

THE SMALL BOUNDARY PROPERTY IN PRODUCTS

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ABSTRACT. For a continuous action $G \curvearrowright X$ of a countable group on a compact metrizable space we show that the following are equivalent: (i) the action $G \curvearrowright X$ has the small boundary property and no finite orbits, (ii) for every continuous action $H \curvearrowright Y$ of a countable group on a compact metrizable space, the product action $G \times H \curvearrowright X \times Y$ has the small boundary property. In particular, (ii) is automatic when X is infinite and the action $G \curvearrowright X$ is minimal and has the small boundary property. The argument relies on a small boundary version of the Urysohn lemma.

1. INTRODUCTION

Over the last decade several analogies have emerged between regularity properties in the classification program for nuclear C^* -algebras and properties of the topological dynamical systems (mainly group actions on compact metrizable spaces) that provide C^* -algebra theory with many of its basic models via the crossed product construction. Prominent among these pairings (see [13, 15, 2]) are

- $\mathcal{Z}$ -stability and almost finiteness,
- the uniform McDuff property (or, equivalently in the simple separable nuclear setting, uniform property Gamma [2]) and almost finiteness in measure.

Almost finiteness and almost finiteness in measure are both Rokhlin-type tiling properties that ask for the existence of open towers with Følner shapes and levels of small diameter [13, 15]. The difference between the two is whether the complement of the towers is small topologically (via a notion of subequivalence using open covers) or uniformly on invariant Borel probability measures. Due to the Følner requirement these properties necessitate that the acting group be amenable, while $\mathcal{Z}$ -stability and the uniform McDuff property are independent of amenability/nuclearity, so that nuclearity is implicitly being assumed when making the above pairings. These analogies run quite far and, together with the property of dynamical comparison, have been instrumental in advancing the project of classifying C^* -crossed products. In each case the backwards implication holds when passing from a minimal action of a countably infinite group on a compact metrizable space to its corresponding crossed product [13, 15], and it is conceivable that one even has an equivalence in the nuclear setting. Indeed $\mathcal{Z}$ -stability is known to fail for the typical examples of minimal actions of countable amenable groups that are not almost finite [6, 9].

At the same time, the structural homologies supporting these relationships are far from perfect and the discrepancies at play are closely tied up with a number of subtle issues in topological dynamics. One can particularly appreciate some of the sticking points when looking

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at behaviour under taking products, or tensor products in the case of C^* -algebras. A product $G \times H \curvearrowright X \times Y$ of two actions $G \curvearrowright X$ and $H \curvearrowright Y$ on compact spaces naturally gives rise to a minimal C^* -tensor product $C(X \times Y) \rtimes_\lambda (G \times H) \cong (C(X) \rtimes_\lambda G) \otimes_{\min} (C(Y) \rtimes_\lambda H)$ of the reduced crossed products of the factors. Now $\mathcal{Z}$ -stability passes to the minimal tensor product with any other C^* -algebra directly by definition, without any nuclearity assumptions, and so $C(X \times Y) \rtimes_\lambda (G \times H)$ will have this property as soon as one of the tensor factors does. The same can be said for the uniform McDuff property granted that one works within the relevant universe of unital tracial C^* -algebras and actions admitting invariant Borel probability measures. On the dynamical side, the tower structure that comes from assuming one of the factors is almost finite or almost finite in measure will naturally pass to the product through amplification but one will lose both the Følner condition on the tower shapes and the small diameter condition on the tower levels, and it is not at all clear how to engineer these features unless one has some subdivisibility mechanism at hand. It turns out however that in this situation one can establish the desired subdivisibility (assuming G and H are infinite) in the form of the small boundary property (SBP). That the product action $G \times H \curvearrowright X \times Y$ has the SBP whenever G and H are infinite and amenable, the actions $G \curvearrowright X$ and $H \curvearrowright Y$ are free, and $G \curvearrowright X$ is almost finite in measure was established independently by Kopsacheilis, Liao, Tikuisis, and Vaccaro [17] and Elliott and Niu [3] by passing through a version of property Gamma for inclusions at the C^* -algebra level. The SBP is actually equivalent, via Ornstein–Weiss tiling technology, to almost finiteness in measure when the acting group is amenable and the action is free [15], and so one obtains the conclusion that if $G \curvearrowright X$ and $H \curvearrowright Y$ are free and one of them is almost finite in measure then the product $G \times H \curvearrowright X \times Y$ is almost finite in measure. Kopsacheilis, Liao, Tikuisis, and Vaccaro showed moreover, by passing through a version of tracial $\mathcal{Z}$ -stability for inclusions and making use of dynamical comparison and almost finiteness in measure via [15, 19], that the corresponding statement also holds for almost finiteness [17].

By definition, the SBP requires that the topology admit a basis of open sets whose boundaries are null for every invariant Borel probability measure, a condition which is meaningful whether or not the acting group is amenable. In the amenable setting it implies mean dimension zero [22] and is equivalent to it when the group is $\mathbb{Z}^d$ and the action has the marker property (e.g., when it is free and minimal) [21, 8], and also more generally whenever the action has the uniform Rokhlin property [24]. It is a simple observation that a product $G \times H \curvearrowright X \times Y$ of actions of infinite amenable groups always has mean dimension zero, and so when G and H are both finitely generated free Abelian groups and the actions are free and minimal one can draw the much stronger and quite surprising conclusion that the product action always has the SBP and hence, via [23], is always almost finite.

