B$. This is the type of norm asymptotic Abelianness that is discussed on pages 401–403 of [6]. In the unital case it implies that the set of invariant states is a simplex ([6], Corollary 4.3.11). Theorem 9.6 of [24] shows that if B is a simple nuclear C^*-algebra (e.g., the CAR algebra) then G-Abelianness is equivalent to an asymptotic tensor product independence property.

Lemma 6.2. *Suppose that G is discrete and admits a nontorsion Abelian quotient. Let F be a finite subset of G and let $\varepsilon > 0$. Then there is an action $\beta : G \to \mathrm{Aut}(M_2)$, a minimal projection $p \in M_2$, and a $t \in G$ such that $\|\beta_s(p) - p\| < \varepsilon$ for all $s \in F$ and $\|p\beta_t(p)\| < \varepsilon$.*

Proof. We may assume that G itself is Abelian and contains a nontorsion element t. By enlarging F we may assume that it contains t. Since a character on a subgroup of G can be extended to a character on the whole group, we can find a character $\gamma \in \widehat{G}$ such that $\gamma(t)$ is of infinite order in $\mathbb{T}$. By Kronecker's theorem we can find an $n \in \mathbb{N}$ such that $|\gamma(s)^n - 1| < \varepsilon/2$ for all $s \in F$ and then an $m \in \mathbb{N}$ such that $|\gamma(t)^{nm} + 1| < \varepsilon$. Let $\pi : G \to \mathcal{U}(M_2)$ be the homomorphism given by $\pi(s) = \mathrm{diag}(\gamma(s)^n, 1)$ for all $s \in G$ and define the action $\beta : G \to \mathrm{Aut}(M_2)$ by $\beta_s = \mathrm{Ad}\,\pi(s)$ for all $s \in G$. Letting p be the minimal projection $\frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ in M_2 we have $\|\beta_s(p) - p\| \leq 2\|\pi(s) - 1\| < \varepsilon$ for all $s \in F$ while

$$\|p\beta_{t^m}(p)\| = \big\|p\big(\beta_{t^m}(p) - 2^{-1}\left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]\big)\big\| = 2^{-1}\Big\|\Big[\begin{smallmatrix} 0 & \gamma(t)^{mn}+1 \\ \gamma(t)^{-mn}+1 & 0 \end{smallmatrix}\Big]\Big\| < \varepsilon,$$

as desired. ∎

Theorem 6.3. *Suppose that G is discrete and admits a nontorsion Abelian quotient. Then the set of G-Abelian actions in $\mathfrak{A}_{A,G}$ is a dense G_δ.*

Proof. Let Ω be a nonempty finite subset of the unit ball of A and let $\varepsilon > 0$. Denote by $W(\Omega, \varepsilon)$ the set of all actions α in $\mathfrak{A}_{A,G}$ such that there exists a $t \in G$ for which $\|[\alpha_t(a), b]\| < \varepsilon$ for all $a, b \in \Omega$. Then $W(\Omega, \varepsilon)$ is clearly open in $\mathfrak{A}_{A,G}$. To show that it is also dense, suppose we are given an $\alpha \in \mathfrak{A}_{A,G}$, a nonempty finite subset Ω' of the unit ball of A, a finite set $F \subseteq G$, and an $\delta > 0$, and let us demonstrate that there exists an $\tilde{\alpha} \in W(\Omega, \varepsilon)$ such that $\|\tilde{\alpha}_s(a) - \alpha_s(a)\| < \delta$ for every $a \in \Omega'$ and $s \in F$. Since $W(\Omega, \varepsilon) \supseteq W(\Omega' \cup \Omega, \varepsilon)$ we may assume for simplicity that $\Omega = \Omega'$. By Lemma 6.2 we can find an action $\beta : G \to \mathrm{Aut}(M_2)$, a minimal projection $p \in M_2$, and a $t \in G$ such that $\|\beta_s(p) - p\| < \delta/12$ for all $s \in F$ and $\|p\beta_t(p)\| < \varepsilon/24$. Set $p_1 = p$ and $p_2 = 1 - p$.

