SUBSHIFTS AND PERFORATION

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ABSTRACT. We demonstrate that the perforative phenomena shown to occur among simple amenable C^*-algebras by Villadsen and Toms can be realized within a dynamical framework. More specifically, we construct a minimal homeomorphism for which the K_0 group of the crossed product fails to be weakly unperforated, and a minimal homeomorphism for which the crossed product has the same Elliott invariant as an A$\mathbb{T}$-algebra but has Cuntz semigroup which fails to be almost unperforated.

1. Introduction

Topological dynamics has long been an important source of examples and motivation for the classification theory of amenable C^*-algebras [21, 8]. The circle algebra decompositions of Putnam for crossed products of minimal $\mathbb{Z}$-systems on the Cantor set [20] and of Elliott and Evans for irrational rotation algebras [7] provided great impetus for the development of classification theory in the 1990s. Giordano, Putnam, and Skau's K-theoretic study of minimal $\mathbb{Z}$-systems on the Cantor set showed that the crossed products of two such systems are isomorphic precisely when the systems are strong orbit equivalent [9]. For crossed products of minimal diffeomorphisms of compact smooth manifolds, Q. Lin and Phillips obtained a direct limit decomposition into recursive subhomogeneous C^*-algebras with no dimension growth [16, 17]. H. Lin and Phillips proved in [15] that, for a minimal homeomorphism of an infinite compact metrizable space X with finite covering dimension such that the image of $K_0(C(X) \rtimes \mathbb{Z})$ in the space of affine functions on the tracial state space of $C(X) \rtimes \mathbb{Z}$ is dense, the crossed product $C(X) \rtimes \mathbb{Z}$ has tracial rank zero and hence is amenable to K-theoretic classification and is a simple AH algebra with no dimension growth and real rank zero.

As reflected by these connections to dynamics in the stably finite case, Elliott's classification program for amenable C^*-algebras in its K-theoretic formulation has enjoyed some spectacular successes. Over the last several years, however, the classification program has had to come to terms with examples of Villadsen [29, 30], Rørdam [22], and Toms [26] that have pointed toward the need for regularity assumptions like $\mathcal{Z}$-stability or the use of invariants of a finer type [8]. Villadsen's pioneering work [29] showed that the perforation that can be observed in the ordered K^0 group of certain manifolds like $\mathbb{T}^4$ can be built into a simple AH algebra by propagating an Euler class obstruction across the building blocks. A necessary condition for this obstruction to survive along the entire sequence is a nonzero lower bound on the asymptotic ratio of the dimension of the base space to the matrix size. As a consequence of the perforation in K_0, Villadsen's C^*-algebra fails to

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be $\mathcal{Z}$-stable [10]. Using the Euler class argument of Villadsen, Toms constructed in [26] a simple AH algebra A whose tensor product with the universal UHF algebra is a simple AI algebra that has the same Elliott invariant (ordered K-theory paired with traces) as A but is not isomorphic to A. The distinguishing feature of Toms's example is the failure of its Cuntz semigroup to be almost unperforated, which means that, like Villadsen's example, it lies outside the class of $\mathcal{Z}$-stable C^*-algebras [23].

In light of these examples and the above results involving topological dynamics, one might ask whether all crossed products of minimal $\mathbb{Z}$-systems exhibit sufficiently regular behaviour to fall within the scope of classification theorems based on the Elliott invariant. The aim of this note is to demonstrate that the noncommutative perforative effects of Euler class obstructions in the work of Villadsen and Toms can be recast in a dynamical framework. More precisely, using a recursive blocking procedure as in the proof of Proposition 3.5 in [18], we first construct in Section 2 a minimal subshift (X, T) which gives rise to a simple crossed product $C(X) \rtimes \mathbb{Z}$ whose Cuntz semigroup is not almost unperforated such that

$$(K_0(C(X) \rtimes \mathbb{Z}), K_0(C(X) \rtimes \mathbb{Z})^+, [1]) \cong (\mathbb{Q}, \mathbb{Q}^+, 1), \qquad K_1(C(X) \rtimes \mathbb{Z}) \cong \mathbb{Z}.$$

It follows by [28] that there is a simple A$\mathbb{T}$-algebra with the same Elliott invariant as $C(X) \rtimes \mathbb{Z}$ but not isomorphic to $C(X) \rtimes \mathbb{Z}$. We in fact show that one can produce such crossed products with arbitrarily large radius of comparison [27]. We don't know however whether the radius of comparison is ever finite within this class of systems. Secondly, in parallel with Villadsen's example from [29], we construct in Section 3 a minimal subshift (X, T) for which $K_0(C(X) \rtimes \mathbb{Z})$ is not weakly unperforated. Like the C^*-algebras of Villadsen and Toms, all of these crossed products fail to be $\mathcal{Z}$-stable on account of perforation.

The basic principle behind our approach is the following. Let Y be a compact Hausdorff space and consider the shift action on $Y^{\mathbb{Z}}$. Since the translates of a single factor Y are independent from each other (in any of the various possible senses), the algebraic effects of any topological phenomenon residing within one of the factors might be expected to survive upon embedding $C(Y^{\mathbb{Z}})$ into the crossed product. Thus if $K^0(Y)$ has perforation on account of an Euler class obstruction we might expect that $K_0(C(Y^{\mathbb{Z}}) \rtimes \mathbb{Z})$ does as well. However the shift on $Y^{\mathbb{Z}}$ is far from being minimal, and so if we desire a simple crossed product exhibiting perforation we should look for a subshift that is "small" enough to be minimal but "large" enough to sustain the Euler class obstruction, which can easily be destroyed by noncommutativity. Largeness in this case will mean that, in the finite blocks which will be used to recursively define the subshift, the factor Y will appear with asymptotically nonzero density, in analogy with the asymptotically nonzero dimension-rank ratio in the examples of Villadsen and Toms. This nonzero density is reflected in nonzero values of mean dimension, which is an entropy-like invariant for dynamical systems that provides a measure of dimension growth (see [18]).

Since the full and reduced crossed products coincide for $\mathbb{Z}$-actions on compact metrizable spaces, our crossed product notation will be tagless throughout.