The uniform McDuff property and the SBP are quite different in nature at the technical level, and perhaps it is unique to amenability, via the connection to almost finiteness in measure, that one is able to compare them in that setting. We show nevertheless that the SBP does continue to share the same permanence behaviour in products beyond the amenable realm. Our approach is completely different from that in [17] and also more direct than that in [3], although it shares the Urysohnian spectral viewpoint guiding the constructions in Section 2 of the latter paper, which were themselves inspired by the level-set analysis of functions in Section 6 of [21] and Section 5 of [8]. It also yields a more general statement in the amenable case that applies in particular to all minimal actions without the hypothesis of freeness that

is present in all of the arguments above involving the SBP and that is also, at least to a certain extent, built into the definitions of almost finiteness and almost finiteness in measure. The latter two concepts are also viable for actions that are essentially free [5], (i.e., free on an invariant set of full measure for every invariant Borel probability measure), but there are minimal actions of amenable groups that are topologically free (i.e., free on a dense invariant subset) but not essentially free, known as *allosteric* actions [10].

Theorem A. Let $G \curvearrowright X$ be a continuous action of a countable group on a compact metrizable space. Then the following are equivalent:

- (1) the action $G \curvearrowright X$ has the SBP and no finite orbits,
- (2) for every continuous action $H \curvearrowright Y$ of a countable group on a compact metrizable space, the product action $G \times H \curvearrowright X \times Y$ has the SBP.

In particular, condition (2) holds whenever X is infinite and the action $G \curvearrowright X$ is minimal and has the SBP.

The arguments in our proof of Theorem A do not actually themselves have much to do with dynamics, which really only enters at the point where we need to know that ergodic measures for the product action are product measures (Lemma 3.1). The crucial tool is a small boundary version of the Urysohn lemma, which we establish in Section 2 as Proposition 2.6. This is related to, but somewhat different from, the recursive constructions used in [25] and [20] to control the value of entropy in factors of actions with the SBP, which was the original motivation for the introduction of the SBP. Similar Urysohn-type results also appear in Section 6 of [21], Section 5 of [8], and Section 2 of [3]. In Section 3 we establish Theorem A by applying our Urysohn-type lemma to produce sets in the product with small boundary. For this we rely on a characterization of the SBP in terms of pairs of points (Proposition 2.2) that, as an offshoot, suggests how the analysis of small boundaries can be localized in the spirit of entropy pairs (see [14]) and mean dimension pairs [4]. We briefly indicate in Section 4 how this localization can be set up through the notion of SBP pair.

That amenability and freeness turn out not to be relevant for Theorem A is in the spirit of [16], where it is shown that the SBP holds whenever the space of invariant Borel probability measures is a Bauer simplex with finite-dimensional extreme boundary. In fact the only essential way in which dynamics gets used in both [16] and the present paper, besides the affine function realization in [16], is in the application of the comparison property for ergodic probability-measure-preserving actions in order to assemble disjoint sets of equal measure into a tower whose shape is a subset of the full group, an operation that requires neither amenability nor freeness.

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2. A SMALL BOUNDARY URYSOHN LEMMA

Throughout X is a compact metrizable space and $\mathcal{N}$ a closed subset of the space $\mathcal{M}(X)$ of Borel probability measures on X equipped with the weak* topology.

Definition 2.1. We say a Borel subset Y of X is $\mathcal{N}$ -small if $\sup_{\mu \in \mathcal{N}} \mu(Y) = 0$. We say X has the $\mathcal{N}$ -SBP ($\mathcal{N}$ -small boundary property) if there is a basis for its topology consisting of open sets whose boundary is $\mathcal{N}$ -small.

Our goal is to establish in Proposition 2.6 a Urysohn lemma for the $\mathcal{N}$ -SBP.

Proposition 2.2. *The following are equivalent:*

- (1) X has the $\mathcal{N}$ -SBP.
- (2) For any two disjoint closed subsets A_1 and A_2 of X there is an open set U such that $A_1 \subseteq U$, $A_2 \cap \overline{U} = \emptyset$, and ∂U is $\mathcal{N}$ -small.
- (3) For any two distinct $x, y \in X$ there is an open set $U \subseteq X$ such that $x \in U$, $y \notin \overline{U}$ and ∂U is $\mathcal{N}$ -small.