Take a simple finite-dimensional unital C^*-subalgebra $N \subseteq A$ such that for each $a \in \bigcup_{s \in F \cup \{e\}} \alpha_s(\Omega)$ there is an element $E(a)$ in the unit ball of N for which $\|a - E(a)\| < \delta/24$. Define a map φ from N to the (in this case unique) C^*-tensor product $M_2 \otimes A \otimes A$ by $\varphi(a) = p_1 \otimes a \otimes \mathbf{1} + p_2 \otimes \mathbf{1} \otimes a$ for all $a \in N$. Then φ is an injective *-homomorphism. Since $M_2 \otimes A \otimes A$ is *-isomorphic to A and any two simple unital C^*-subalgebras of A of the same finite dimension are unitarily equivalent by classification theory (see for example Chapter 1 of [27]), we can extend φ to a *-isomorphism $\Phi : A \to M_2 \otimes A \otimes A$. Set $\theta = \beta \otimes \alpha \otimes \alpha$ and define the action $\tilde{\alpha} \in \mathfrak{A}_{A,G}$ by $\tilde{\alpha}_s = \Phi^{-1} \circ \theta_s \circ \Phi$ for all $s \in G$.

Using the fact that $\|\beta_s(p_i) - p_i\| < \delta/12$ for all $s \in F$ and $i = 1, 2$, we estimate as in the proof of Theorem 3.7 to obtain $\|\tilde{\alpha}_s(a) - \alpha_s(a)\| < \delta$ for all $a \in \Omega$ and $s \in F$. Moreover, noting that

$$\|[\beta_t(p_2), p_1]\| = \|[\beta_t(p_1), p_2]\| = \|[\beta_t(p_1), p_1]\| \leq 2\|p_1\beta_t(p_1)\| < \frac{\varepsilon}{12},$$

for all $a, b \in \Omega$ we have

$$\begin{aligned} \|[\tilde{\alpha}_t(E(a)), E(b)]\| &= \|[\beta_t \otimes \alpha \otimes \alpha(\Phi(E(a))), \Phi(E(b))]\| \\ &\leq \|[\beta_t(p_1), p_2] \otimes \alpha_t(E(a)) \otimes E(b)\| + \|[\beta_t(p_2), p_1] \otimes E(b) \otimes \alpha_t(E(a))\| \\ &\qquad + \|[\beta_t(p_1) \otimes \alpha_t(E(a)) \otimes \mathbf{1}, p_1 \otimes E(b) \otimes \mathbf{1}]\| \\ &\qquad + \|[\beta_t(p_2) \otimes \mathbf{1} \otimes \alpha_t(E(a)), p_2 \otimes \mathbf{1} \otimes E(b)]\| \\ &< \frac{\varepsilon}{12} + \frac{\varepsilon}{12} + 2\|p_1\beta(p_1)\| + 2\|p_2\beta(p_2)\| < \frac{\varepsilon}{3} \end{aligned}$$

and hence

$$\|[\tilde{\alpha}_t(a), b]\| \leq \|[\tilde{\alpha}_t(E(a)), E(b)]\| + 2\|b - E(b)\| + 2\|a - E(a)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus $\tilde{\alpha} \in W(\Omega, \varepsilon)$, and so we conclude that $W(\Omega, \varepsilon)$ is dense in $\mathfrak{A}_{A,G}$.

Now take an increasing sequence $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ of nonempty finite subsets of A whose union has dense linear span in A. Then the set of G-Abelian actions in $\mathfrak{A}_{A,G}$ is equal to $\bigcap_{j=1}^{\infty} W(\Omega_j, 1/j)$, which is a dense G_δ. □

Theorem 6.3 for $G = \mathbb{Z}$ was established in [24] using some facts about the Rokhlin property. In addition to enlarging the class of groups, we have given here a more elementary argument.

We remark finally that an argument similar to that for Lemma 3.5 shows that, for every second countable locally compact G, the elements of $\mathfrak{A}_{A,G}$ with dense orbit under the conjugation action of the *-automorphism group $\mathrm{Aut}(A)$ form a dense G_δ subset.

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