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2. Subshifts and perforation in the Cuntz semigroup

Let Y be a compact metrizable space which contains more than one point. Denote by I the closed unit interval. Write T for the shift $(x_k)_k \mapsto (x_{k+1})_k$ on $(Y \times I)^{\mathbb{Z}}$ with the product topology. Let ρ be a compatible metric on $Y \times I$ such that $Y \times I$ has ρ-diameter at most one and $\rho((y,z),(y,z')) \leq |z - z'|$ for all $y \in Y$ and $z, z' \in I$. Setting $d(x,w) = \sum_{k\in\mathbb{Z}} 2^{-|k|}\rho(x_k, w_k)$ for all $x = (x_k)_k$ and $w = (w_k)_k$ in $(Y \times I)^{\mathbb{Z}}$ we obtain a compatible metric on $(Y \times I)^{\mathbb{Z}}$.

We will construct our minimal subsystem of $((Y \times I)^{\mathbb{Z}}, T)$ by a recursive blocking procedure of the type used in the proof of Proposition 3.5 in [18]. By a *block* we mean a subset of $(Y \times I)^l$ for some positive integer l that has the form $D_1 \times \cdots \times D_l$ for some closed subsets $D_1, \dots, D_l$ of $Y \times I$. For a block $B \subseteq (Y \times I)^l$ and an $i \in \{1, \dots, l\}$ we write $X_{B,i}$ for the set of all $(x_k)_k \in (Y \times I)^{\mathbb{Z}}$ such that $(x_{i+sl}, x_{i+sl+1}, \dots, x_{i+sl+l-1}) \in B$ for every $s \in \mathbb{Z}$. Thus $X_{B,i}$ represents the set of points which can be blocked off by B with a certain phase as described by i. Note that T cyclically permutes the sets $X_{B,i}$, with $X_{B,i} = T^{-i+j+sl}X_{B,j}$ for all $i, j = 1, \dots, l$ and $s \in \mathbb{Z}$. We define X_B to be the closed T-invariant subset $\bigcup_{i=1}^{l} X_{B,i}$ of $(Y \times I)^{\mathbb{Z}}$. In general the sets $X_{B,1}, \dots, X_{B,l}$ need not be pairwise disjoint (i.e., it might be possible to block off a sequence by B in more than one way), but here we will want to arrange for this to be the case in order for the K_0 group of the crossed product to be isomorphic to the rationals. This is the reason for the second factor in $Y \times I$, which will serve as a spacing device.

Let ω be the continuous injection $B \to X_B$ defined by specifying the $sl + i$ coordinate of the image of a point $(x_1, \dots, x_l) \in B$ to be equal to x_i for $s \in \mathbb{Z}$ and $i = 1, \dots, l$. Note that the image of ω is the set of l-periodic sequences in $X_{B,1}$. Writing $\{e_{i,j}\}_{i,j}$ for matrix units and u for the canonical unitary in $C(X_B) \rtimes \mathbb{Z}$ implementing the action via $ufu^* = f \circ T^{-1}$ for $f \in C(X_B)$, by the universal property of the full crossed product there exists a *-homomorphism $\varphi : C(X_B) \rtimes \mathbb{Z} \to M_l \otimes C(B)$ such that $\varphi(f) = \sum_{i=1}^{l} e_{i,i} \otimes (f \circ T^{1-i} \circ \omega)$ for $f \in C(X_B)$ and $\varphi(u) = e_{l,1} + \sum_{i=1}^{l-1} e_{i,i+1}$, as it is easily checked with these prescriptions that $\varphi(u)\varphi(f)\varphi(u)^* = \varphi(f \circ T^{-1})$.

Let $0 < d < 1$. We will recursively construct a decreasing sequence $X_1 \supseteq X_2 \supseteq \dots$ of closed T-invariant subsets of $(Y \times I)^{\mathbb{Z}}$ such that for each $n \in \mathbb{N}$ the set X_n is defined as X_{B_n} for a block $B_n \subseteq (Y \times I)^{l_n}$ of a certain length l_n and of the form $(Y_{n,1} \times I_{n,1}) \times \cdots \times (Y_{n,l_n} \times I_{n,l_n})$ where l_n divides l_{n+1} and

1. for all $x, w \in X_n$ there is a $k \in \mathbb{Z}$ such that $d(T^k x, w) \leq 2^{-n+3}$,
2. $I_{n,1}, \dots, I_{n,l_n}$ are pairwise disjoint closed subintervals of I each with nonempty interior and length at most 2^{-n-2},
3. $Y_{n,i} = Y$ for all i in a subset of $\{1, \dots, l_n\}$ of cardinality greater than dl_n, and $Y_{n,i}$ is a singleton for all other i, and
4. n divides l_n.

Thus d is a lower bound on the density of the appearance of Y in the first components of the factors of a block. The block B_{n+1} will be constructed as a subset of $B_n^{l_{n+1}/l_n}$ by taking a large number of copies of B_n, trimming these by shrinking the subintervals in the second factor at each coordinate, and then forming the product of the resulting blocks together with a bunch of sets of the form $\{y\} \times J$ where J is a small subinterval of I. The presence of the latter sets with a singleton in the first factor is necessary to arrange condition (1), which will guarantee minimality on the intersection of the X_n. For brevity we will write $X_{n,i}$ for $X_{B_n,i}$. Condition (2) implies that the sets $X_{n,1}, \dots, X_{n,l_n}$ are pairwise disjoint. In conjunction with condition (4) this will have the consequence that the system (X, T) will act like the universal odometer at the level of K-theory, as we will demonstrate after describing the construction.