Proof. (1) $\Rightarrow$ (2). By (1), for each $x \in A_1$ there is an open neighbourhood U_x of x such that $\overline{U_x} \subseteq X \setminus A_2$ and ∂U_x is $\mathcal{N}$ -small. Since A_1 is compact, there is a finite subset W of A_1 such that the sets U_x for $x \in W$ cover A_1 . Set $U = \bigcup_{x \in W} U_x$. Then U is open, $A_1 \subseteq U$, and $\overline{U} = \bigcup_{x \in W} \overline{U_x} \subseteq X \setminus A_2$. Note that $\partial U \subseteq \bigcup_{x \in W} \partial U_x$ is $\mathcal{N}$ -small. Thus (2) holds.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Let $x \in X$ and let V be an open neighbourhood of x . By (3), for each $y \in X \setminus V$ we can find an open neighbourhood U_y of x such that $y \in X \setminus \overline{U_y}$ and ∂U_y is $\mathcal{N}$ -small. Since $X \setminus V$ is compact, there is a finite subset W of $X \setminus V$ such that the sets $X \setminus \overline{U_y}$ for $y \in W$ cover $X \setminus V$. Then $U := \bigcap_{y \in W} U_y$ is an open neighbourhood of x , and

$$\overline{U} \subseteq \bigcap_{y \in W} \overline{U_y} = X \setminus \left(X \setminus \bigcap_{y \in W} \overline{U_y} \right) = X \setminus \bigcup_{y \in W} (X \setminus \overline{U_y}) \subseteq X \setminus (X \setminus V) = V.$$

Note that $\partial U \subseteq \bigcup_{y \in W} \partial U_y$ is $\mathcal{N}$ -small. Thus (1) holds. $\square$

Lemma 2.3. *Suppose that X has the $\mathcal{N}$ -SBP. Let A be an $\mathcal{N}$ -small closed subset of X and let V be an open subset of X such that $A \subseteq V$. Let $\varepsilon > 0$. Then there is an open set $U \subseteq X$ such that $A \subseteq U \subseteq \overline{U} \subseteq V$, ∂U is $\mathcal{N}$ -small, and $\mu(U) < \varepsilon$ for all $\mu \in \mathcal{N}$.*

Proof. Let $\mu \in \mathcal{N}$. Since $\mu(A) = 0$, we can find an open set U_μ such that $A \subseteq U_\mu \subseteq \overline{U_\mu} \subseteq V$ and $\mu(\overline{U_\mu}) < \varepsilon$. Then we can find an open neighbourhood O_μ of μ in $\mathcal{N}$ such that $\nu(\overline{U_\mu}) < \varepsilon$ for all $\nu \in O_\mu$.

Since $\mathcal{N}$ is compact, we can find a finite subset W of $\mathcal{N}$ such that the sets O_μ for $\mu \in W$ cover $\mathcal{N}$. Set $U = \bigcap_{\mu \in W} U_\mu$. Then U is open, $A \subseteq U \subseteq V$, and $\nu(U) < \varepsilon$ for all $\nu \in \mathcal{N}$. By condition (2) in Proposition 2.2 we can find an open set U' such that $A \subseteq U' \subseteq \overline{U'} \subseteq U$ and $\partial U'$ is $\mathcal{N}$ -small. $\square$

Lemma 2.4. *Suppose that X has the $\mathcal{N}$ -SBP. Let A_0 and A_1 be disjoint closed subsets of X , let $\mu \in \mathcal{M}(X)$ be atomless, and let $\varepsilon, C > 0$. Then there are an integer $M \geq C$ and open sets $U_1, \dots, U_M \subseteq X$ satisfying the following conditions:*

- (1) $A_0 \subseteq U_1$,
- (2) $\overline{U_j} \subseteq U_{j+1}$ for all $j = 1, \dots, M-1$,
- (3) $U_M = X \setminus A_1$,
- (4) ∂U_j is $\mathcal{N}$ -small for all $j = 1, \dots, M-1$,
- (5) $\mu(U_1 \setminus A_0) < \varepsilon$,

(6) $\mu(U_{j+1} \setminus \overline{U_j}) < \varepsilon$ for all $j = 1, \dots, M-1$.

Proof. Take an integer $M \geq C$ such that $\frac{1}{M}\mu(X \setminus (A_0 \cup A_1)) < \frac{1}{2}\varepsilon$. Since μ is atomless and X is metrizable, (X, μ) is measure-isomorphic to the unit interval with Lebesgue measure [14, Theorem A.20] and so we can find a Borel partition $\mathcal{P} = \{P_1, \dots, P_M\}$ of $X \setminus (A_0 \cup A_1)$ such that $\mu(P_j) = \frac{1}{M}\mu(X \setminus (A_0 \cup A_1))$ for every $j = 1, \dots, M$. For each $1 \leq j \leq M$, take a compact subset Y_j of P_j such that $\mu(P_j \setminus Y_j) < \varepsilon/(2M)$.

By condition (2) in Proposition 2.2, we can find an open set U_1 such that $A_0 \cup Y_1 \subseteq U_1$, $\overline{U_1}$ is disjoint from $A_1 \cup \bigcup_{k=2}^M Y_k$, and ∂U_1 is $\mathcal{N}$ -small. Again using condition (2) in Proposition 2.2 we can find an open set U_2 such that $\overline{U_1} \cup Y_2 \subseteq U_2$, $\overline{U_2}$ is disjoint from $A_1 \cup \bigcup_{k=3}^M Y_k$, and ∂U_2 is $\mathcal{N}$ -small. Continuing in this way, we produce open sets $U_1, \dots, U_{M-1}$ of X such that $\overline{U_{j-1}} \cup Y_j \subseteq U_j$, $\overline{U_j}$ is disjoint from $A_1 \cup \bigcup_{k=j+1}^M Y_k$, and ∂U_j is $\mathcal{N}$ -small for all $2 \leq j \leq M-1$. Set $U_M = X \setminus A_1$. Then conditions (1) to (4) in the lemma statement hold.

Since the sets $U_1 \setminus A_0$ and Y_k for $k \in \{2, \dots, M\}$ are pairwise disjoint subsets of $X \setminus (A_0 \cup A_1)$, we have

$$\begin{aligned} \mu(U_1 \setminus A_0) &\leq \mu(X \setminus (A_0 \cup A_1)) - \sum_{2 \leq k \leq M} \mu(Y_k) \\ &= \mu(X \setminus (A_0 \cup A_1)) - \sum_{2 \leq k \leq M} \mu(P_k) + \sum_{2 \leq k \leq M} \mu(P_k \setminus Y_k) \\ &= \frac{1}{M}\mu(X \setminus (A_0 \cup A_1)) + \sum_{2 \leq k \leq M} \mu(P_k \setminus Y_k) \\ &\leq \frac{1}{2}\varepsilon + (M-1)\frac{\varepsilon}{2M} < \varepsilon. \end{aligned}$$

This establishes (5).

Let $1 \leq j \leq M-1$. Then the sets $U_{j+1} \setminus \overline{U_j}$ and Y_k for $k \in \{1, \dots, M\} \setminus \{j+1\}$ are pairwise disjoint subsets of $X \setminus (A_0 \cup A_1)$, in which case

$$\begin{aligned} \mu(U_{j+1} \setminus \overline{U_j}) &\leq \mu(X \setminus (A_0 \cup A_1)) - \sum_{\substack{1 \leq k \leq M \\ k \neq j+1}} \mu(Y_k) \\ &= \mu(X \setminus (A_0 \cup A_1)) - \sum_{\substack{1 \leq k \leq M \\ k \neq j+1}} \mu(P_k) + \sum_{\substack{1 \leq k \leq M \\ k \neq j+1}} \mu(P_k \setminus Y_k) \\ &= \frac{1}{M}\mu(X \setminus (A_0 \cup A_1)) + \sum_{\substack{1 \leq k \leq M \\ k \neq j+1}} \mu(P_k \setminus Y_k) \\ &\leq \frac{1}{2}\varepsilon + (M-1)\frac{\varepsilon}{2M} < \varepsilon. \end{aligned}$$

This establishes (6). □

Lemma 2.5. *Suppose that $\mathcal{N}$ consists of atomless measures and that X has the $\mathcal{N}$ -SBP. Let A_0 and A_1 be disjoint closed subsets of X , and let $\varepsilon, C > 0$. Then there are a positive integer $M \geq C$ and open sets $U_1, \dots, U_M \subseteq X$ satisfying the following conditions:*

(1) $A_0 \subseteq U_1$,

- (2) $\overline{U_j} \subseteq U_{j+1}$ for all $j = 1, \dots, M-1$,
- (3) $U_M = X \setminus A_1$,
- (4) ∂U_j is $\mathcal{N}$ -small for all $j = 1, \dots, M-1$,
- (5) $\mu(U_1 \setminus A_0) < \varepsilon$ for all $\mu \in \mathcal{N}$,
- (6) $\mu(U_{j+1} \setminus \overline{U_j}) < \varepsilon$ for all $\mu \in \mathcal{N}$ and $j = 1, \dots, M-1$.

Proof. We may assume that $\mathcal{N} \neq \emptyset$, for otherwise the statement is trivial.

Let $\mu \in \mathcal{N}$. By Lemma 2.4 there are a positive integer $M_\mu \geq C + 2$ and open sets $U_{\mu,1}, \dots, U_{\mu,M_\mu} \subseteq X$ satisfying the following conditions:

- (i) $A_0 \subseteq U_{\mu,1}$,
- (ii) $\overline{U_{\mu,j}} \subseteq U_{\mu,j+1}$ for all $j = 1, \dots, M_\mu - 1$,
- (iii) $U_{M_\mu} = X \setminus A_1$,
- (iv) $\partial U_{\mu,j}$ is $\mathcal{N}$ -small for all $j = 1, \dots, M_\mu - 1$,
- (v) $\mu(U_{\mu,1} \setminus A_0) < \varepsilon/2$,
- (vi) $\mu(U_{\mu,j+1} \setminus \overline{U_{\mu,j}}) < \varepsilon/2$ for all $j = 1, \dots, M_\mu - 1$.