We start with $l_1 = 1$, $I_{1,1} = I$, $B_1 = Y \times I$, and $X_1 = (Y \times I)^{\mathbb{Z}}$. Suppose then that we have constructed l_n and $B_n = (Y_{n,1} \times I_{n,1}) \times \cdots \times (Y_{n,l_n} \times I_{n,l_n})$ such that (1), (2), (3), and (4) are satisfied, with X_n defined as X_{B_n}. Take an $\tilde{x} = (\tilde{x}_k)_k \in X_{n,1}$ which contains as a substring the concatenation of a finite collection of $3l_n$-tuples in $B_n \times B_n \times B_n$ that is sufficiently dense to ensure the existence of an integer $b \geq 2$ such that for all $w = (w_k)_k \in X_n$ there are an $s \in \{1, \dots, b-2\}$ and a $j \in \{1, \dots, l_n\}$ for which $\rho(\tilde{x}_{sl_n+j+k}, w_k) \leq 2^{-n-3}$ for all k in the interval $E_n = \{-l_n, -l_n + 1, \dots, l_n\}$. Writing $\tilde{x}_k = (\tilde{y}_k, \tilde{z}_k) \in Y \times I$ for each $k \in \mathbb{Z}$, we may assume by perturbing as necessary that $\tilde{z}_k \neq \tilde{z}_{k'}$ when $k \neq k'$, since each $I_{n,i}$ contains no isolated points. We will use $\tilde{x}$ to define singleton factors in the construction of B_{n+1} so as to arrange the approximate density of all orbits in X_{n+1}.

Let a be a positive integer whose size will be specified in a moment. Since the intervals $I_{n,1}, \dots, I_{n,l_n}$ are pairwise disjoint and the coordinates of $\tilde{x}$ are all distinct, we can find pairwise disjoint closed subintervals $I_{n+1,1}, \dots, I_{n+1,l_{n+1}}$ of I each with nonempty interior and length at most 2^{-n-3} such that $I_{n+1,sl_n+i} \subseteq I_{n,i}$ for all $s = 0, \dots, a+b-1$ and $i = 1, \dots, l_n$ and $\tilde{z}_{sl_n+i} \in I_{n+1,(a+s)l_n+i}$ for all $s = 0, \dots, b-1$ and $i = 1, \dots, l_n$. Set $Y_{n+1,sl_n+i} = Y_{n,i}$ for $s = 0, \dots, a-1$ and $i = 1, \dots, l_n$, and $Y_{n+1,(a+s)l_n+i} = \{\tilde{y}_{sl_n+i}\}$ for $s = 0, \dots, b-1$ and $i = 1, \dots, l_n$, and then put $l_{n+1} = (a+b)l_n$, $B_{n+1} = (Y_{n+1,1} \times I_{n+1,1}) \times \cdots \times (Y_{n+1,l_{n+1}} \times I_{n+1,l_{n+1}})$, and $X_{n+1} = X_{B_{n+1}}$. Now we observe that by choosing a large enough relative to b we can ensure that condition (3) holds for $Y_{n+1,1}, \dots, Y_{n+1,l_{n+1}}$ given that it holds for $Y_{n,1}, \dots, Y_{n,l_n}$. Moreover, to ensure that condition (4) holds we may increase a so that $n+1$ will divide $a+b$.

It remains to verify condition (1). Let $x = (x_k)_k$ and $w = (w_k)_k$ be elements of X_{n+1}. Since w is contained in X_n there exist an $s \in \{1, \dots, b-2\}$ and a $j \in \{1, \dots, l_n\}$ for which $\rho(\tilde{x}_{sl_n+j+k}, w_k) \leq 2^{-n-3}$ for every $k \in E_n$. Since x lies in one of the sets $X_{n+1,1}, \dots, X_{n+1,l_{n+1}}$ there is an integer m such that $x_{m+sl_n+j+k} \in Y_{n+1,(a+s)l_n+j+k} \times I_{n+1,(a+s)l_n+j+k}$ for all $k \in E_n$. For $k \in E_n$ the set $Y_{n+1,(a+s)l_n+j+k}$ is equal to the singleton $\{\tilde{y}_{sl_n+j+k}\}$, and so by our choice of ρ the distance $\rho(x_{m+sl_n+j+k}, \tilde{x}_{sl_n+j+k})$ is at most the length of the interval $I_{n+1,(a+s)l_n+j+k}$, which is bounded above by 2^{-n-3}. Thus, since $l_n \geq n$ and $Y \times I$ has ρ-diameter at most one, we have

$$d(T^{m+sl_n+j}x, w) \leq \sum_{k \in E_n} 2^{-|k|} \big(\rho(x_{m+sl_n+j+k}, \tilde{x}_{sl_n+j+k}) + \rho(\tilde{x}_{sl_n+j+k}, w_k) \big) + \sum_{k \in \mathbb{Z} \setminus E_n} 2^{-|k|}$$

$$\leq 3 \cdot 2^{-n-2} + 2 \cdot 2^{-l_n} \leq 2^{-n+2},$$

as desired. This completes the recursive construction.

Set $X = \bigcap_{n=1}^{\infty} X_n$, which is a closed T-invariant subset of $(Y \times I)^{\mathbb{Z}}$. The restriction of T to X will again be denoted by T. It follows from condition (1) that the system (X,T) is minimal. It is also free, since it is clear from the description of the X_n in terms of blocks that X contains no periodic points. It follows that $C(X) \rtimes \mathbb{Z}$ is simple (see [6, Thm. VIII.3.9]).

Suppose now that Y is contractible. Then for a given $n \in \mathbb{N}$ the sets $X_{n,1}, \dots, X_{n,l_n}$ are contractible, and since they are pairwise disjoint by (2) it follows that $K^1(X_n) = 0$ and

$$K^0(X_n) \cong K^0(X_{n,1}) \oplus \cdots \oplus K^0(X_{n,l_n}) \cong \mathbb{Z}^{l_n}.$$

Assuming these isomorphisms to be the canonical ones (so that the unit in $C(X_{n,i})$ is associated to 1 in the ith factor of $\mathbb{Z}^{l_n}$), the map $K^0(X_n) \to K^0(X_{n+1})$ arising from the inclusion $X_{n+1} \subseteq X_n$ can be described by $k \mapsto (k, \dots, k) \in (\mathbb{Z}^{l_n})^{l_{n+1}/l_n} \cong \mathbb{Z}^{l_{n+1}}$, and we have

$$K^0(X) \cong \varinjlim K^0(X_n) \cong \varinjlim \mathbb{Z}^{l_n}.$$

Let $\iota : C(X) \hookrightarrow C(X) \rtimes \mathbb{Z}$ be the canonical embedding. We write α for the *-automorphism of $C(X)$ given by $\alpha(f) = f \circ T^{-1}$. The Pimsner-Voiculescu exact sequence for α reads

$$\begin{tikzcd}
K^*(X) \arrow[rr, "\mathrm{id}-\alpha_*"] & & K^*(X) \arrow[dl, "\iota_*"] \\
& K_*(C(X) \rtimes \mathbb{Z}) \arrow[ul] &
\end{tikzcd}$$