Consider the pairwise disjoint collection

$$\mathcal{V}_\mu := \{U_{\mu,1} \setminus A_0\} \cup \{U_{\mu,j+1} \setminus \overline{U_{\mu,j}} : j = 1, \dots, M_\mu - 1\}$$

of open subsets of $X \setminus (A_0 \cup A_1)$. By (iv) the set

$$(X \setminus (A_0 \cup A_1)) \setminus \bigcup_{j=1}^{M_\mu-1} \partial U_{\mu,j}$$

is $\mathcal{N}$ -small, and by (v) and (vi) we have $\mu(V) < \varepsilon/2$ for all $V \in \mathcal{V}_\mu$. By (iv) and the portmanteau theorem, the functions sending ν to $\nu(U_{\mu,j})$ for $j = 1, \dots, M_\mu - 1$ and $\nu(X \setminus \overline{U_{\mu,M_\mu-1}})$ are continuous on $\mathcal{N}$. It follows that for each $V \in \mathcal{V}_\mu$ the function on $\mathcal{N}$ sending ν to $\nu(V)$ is continuous. We can therefore find an open neighbourhood O_μ of μ in $\mathcal{N}$ such that

$$\nu(V) < \frac{\varepsilon}{2}$$

for all $\nu \in O_\mu$ and $V \in \mathcal{V}_\mu$. Enumerate the elements of $\mathcal{V}_\mu$ as $V_{\mu,1}, \dots, V_{\mu,M_\mu}$ with $V_{\mu,1} = U_{\mu,1} \setminus A_0$ and $V_{\mu,M_\mu} = U_{\mu,M_\mu} \setminus \overline{U_{\mu,M_\mu-1}}$. Then $V_{\mu,1} \cup A_0 = U_{\mu,1}$ and $V_{\mu,M_\mu} \cup A_1 = X \setminus \overline{U_{\mu,M_\mu-1}}$ are open.

Since $\mathcal{N}$ is compact, we can find a finite subset W of $\mathcal{N}$ such that the sets O_μ for $\mu \in W$ cover $\mathcal{N}$. Set $M = \prod_{\mu \in W} M_\mu \geq C + 2$. Then $M \geq 3$. Denote by $\mathcal{V}$ the join of the sets $\mathcal{V}_\mu$ for $\mu \in W$, i.e., the family of sets of the form $\bigcap_{\mu \in W} V_\mu$ with $V_\mu \in \mathcal{V}_\mu$ for all $\mu \in W$. Then $\mathcal{V}$ consists of M pairwise disjoint open subsets of $X \setminus (A_0 \cup A_1)$, some of which may be empty. Note that the set

$$(X \setminus (A_0 \cup A_1)) \setminus \bigcup \mathcal{V} = \bigcup_{\mu \in W} \left((X \setminus (A_0 \cup A_1)) \setminus \bigcup \mathcal{V}_\mu \right)$$

is $\mathcal{N}$ -small. Furthermore,

$$\nu(V) < \frac{\varepsilon}{2}$$

for all $V \in \mathcal{V}$ and $\nu \in \mathcal{N}$. Enumerate the elements of $\mathcal{V}$ as $V_1, \dots, V_M$ with

$$V_1 = \bigcap_{\mu \in W} V_{\mu,1} \quad \text{and} \quad V_M = \bigcap_{\mu \in W} V_{\mu,M_\mu}.$$

Then $V_1 \cup A_0 = \bigcap_{\mu \in W} (V_{\mu,1} \cup A_0)$ and $V_M \cup A_1 = \bigcap_{\mu \in W} (V_{\mu,M_\mu} \cup A_1)$ are open. For each $1 < j < M$ the boundary ∂V_j is a subset of $(X \setminus (A_0 \cup A_1)) \setminus \bigcup \mathcal{V}$ and hence is $\mathcal{N}$ -small. Set $U_1 = V_1 \cup A_0$. Since $V_1 \cup A_0$ and $V_M \cup A_1$ are disjoint open sets, ∂U_1 is a subset of $(X \setminus (A_0 \cup A_1)) \setminus \bigcup \mathcal{V}$ and hence is $\mathcal{N}$ -small.

By Lemma 2.3 we can find an open set $R_1 \subseteq X$ such that $\partial U_1 \subseteq R_1 \subseteq \overline{R_1} \subseteq X \setminus (A_0 \cup A_1)$, ∂R_1 is $\mathcal{N}$ -small, and $\mu(R_1) < \varepsilon/2$ for all $\mu \in \mathcal{N}$. Set $U_2 = U_1 \cup R_1 \cup V_2$. Then $\overline{U_2} = \overline{U_1} \cup \overline{R_1} \cup \overline{V_2}$ is disjoint from A_1 , and $\overline{U_1} \subseteq U_2$. Note that $\partial U_2 \subseteq \partial U_1 \cup \partial R_1 \cup \partial V_2$ is $\mathcal{N}$ -small. For each $\mu \in \mathcal{N}$, we have

$$\mu(U_2 \setminus \overline{U_1}) \leq \mu(R_1 \cup V_2) \leq \mu(R_1) + \mu(V_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

If $M = 3$, then setting $U_3 = X \setminus A_1$ we have $\mu(U_3 \setminus \overline{U_2}) \leq \mu(V_3) < \varepsilon/2$ for all $\mu \in \mathcal{N}$, in which case (1) to (6) hold. Thus we may assume that $M > 3$.