Since $K^1(X) = \varinjlim K^1(X_n) = 0$, we have the exact sequence

$$0 \longrightarrow K_1(C(X) \rtimes \mathbb{Z}) \longrightarrow K^0(X) \xrightarrow{\mathrm{id}-\alpha_*} K^0(X) \xrightarrow{\iota_*} K_0(C(X) \rtimes \mathbb{Z}) \longrightarrow 0.$$

Notice that, since for each n the sets $X_{n,1}, \dots, X_{n,l_n}$ are pairwise disjoint and are cyclically permuted by T and n divides l_n, the system (X,T) is an extension of the universal odometer, which can be viewed as addition of $(1, 0, 0, \dots)$ with carry over to the right on the sequence space $W = \prod_{n=1}^{\infty} \{1, \dots, l_{n+1}/l_n\}$ with the product topology (see Section VIII.4 of [6]). Moreover, since for each n the sets $X_{n,1}, \dots, X_{n,l_n}$ are contractible, the above exact sequence is identical to that associated to the universal odometer and hence of the type arising in Giordano, Putnam, and Skau's K-theoretic classification of minimal Cantor systems up to strong orbit equivalence [9]. The map α_* acts on $K^0(X) \cong \varinjlim \mathbb{Z}^{l_n}$ by cyclically permuting the summands in each $\mathbb{Z}^{l_n}$ and the kernel of $\mathrm{id} - \alpha_*$ is isomorphic to $\mathbb{Z}$.

To compute the K_0 group of the crossed product of the universal odometer system (W, S), one uses the identification of $K^0(W)$ with the additive group $C(W, \mathbb{Z})$ of $\mathbb{Z}$-valued continuous functions on W to obtain

$$K_0(C(W) \rtimes \mathbb{Z}) \cong C(W, \mathbb{Z}) / \{f - f \circ S^{-1} : f \in C(W, \mathbb{Z})\}$$

by the Pimsner-Voiculescu sequence. By unique ergodicity one shows that the equivalence class of a function in $C(W,\mathbb{Z})$ is determined by the value of f on the unique S-invariant state μ, which arises as the product of uniform probability measures on the factors $\{1,\dots,l_{n+1}/l_n\}$ (see Section 2 of [9]). Since $\mu(C(W,\mathbb{Z}))=\mathbb{Q}$, it follows that the K_0 groups of the crossed products of the universal odometer and of the system (X,T) can both be identified with $\mathbb{Q}$, with the set of projections in matrix algebras over the crossed product in each case having image $\mathbb{Q}^+$ in K_0. We thus have

$$(K_0(C(X)\rtimes\mathbb{Z}),K_0(C(X)\rtimes\mathbb{Z})^+,[1])\cong(\mathbb{Q},\mathbb{Q}^+,1),\qquad K_1(C(X)\rtimes\mathbb{Z})\cong\mathbb{Z}.$$

We point out that by dropping condition (4) in the construction of (X,T) and alternatively arranging for the numbers l_n to have suitable prime factorizations, we could have actually produced any given odometer as a factor and hence any triple of the form $(G,G^+,1)$ for a dense subgroup $G\subseteq\mathbb{Q}$ as the K_0 data of the crossed product.

We will show at the end of the section that if Y is infinite then the tracial state space of $C(X)\rtimes\mathbb{Z}$ has infinitely many extreme points and consequently $C(X)\rtimes\mathbb{Z}$ does not have real rank zero. Note in contrast that the crossed product of the universal odometer has a unique tracial state.

Adapting the argument from [26], we now proceed to show that the Cuntz semigroup of $C(X)\rtimes\mathbb{Z}$ is not almost unperforated for certain Y. In fact we will obtain nonzero lower bounds on the radius of comparison of $C(X)\rtimes\mathbb{Z}$ when Y has the form I^{3q} with $q\in\mathbb{N}$. By Corollary 4.6 of [23], in the case of simple unital stably finite exact C^*-algebras, which includes all of our crossed products, the Cuntz semigroup can only be almost unperforated if the radius of comparison is zero.

The following lemma is well known but we have not been able to find a reference. We thank Hanfeng Li for indicating the injectivity argument in the proof.

Lemma 2.1. *Let (X_1,T) be a topological dynamical system and let $X_2\supseteq X_3\supseteq\dots$ be closed T-invariant subsets of X_1. Set $X=\bigcap_{n=1}^\infty X_n$. Let*

$$C(X_1)\rtimes\mathbb{Z}\xrightarrow{\varphi_1}C(X_2)\rtimes\mathbb{Z}\xrightarrow{\varphi_2}C(X_3)\rtimes\mathbb{Z}\xrightarrow{\varphi_3}\dots$$

be the inductive system with connecting maps induced from the quotients $C(X_n)\to C(X_{n+1})$ via the universal property of the full crossed product, and let $\gamma:\varinjlim C(X_n)\rtimes\mathbb{Z}\to C(X)\rtimes\mathbb{Z}$ be the map arising from the maps $\gamma_n:C(X_n)\rtimes\mathbb{Z}\to C(X)\rtimes\mathbb{Z}$ induced by the universal property of the full crossed product from the quotients $C(X_n)\to C(X)$. Then γ is a *-isomorphism.

Proof. Write u for the canonical unitary in $C(X)\rtimes\mathbb{Z}$ and u_n for the canonical unitary in $C(X_n)\rtimes\mathbb{Z}$. Given a finite sum $\sum_{k\in I}f_ku^k\in C(X)\rtimes\mathbb{Z}$ where each f_k is a function in $C(X)$, for every $k\in I$ we can find by Tietze's extension theorem a $g_k\in C(X_1)$ with $g_k|_X=f_k$. Then for each $n\geq1$ we have $\gamma_n(a_n)=\sum_{k\in I}f_ku^k$ and $\varphi_n(a_n)=a_{n+1}$ where $a_n=\sum_{k\in I}(g_k|_{X_n})u_n^k$. Thus $\sum_{k\in I}f_ku^k$ lies in the image of γ, and since such finite sums are norm dense in $C(X)\rtimes\mathbb{Z}$ and the image of γ is closed we conclude that γ is surjective.

It remains to demonstrate that γ is injective. Note that $C(X)$ can be expressed as the inductive limit $\varinjlim C(X_n)$. On each of our crossed products we have the dual action of the circle [19, Prop. 7.8.3], which is given on the canonical unitary by $(\lambda,u)\mapsto\lambda u$ and has the canonical commutative C^*-subalgebra as its fixed point subalgebra. The dual actions on the crossed products $C(X_n) \rtimes \mathbb{Z}$ also induce a circle action on $\varinjlim C(X_n) \rtimes \mathbb{Z}$.

Now suppose that we are given a positive element a in the kernel of γ. Since γ intertwines the circle actions, integrating the orbit of a over the circle yields a positive element b which lies in both the fixed point subalgebra of $\varinjlim C(X_n) \rtimes \mathbb{Z}$ and the kernel of γ. Since integration with respect to the dual action is faithful, to conclude that $a = 0$ and hence that γ is injective it is enough to show that $b = 0$. Given an $\varepsilon > 0$, we can find an $m \in \mathbb{N}$ and a $c \in C(X_m) \rtimes \mathbb{Z}$ such that $\|\gamma_m(c) - b\| < \varepsilon$. By integrating with respect to the dual action we may assume that $c \in C(X_m)$. Then $\gamma_m(c)$ lies in $\varinjlim C(X_n)$ viewed as a C^*-subalgebra of $\varinjlim C(X_n) \rtimes \mathbb{Z}$, and so

$$\|b\| \leq \|\gamma_m(c)\| + \varepsilon = \|\gamma(\gamma_m(c) - b)\| + \varepsilon < 2\varepsilon.$$

Since ε is arbitrary it follows that $b = 0$, as desired. $\square$

We next recall the definitions of the Cuntz semigroup [5] (see also [23, 26]) and the radius of comparison [27]. Let A be a C^*-algebra. For elements a, b in $M_\infty(A)^+ = \bigcup_{n=1}^\infty M_n(A)^+$ (viewing $M_n(A)^+$ as an upper left-hand corner in $M_m(A)^+$ for $m > n$), we write $a \precsim b$ if there is a sequence $\{t_k\}_k$ in $M_{m,n}(A)$ such that $\lim_{k\to\infty} t_k^* b t_k = a$, and $a \sim b$ if $a \precsim b$ and $b \precsim a$. Set $W(A) = M_\infty(A)^+/\sim$ and write $\langle a \rangle$ for the equivalence class of a. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ we set $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ where $a \oplus b = \mathrm{diag}(a, b) \in M_{n+m}(A)^+$, and declare that $\langle a \rangle \leq \langle b \rangle$ when $a \precsim b$. This endows $W(A)$ with the structure of a positively ordered Abelian semigroup.

Associated to a quasitrace τ on A is the lower semicontinuous map $s_\tau : M_\infty(A)^+ \to \mathbb{R}^+$ given by $s_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$. The value $s_\tau(a)$ depends only on the Cuntz equivalence class of a, and so s_τ can be regarded as a state on $W(A)$. Such states are called *lower semicontinuous dimension functions*. When A is exact, which is the case for all of our crossed products, the states on $W(A)$ can be identified with the quasitraces on A [2] and hence with the tracial states on A [12, 13].

The radius of comparison for a unital stably finite C^*-algebra A was introduced in [27] as an abstract version of dimension-rank ratio. We say that A has *r-comparison* if for all $a, b \in M_\infty(A)^+$ we have $\langle a \rangle \leq \langle b \rangle$ whenever $s(\langle a \rangle) + r < s(\langle b \rangle)$ for all lower semicontinuous dimension functions s on $W(A)$. The *radius of comparison* of A is defined as the infimum of the set of all $r \in \mathbb{R}^+$ for which A has r-comparison, unless this set is empty, in which case it is defined to be ∞.

In the proof of the following theorem we use the notation θ_r for the trivial bundle of rank r over whatever space is in question and $\xi^{\times r}$ for the r-fold Cartesian product of a given bundle ξ.

Theorem 2.2. *Let $q \geq 2$ and suppose that $1 - 1/q < d < 1$. Then for $Y = I^{3q}$ the radius of comparison of $C(X) \rtimes \mathbb{Z}$ is bounded below by $q - 1$.*

Proof. Let ξ be a line bundle on S^2 with nonzero Euler class. Fix a continuous embedding ε of $(S^2 \times [0, 1])^q$ into Y. Take a function $f \in C(Y)$ that takes the constant value one on $\varepsilon((S^2 \times \{\frac{1}{2}\})^q)$ and is zero outside of $\varepsilon((S^2 \times (0, 1))^q)$. Let $\pi : S^2 \times [0, 1] \to S^2$ be the projection onto the first coordinate. Regarding bundles as projections, we define a and b to be the positive elements θ_1 and $(\mathbf{1} - f)\theta_q + f\pi^*(\xi)^{\times q}$, respectively, of $M_{2q}(C(Y))$.

Given an $n \in \mathbb{N}$, let ψ_n be the composition $C(Y) \to C(X_n) \hookrightarrow C(X_n) \rtimes \mathbb{Z}$ where the second *-homomorphism is the canonical embedding and the first *-homomorphism arises from the surjective composition $X_n \to Y \times I \to Y$ where the first map is the projection onto the zeroeth coordinate and the second map is the projection onto the first coordinate. Write E for the set of all $k \in \{1, \dots, l_n\}$ such that $Y_{n,k} = Y$. Fix a point $(z_k)_{1 \leq k \leq l_n} \in \prod_{k=1}^{l_n} I_{n,k}$. Let $\gamma_n : C(B_n) \to C(((S^2)^q)^E)$ be the *-homomorphism induced from the continuous embedding $((S^2)^q)^E \to B_n$ under which the kth coordinate of the image of $(x_j)_{j \in E} \in ((S^2)^q)^E$ is $(\varepsilon(x_k, \frac{1}{2}), z_k)$ if $k \in E$ and (y_k, z_k) otherwise, where in the latter case y_k is the unique point contained in $Y_{n,k}$. Let $\varphi_n : C(X_n) \rtimes \mathbb{Z} \to M_{l_n} \otimes C(B_n)$ be the *-homomorphism defined via periodic sequences and the universal property of the crossed product as described at the beginning of the section.