By Lemma 2.3 we can find an open set R_2 of X such that $\partial U_2 \subseteq R_2 \subseteq \overline{R_2} \subseteq X \setminus (A_0 \cup A_1)$, ∂R_2 is $\mathcal{N}$ -small, and $\mu(R_2) < \varepsilon/2$ for all $\mu \in \mathcal{N}$. Set $U_3 = U_2 \cup R_2 \cup V_3$. Then $\overline{U_3} = \overline{U_2} \cup \overline{R_2} \cup \overline{V_3}$ is disjoint from A_1 , and $\overline{U_2} \subseteq U_3$. Note that $\partial U_3 \subseteq \partial U_2 \cup \partial R_2 \cup \partial V_3$ is $\mathcal{N}$ -small. For each $\mu \in \mathcal{N}$, we have

$$\mu(U_3 \setminus \overline{U_2}) \leq \mu(R_2 \cup V_3) \leq \mu(R_2) + \mu(V_3) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Continuing in this way, we construct open sets $U_2, \dots, U_{M-1}$ such that (4) holds and $\overline{U_j} \cup V_{j+1} \subseteq U_{j+1}$ and $\mu(U_{j+1} \setminus \overline{U_j}) < \varepsilon$ for all $j = 1, \dots, M-2$ and $\mu \in \mathcal{N}$, and $\overline{U_{M-1}}$ is disjoint from A_1 . Set $U_M = X \setminus A_1$. Then $\overline{U_{M-1}} \subseteq U_M$, and $\mu(U_M \setminus \overline{U_{M-1}}) \leq \mu(V_M) < \varepsilon/2$ for all $\mu \in \mathcal{N}$. Therefore (1) to (6) hold. $\square$

Proposition 2.6. *Suppose that $\mathcal{N}$ consists of atomless measures and X has the $\mathcal{N}$ -SBP. Let A_0 and A_1 be disjoint closed subsets of X . Then there is a continuous function $f : X \rightarrow [0, 1]$ such that $f = t$ on A_t for $t \in \{0, 1\}$ and $f^{-1}(t) \setminus (A_0 \cup A_1)$ is $\mathcal{N}$ -small for all $t \in [0, 1]$.*

Proof. We will construct a sequence $\{Q_n\}_{n \in \mathbb{N}}$ of finite subsets of $(0, 1]$ and an open set $U_r \subseteq X$ for each $r \in Q := \bigcup_{n \in \mathbb{N}} Q_n$ such that the following hold:

- (1) $1 \in Q_n \subseteq Q_{n+1}$ for all n ,
- (2) $\overline{U_r} \subseteq U_{r'}$ for all $r < r'$ in Q ,
- (3) $A_0 \subseteq U_r$ for all $r \in Q$ and $U_1 = X \setminus A_1$,
- (4) ∂U_r is $\mathcal{N}$ -small for each $r \in Q$,
- (5) setting $U_0 = A_0$, for each n , if we list the elements of $Q_n \cup \{0\}$ in increasing order, then for any consecutive $r < r'$ in $Q_n \cup \{0\}$ we have $r' - r \leq 2^{-n}$ and $\mu(U_{r'} \setminus \overline{U_r}) < 2^{-n}$ for all $\mu \in \mathcal{N}$.

Assume for now that we have found such $\{Q_n\}_{n \in \mathbb{N}}$ and U_r . From (1) and (5) we know that Q is dense in $[0, 1]$. From (2) we know that the boundaries ∂U_r for $r \in Q$ are pairwise disjoint.

Let $x \in X \setminus (A_0 \cup A_1)$. If $x \in U_{r'}$ and $x \notin \overline{U_r}$ for some $r, r' \in Q \cup \{0\}$, then by (2) and (3) we have $r < r'$. Set

$$Q_{>x} = \{r \in Q \cup \{0\} : x \in U_r\},$$

$$Q_{<x} = \{r \in Q \cup \{0\} : x \notin \overline{U_r}\}.$$

Then $1 \in Q_{>x}$ and $0 \in Q_{<x}$. We claim that $\sup Q_{<x} = \inf Q_{>x}$. Suppose instead that $\sup Q_{<x} < \inf Q_{>x}$. For any $r \in Q$ satisfying $\sup Q_{<x} < r < \inf Q_{>x}$, we have $x \in \overline{U_r}$ and $x \notin U_r$, whence $x \in \partial U_r$. Since Q is dense in $(0, 1)$, there are infinitely many $r \in Q$ satisfying $\sup Q_{<x} < r < \inf Q_{>x}$, and the corresponding ∂U_r are disjoint. Thus we obtain a contradiction, proving our claim.

Now we define $f : X \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A_0, \\ \sup Q_{<x} = \inf Q_{>x} & \text{if } x \notin A_0 \cup A_1, \\ 1 & \text{if } x \in A_1. \end{cases}$$

Then $f = t$ on A_t for $t \in \{0, 1\}$.

Let $x \in X$ and let us check that f is continuous at x . Suppose first that $f(x) \in (0, 1)$. For every $\varepsilon > 0$, since Q is dense in $[0, 1]$ we can find $r, r' \in Q$ such that $f(x) - \varepsilon < r < f(x) < r' < f(x) + \varepsilon$. Then $x \in U_{r'} \setminus \overline{U_r}$. Note that $r \leq f(y) \leq r'$ for all $y \in U_{r'} \setminus \overline{U_r}$. Therefore f is continuous at x .