Set $a_n = (\mathrm{id}_{M_{2q}} \otimes \psi_n)(a)$ and $b_n = (\mathrm{id}_{M_{2q}} \otimes \psi_n)(b)$. Consider the map ζ_n given by the composition

$$C(Y) \xrightarrow{\psi_n} C(X_n) \rtimes \mathbb{Z} \xrightarrow{\varphi_n} M_{l_n} \otimes C(B_n) \xrightarrow{\mathrm{id} \otimes \gamma_n} M_{l_n} \otimes C(((S^2)^q)^E).$$

Viewing bundles as projections in matrix algebras, we have $(\mathrm{id}_{M_{2q}} \otimes \zeta_n)(a) = \theta_{l_n}$ and $(\mathrm{id}_{M_{2q}} \otimes \zeta_n)(b) = \xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}$. Since ξ has nonzero Euler class and $q(l_n - |E|) \leq ql_n(1-d) < l_n$ by our hypothesis on d, by Lemma 1 of [29] the trivial bundle θ_{l_n} on $((S^2)^q)^E$ is not subequivalent to $\xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)}$. It follows by Lemma 2.1 of [26] that $\|t^*(\xi^{\times q|E|} \oplus \theta_{q(l_n - |E|)})t - \theta_{l_n}\| \geq 1/2$ for all $t \in M_{2q} \otimes M_{l_n} \otimes C(((S^2)^q)^E)$. Since *-homomorphisms are contractive, we thus obtain $\|t^* b_n t - a_n\| \geq 1/2$ for all $t \in M_{2q} \otimes (C(X_n) \rtimes \mathbb{Z})$.

Now let λ be the *-homomorphism $C(Y) \to C(X) \rtimes \mathbb{Z}$ arising from the surjective composition $X \to Y \times I \to Y$ where the first map is the projection onto the zeroeth coordinate and the second map is the projection onto the first coordinate. Consider the elements $a_\infty = (\mathrm{id}_{M_{2q}} \otimes \lambda)(a)$ and $b_\infty = (\mathrm{id}_{M_{2q}} \otimes \lambda)(b)$ in $M_{2q} \otimes (C(X) \rtimes \mathbb{Z})$. Note that for each $n \geq 1$ the elements a_n and b_n map to a_∞ and b_∞, respectively, under the canonical quotient $M_{2q} \otimes (C(X_n) \rtimes \mathbb{Z}) \to M_{2q} \otimes (C(X) \rtimes \mathbb{Z})$. In view of the previous paragraph and Lemma 2.1 we therefore have $\|t^* b_\infty t - a_\infty\| \geq 1/2$ for all $t \in M_{2q} \otimes (C(X) \rtimes \mathbb{Z})$ and hence $\langle a_\infty \rangle \not\leq \langle b_\infty \rangle$.

Finally, note that by the minimality of (X, T) the tracial states on $C(X) \rtimes \mathbb{Z}$ are precisely the compositions of the canonical conditional expectation $C(X) \rtimes \mathbb{Z} \to C(X)$ with α-invariant states on $C(X)$ (see Section VIII.3 in [6]). We thus see that $s(\langle a_\infty \rangle) = 1$ and $s(\langle b_\infty \rangle) \geq q$ for every state s on $W(C(X) \rtimes \mathbb{Z})$. It follows that $C(X) \rtimes \mathbb{Z}$ fails to have r-comparison for every $0 \leq r < q - 1$, and so the radius of comparison of $C(X) \rtimes \mathbb{Z}$ is at least $q - 1$, as desired. □

The final goal of this section is to show that if Y is infinite then $C(X) \rtimes \mathbb{Z}$ does not have real rank zero.

Note that by minimality the tracial state space of $C(X) \rtimes \mathbb{Z}$ can be identified with the space $S_\alpha(C(X))$ of α-invariant states on $C(X)$ and hence also with the space of T-invariant Borel probability measures on X (see Section VIII.3 in [6]).

Lemma 2.3. *Suppose that Y is infinite. Then the tracial state space of $C(X) \rtimes \mathbb{Z}$ has infinitely many extreme points.*

Proof. We will prove the result by showing that $S_\alpha(C(X))$ has infinitely many extreme points. Consider the closed subset $W = \bigcap_{n=1}^\infty X_{n,1}$ of X. Then W can be expressed as $\prod_{k\in\mathbb{Z}} D_k$ where for all k in a set $L \subseteq \mathbb{N}$ of density greater than d the set D_k is of the form $Y \times \{z\}$ and for all $k \in \mathbb{Z} \setminus L$ the set D_k is a singleton. Let $\gamma : C(X) \to C(Y)$ be the *-homomorphism arising from the embedding $Y \to W \subseteq X$ determined by setting the first element in the pair describing the kth coordinate of the image of y to be y for each $k \in L$. For each $y \in Y$ take a weak* limit point σ_y of the sequence of states $\{\frac{1}{m}\sum_{j=0}^{m-1} \delta_y \circ \gamma \circ \alpha^j\}_{m=1}^\infty$ where δ_y is point evaluation at y. Then each σ_y is α-invariant.

Let $\psi : C(Y) \to C(X)$ be the map induced by the restriction to X of the projection of $(Y \times I)^{\mathbb{Z}}$ onto the first factor of the zeroeth coordinate. To complete the proof it suffices to show that the compact convex subset $K = \{\sigma \circ \psi : \sigma \in S_\alpha(C(X))\}$ of the state space of $C(Y)$ has infinitely many extreme points. Notice that the lower bound d on the density of the set L in $\mathbb{N}$ means that for each $y \in Y$ and nonnegative $f \in C(Y)$ we will have $(\sigma_y \circ \psi)(f) \geq d \cdot f(y)$, so that $\sigma_y \circ \psi$, viewed as a probability measure on Y, has an atom at y with weight at least d. By the Krein-Milman theorem, for every $y \in Y$ we can find an extreme point ω_y of K which has an atom at y with weight at least d. Since each ω_y has total weight one and Y is infinite, the set $\{\omega_y : y \in Y\}$ must be infinite, completing the proof. □

Proposition 2.4. *Suppose that Y is infinite. Then $C(X) \rtimes \mathbb{Z}$ does not have real rank zero.*

Proof. Recall that real rank zero implies that the linear span of projections is norm dense and hence that K_0 separates tracial states [4]. Since $K_0(C(X) \rtimes \mathbb{Z})$ has a unique state, we obtain the conclusion by Lemma 2.3. □

The proof of Lemma 2.3 also shows that if Y is uncountable then the tracial state space of $C(X) \rtimes \mathbb{Z}$ has uncountably many extreme points. When Y has nonzero topological dimension the system (X, T) has nonzero mean dimension (see Section 3 of [18]), and in this case the uncountability of the set of extreme points also follows from the fact that $\mathbb{Z}$-systems with nonzero mean dimension do not possess the small boundary property [18, Thm. 5.4] and hence cannot have only countably many extreme invariant states [25].