Next suppose that $f(x) = 1$. For every $\varepsilon > 0$, since Q is dense in $[0, 1]$ we can find an $r \in Q$ such that $1 - \varepsilon < r < 1$. Then $x \in X \setminus \overline{U_r}$. Note that $r \leq f(y)$ for all $y \in X \setminus \overline{U_r}$. Therefore f is continuous at x .

Finally, suppose that $f(x) = 0$. For every $\varepsilon > 0$, since Q is dense in $[0, 1]$ we can find an $r \in Q$ such that $0 < r < \varepsilon$. Then $x \in U_r$. Note that $f(y) \leq r$ for all $y \in U_r$. Therefore f is continuous at x . In summary, we have shown that f is continuous on all of X .

Let $t \in (0, 1)$. When n is sufficiently large, by (5) we can find some consecutive $r_1 < r_2 < r_3$ in Q_n such that $r_1 < t < r_3$. Then $f^{-1}(t)$ is disjoint from $\overline{U_{r_1}}$, and $f^{-1}(t) \subseteq U_{r_3}$. Thus for every $\mu \in \mathcal{N}$ we have

$$\begin{aligned} \mu(\{f^{-1}(t)\}) &\leq \mu(U_{r_3} \setminus \overline{U_{r_1}}) = \mu(U_{r_3} \setminus \overline{U_{r_2}}) + \mu(\partial U_{r_2}) + \mu(U_{r_2} \setminus \overline{U_{r_1}}) \\ &< \frac{1}{2^n} + 0 + \frac{1}{2^n} = \frac{1}{2^{n-1}}. \end{aligned}$$

Letting $n \rightarrow \infty$, we see that $f^{-1}(t)$ is $\mathcal{N}$ -small.

For each n , denoting by r the largest element in $Q_n \setminus \{1\}$ we have $f^{-1}(0) \setminus A_0 \subseteq U_{\min Q_n} \setminus U_0$ and $f^{-1}(1) \setminus A_1 \subseteq U_1 \setminus \overline{U_r}$, so that by (5) we have, for each $\mu \in \mathcal{N}$,

$$\mu(f^{-1}(0) \setminus A_0) \leq \mu(U_{\min Q_n} \setminus U_0) < \frac{1}{2^n}$$

and

$$\mu(f^{-1}(1) \setminus A_1) \leq \mu(U_1 \setminus \overline{U_r}) < \frac{1}{2^n}.$$

Letting $n \rightarrow \infty$, we conclude that $f^{-1}(0) \setminus A_0$ and $f^{-1}(1) \setminus A_1$ are $\mathcal{N}$ -small.

Now we turn to the construction of the sequence $\{Q_n\}_{n \in \mathbb{N}}$ and the sets U_r satisfying (1) to (5). Set $U_0 = A_0$.

By Lemma 2.5 we can find an integer $M \geq 2$ and open sets U_r for $r \in Q_1 := \{\frac{1}{M}, \frac{2}{M}, \dots, \frac{M}{M}\}$ such that the following hold:

- (i) $\overline{U_{j/M}} \subseteq U_{(j+1)/M}$ for all $j = 0, 1, \dots, M-1$,

- (ii) $U_1 = X \setminus A_1$,
- (iii) $\partial U_{j/M}$ is $\mathcal{N}$ -small for all $j = 1, \dots, M-1$,
- (iv) $\mu(U_{(j+1)/M} \setminus \overline{U_{j/M}}) < 2^{-1}$ for all $\mu \in \mathcal{N}$ and $j = 0, 1, \dots, M-1$.

This defines Q_1 and U_r for $r \in \{0\} \cup Q_1$ satisfying (1) to (5).

Let $r_0 < r_1$ be two consecutive elements in $\{0\} \cup Q_1$. By Lemma 2.5 we can find an integer $M \geq 2$ and open sets U_r for $r \in Q_{2,r_0} := \{r_0 + (r_1 - r_0)/M, r_0 + 2(r_1 - r_0)/M, \dots, r_0 + M(r_1 - r_0)/M\}$ such that the following hold:

- (i) $\overline{U_{r_0+j(r_1-r_0)/M}} \subseteq U_{r_0+(j+1)(r_1-r_0)/M}$ for all $j = 0, 1, \dots, M-1$,
- (ii) $\partial U_{r_0+j(r_1-r_0)/M}$ is $\mathcal{N}$ -small for all $j = 1, \dots, M-1$,
- (iii) $\mu(U_{r_0+(j+1)(r_1-r_0)/M} \setminus \overline{U_{r_0+j(r_1-r_0)/M}}) < 2^{-2}$ for all $\mu \in \mathcal{N}$ and $j = 0, 1, \dots, M-1$.

Note that $U_{r_0+M(r_1-r_0)/M} = U_{r_1}$ was chosen before already.

Now set $Q_2 = \bigcup_{r \in (\{0\} \cup Q_1) \setminus \{1\}} Q_{2,r}$. This defines Q_2 and U_r for $r \in \{0\} \cup Q_2$ satisfying (1) to (5).