3. Subshifts and perforation in K_0

Here we will show how to produce perforation in the ordered K_0 group of a crossed product arising from a minimal subshift.

Let Y be a compact metric space and let T be the shift $(x_k)_k \mapsto (x_{k+1})_k$ on $Y^{\mathbb{Z}}$. Let $0 < d < 1$. As in Section 2 but without the extra spacing factor I in the seed space (cf. the proof of Proposition 3.5 in [18]), we recursively construct a decreasing sequence $X_1 \supseteq X_2 \supseteq \dots$ of closed T-invariant subsets of $Y^{\mathbb{Z}}$ such that for each $n \in \mathbb{N}$ the set X_n is defined as X_{B_n} for a block $B_n = D_{n,1} \times \cdots \times D_{n,l_n} \subseteq Y^{l_n}$ of a certain length l_n where

1. for all $x, w \in X_n$ there is a $k \in \mathbb{Z}$ such that $d(T^k x, w) \leq 2^{-n+2}$, and
2. $D_{n,i} = Y$ for all i in a subset of $\{1, \dots, l_n\}$ of cardinality greater than dl_n, and $D_{n,i}$ is a singleton for all other i.

Thus d is a lower bound on the density of the appearance of Y in the factors of a block. As in Section 2, we will write $X_{n,i}$ for the set of all $(x_k)_k \in Y^{\mathbb{Z}}$ such that $(x_{i+sl_n}, x_{i+sl_n+1}, \dots, x_{i+sl_n+l_n-1}) \in B_n$ for every $s \in \mathbb{Z}$.

Set $X = \bigcap_{n=1}^{\infty} X_n$, which is a closed T-invariant subset of $Y^{\mathbb{Z}}$, and denote the restriction of T to X again by T. The system (X, T) is minimal by condition (1), and it is free since X contains no periodic points by the definition of the X_n in terms of blocks. As the argument in Section 3 of [18] demonstrates, if $d > 0$ then the mean dimension of (X, T) is nonzero.

As before θ_r denotes the trivial bundle of rank r over whatever space is in question and $\xi^{\times r}$ denotes the r-fold Cartesian product of a given bundle ξ.

Theorem 3.1. *Suppose that $Y = Z \times Z$ where Z is a finite CW-complex admitting a line bundle ξ for which no tensor power of the Euler class is zero (for example, S^2). Then $K_0(C(X) \rtimes \mathbb{Z})$ is not weakly unperforated.*

Proof. We will first describe the situation relative to a fixed blocking as described at the beginning of Section 2, to which we refer the reader for notation and terminology. Let $B = D_1 \times \cdots \times D_l \subseteq Y^l$ be a block. Let $\varphi : C(X_B) \rtimes \mathbb{Z} \to M_l \otimes C(B)$ be the *-homomorphism defined via periodic sequences and the universal property of the crossed product as described at the beginning of Section 2. Denote by ψ the composition $C(Y) \to C(X_B) \hookrightarrow C(X_B) \rtimes \mathbb{Z}$ where the first map is composition with the restriction to X_B of the projection of $Y^{\mathbb{Z}}$ onto the zeroeth coordinate and the second map is the canonical embedding. Set $\gamma = \varphi \circ \psi : C(Y) \to M_l \otimes C(B)$. For each $i = 1, \dots, l$ let $\pi_i : B = D_1 \times \cdots \times D_l \to D_i$ be the projection onto the ith coordinate.

Viewing vector bundles as projections in a suitable matrix algebra and γ as also being defined over matrices in the standard way, we have $\gamma(\xi^{\times 2}) = \sum_{i=1}^{l} e_{i,i} \otimes \pi_i^*(\xi^{\times 2})$, so that as bundles $\gamma(\xi^{\times 2}) \cong \pi_1^*(\xi^{\times 2}) \oplus \cdots \oplus \pi_l^*(\xi^{\times 2})$. Let E be the set of $i \in \{1, \dots, l\}$ such that $D_i = Y$, and suppose that D_i is a singleton for all other i. Since ξ has rank one we have $\gamma(\xi^{\times 2}) \cong \xi^{\times 2|E|} \oplus \theta_{2(l-|E|)}$.

Put $g = [\xi^{\times 2}] - [\theta_1] \in K^0(Y)$. The image of this element under ψ^* witnesses perforation in $K_0(C(X_B) \rtimes \mathbb{Z})$ granted that $|E| > l/2$. Indeed in this case we have

$$K_0(\gamma)(g) = [\xi^{\times 2|E|} \oplus \theta_{2(l-|E|)}] - [\theta_l] = [\xi^{\times 2|E|}] - [\theta_{2|E|-l}] \leq [\xi^{\times 2|E|}] - [\theta_1]$$

so that $K_0(\gamma)(g) \notin K^0(B)^+$ by Lemma 1 of [29], and hence $\psi^*(g) \notin K_0(C(X_B) \rtimes \mathbb{Z})^+$. On the other hand $\dim(Z) g \in K^0(Y)^+$ by Theorem 8.1.2 of [14] and thus $\dim(Z)\psi^*(g) \in K_0(C(X_B) \rtimes \mathbb{Z})^+$.

Now we turn to the systems in our recursive construction. For each $n \geq 1$ consider the maps $\varphi_n : C(X_n) \rtimes \mathbb{Z} \to M_{l_n} \otimes C(B_n)$, $\psi_n : C(Y) \to C(X_n) \rtimes \mathbb{Z}$, and $\gamma_n = \varphi_n \circ \psi_n : C(Y) \to M_{l_n} \otimes C(B_n)$ as defined in the previous paragraph with respect to B_n (with everything here carrying a subscript n for specificity), where X_{B_n} has been abbbreviated to X_n. Assuming that the lower density bound d is greater than $1/2$, the number of factors in $B_n = D_1 \times \cdots \times D_{l_n}$ which are equal to Y is strictly greater than $l_n/2$, and so $\psi_n^*(g) \notin K_0(C(X_n) \rtimes \mathbb{Z})^+$. Let $\psi_\infty : C(Y) \to C(X) \rtimes \mathbb{Z}$ be the composition of the map $C(Y) \to C(X)$ induced by the restriction to X of the projection of $Y^{\mathbb{Z}}$ onto the zeroeth coordinate with the canonical embedding $C(X) \hookrightarrow C(X) \rtimes \mathbb{Z}$. Then $\psi_n^*(g)$ is sent to $\psi_\infty^*(g)$ under the map induced by the canonical quotient $C(X_n) \rtimes \mathbb{Z} \to C(X) \rtimes \mathbb{Z}$ at the level of K_0. It follows by Lemma 2.1 that $\psi^*_\infty(g) \notin K_0(C(X) \rtimes \mathbb{Z})^+$. On the other hand $\dim(Z)\psi^*_\infty(g) \in K_0(C(X) \rtimes \mathbb{Z})^+$, showing that $K_0(C(X) \rtimes \mathbb{Z})^+$ fails to be weakly unperforated. □

The proof of Lemma 2.3 shows that the tracial state space of $C(X) \rtimes \mathbb{Z}$ has infinitely many extreme points when Y is infinite, and uncountably many extreme points when Y is uncountable. However, for Y as in the above theorem the K_0 group of $C(X) \rtimes \mathbb{Z}$ will be much more complicated than the group of rationals that appeared in Section 2, and so a separate argument is needed to conclude that $C(X) \rtimes \mathbb{Z}$ does not have real rank zero. For this we will assume that Y has an infinite path component and $K^1(Y) = 0$, which is the case for $Y = S^2 \times S^2$.

Proposition 3.2. *Suppose that Y has an infinite path component and $K^1(Y) = 0$. Then $C(X) \rtimes \mathbb{Z}$ does not have real rank zero.*

Proof. For each $y \in Y$ we will define a tracial state σ_y on $C(X) \rtimes \mathbb{Z}$ as in the proof of Lemma 2.3. Let W be the closed subset $\bigcap_{n=1}^\infty X_{n,1}$ of X, which can be expressed as $\prod_{k\in\mathbb{Z}} D_k$ where D_k is equal to Y for all k in a set $L \subseteq \mathbb{N}$ of positive density and is a singleton otherwise. Let $\varepsilon : Y \to W \subseteq X$ be the continuous map determined by setting the kth coordinate of $\varepsilon(y)$ to be y for all $k \in L$. Define the *-homomorphism $\gamma : C(X) \to C(Y)$ by $\gamma(f) = f \circ \varepsilon$.

By hypothesis, Y has a path component containing a finite set F of cardinality greater than $1/d$. By a diagonal argument there is an increasing sequence $\{m_j\}_{j=1}^\infty$ of positive integers such that $\frac{1}{m_j}\sum_{i=0}^{m_j-1} \delta_y \circ \gamma \circ \alpha^i$ converges to some state σ_y on $C(X)$ for every $y \in F$, where δ_y is the point mass at y. Note that each σ_y is α-invariant.

Let $\psi : C(Y) \to C(X) \rtimes \mathbb{Z}$ be the composition of the map $C(Y) \to C(X)$ induced by restriction to X of the projection of $Y^{\mathbb{Z}}$ onto the zeroeth coordinate with the canonical embedding $C(X) \hookrightarrow C(X) \rtimes \mathbb{Z}$. Then the lower bound d on the density of the set L in $\mathbb{N}$ implies that for each $y \in Y$ and nonnegative $f \in C(Y)$ we have $(\sigma_y \circ \psi)(f) \geq d \cdot f(y)$, which means that the state $\sigma_y \circ \psi$, regarded as probability measure on Y, has an atom at y with weight at least d. Since $|F| > 1/d$ it follows that there exist $y_0, y_1 \in F$ such that $\sigma_{y_0} \neq \sigma_{y_1}$. Since F lies in a path component we can connect y_0 and y_1 by a continuous path $g : [0,1] \to Y$. Then $\varepsilon(y_0)$ and $\varepsilon(y_1)$ can be connected by the continuous path $\tilde{g} : [0,1] \to X$ where the kth coordinate of $\tilde{g}(t)$ is independent of t for $k \in \mathbb{Z} \setminus L$ and is equal to $g(t)$ for $k \in L$. This implies that the states $\frac{1}{m_j}\sum_{i=0}^{m_j-1} \delta_{y_0} \circ \gamma \circ \alpha^i$ and $\frac{1}{m_j}\sum_{i=0}^{m_j-1} \delta_{y_1} \circ \gamma \circ \alpha^i$ agree on $K^0(X)$ for each $j \in \mathbb{N}$ and consequently that the states σ_{y_0} and σ_{y_1} agree on $K^0(X)$.

Let $\iota : C(X) \hookrightarrow C(X) \rtimes \mathbb{Z}$ be the canonical embedding and $E : C(X) \rtimes \mathbb{Z} \to C(X)$ the canonical conditional expectation. Since $K^1(Y) = 0$ we have $K^1(X) = 0$, which is clear if for each $n \geq 1$ the sets $X_{n,i}$ for $i = 1, \dots, l_n$ are pairwise disjoint, and otherwise follows from the Mayer-Vietoris sequence (see Section 21.2 of [1]). Thus $\iota_*(K^0(X)) = K_0(C(X) \rtimes \mathbb{Z})$ by the Pimsner-Voiculescu exact sequence, and so no element of $K_0(C(X) \rtimes \mathbb{Z})$ separates the tracial states $\sigma_{y_0} \circ E$ and $\sigma_{y_1} \circ E$. Since real rank zero implies that the linear span of projections is norm dense and hence that K_0 separates tracial states [4], we obtain the desired conclusion. □

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