Let $r_0 < r_1$ be two consecutive elements in $\{0\} \cup Q_2$. By Lemma 2.5 we can find an integer $M \geq 2$ and open sets U_r for $r \in Q_{3,r_0} := \{r_0 + (r_1 - r_0)/M, r_0 + 2(r_1 - r_0)/M, \dots, r_0 + M(r_1 - r_0)/M\}$ such that the following hold:

- (i) $\overline{U_{r_0+j(r_1-r_0)/M}} \subseteq U_{r_0+(j+1)(r_1-r_0)/M}$ for all $j = 0, 1, \dots, M-1$,
- (ii) $\partial U_{r_0+j(r_1-r_0)/M}$ is $\mathcal{N}$ -small for all $j = 1, \dots, M-1$,
- (iii) $\mu(U_{r_0+(j+1)(r_1-r_0)/M} \setminus \overline{U_{r_0+j(r_1-r_0)/M}}) < 2^{-3}$ for all $\mu \in \mathcal{N}$ and $j = 0, 1, \dots, M-1$.

Note that $U_{r_0+M(r_1-r_0)/M} = U_{r_1}$ was chosen before already.

Now set $Q_3 = \bigcup_{r \in (\{0\} \cup Q_2) \setminus \{1\}} Q_{3,r}$. This defines Q_3 and U_r for $r \in \{0\} \cup Q_3$ satisfying (1) to (5).

Continuing in this way, we find $\{Q_n\}_{n \in \mathbb{N}}$ and U_r satisfying (1) to (5). $\square$

3. PRODUCT ACTIONS AND THE PROOF OF THEOREM A

Let G and H be countable groups. The product $G \times H \curvearrowright X \times Y$ of two actions $G \curvearrowright X$ and $H \curvearrowright Y$ is defined by $(g, h)(x, y) = (gx, hy)$.

Let $G \curvearrowright X$ be a continuous action on a compact metrizable space. Write $\mathcal{M}_G(X)$ for the space of G -invariant Borel probability measures on X , which is a weak* closed subset of the space $\mathcal{M}(X)$ of all Borel probability measures on X . The action $G \curvearrowright X$ is said to have the *SBP* if X has the $\mathcal{M}_G(X)$ -SBP in the sense of Definition 2.1.

The next lemma makes use of the well-known comparison property for ergodic probability-measure-preserving actions $G \curvearrowright (X, \mu)$, which says that for any two sets $A, B \subseteq X$ of equal measure there exists an element T of the full group (i.e., a measurable automorphism T of X for which there are a countable measurable partition $\{A_i\}_{i \in I}$ of X modulo null sets and group elements s_i for $i \in I$ with $Tx = s_i x$ for all $i \in I$ and $x \in A_i$) such that $TA = B$ modulo null sets (see for example Lemma 7.10 of [12]). This is easily deduced using a greedy algorithm with respect to some fixed enumeration of G .

Lemma 3.1. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be continuous actions on compact metrizable spaces. Then every ergodic measure $\mu \in \mathcal{M}_{G \times H}(X \times Y)$ for the product action $G \times H \curvearrowright X \times Y$ is of the form $\mu_X \times \mu_Y$ for some $\mu_X \in \mathcal{M}_G(X)$ and $\mu_Y \in \mathcal{M}_H(Y)$.*

Proof. Let μ be an ergodic measure in $\mathcal{M}_{G \times H}(X \times Y)$ and write μ_X and μ_Y for the marginals, which are invariant and ergodic for the actions of G and H , respectively. If neither μ_X and μ_Y are atomless then by ergodicity they are each supported and uniform on a single finite orbit, which means that μ must be supported and uniform on the product of these finite orbits and therefore be equal to $\mu_X \times \mu_Y$. We may thus assume that one of μ_X and μ_Y is atomless, say μ_X without loss of generality.

Let $A \subseteq X$ and $B \subseteq Y$ be Borel sets. Suppose first that $\mu_X(A)$ is rational, say k/n . The fact that μ is atomless and X is metrizable implies that (X, μ) is measure-isomorphic to the unit interval with Lebesgue measure [14, Theorem A.20] and so we can find a μ_X -uniform Borel partition $\{P_1, \dots, P_n\}$ of X such that $A = \bigsqcup_{i=1}^k P_i$. Since $G \curvearrowright X$ is μ_X -ergodic, by the comparison property recalled before Lemma 3.1 there exist elements $S_2, \dots, S_n$ in the full group such that, modulo μ_X -null sets, we have $P_i = S_i P_1$ for every $i = 2, \dots, n$. For $i = 2, \dots, n$ the product transformation $T_i := S_i \times \text{id}_Y$ of $X \times Y$, defined modulo a μ -null set, belongs to the full group of the $G \times H$ action, and so all of the sets $P_i \times B$ have the same μ -measure. Since these sets partition $X \times B$, which has μ -measure $\mu_Y(B)$, we therefore have $\mu(P_i \times B) = \mu_Y(B)/n$ for every i . It follows that

$$\mu(A \times B) = \sum_{i=1}^k \mu(P_i \times B) = \sum_{i=1}^k \frac{1}{n} \mu_Y(B) = \frac{k}{n} \mu_Y(B) = \mu_X(A) \mu_Y(B).$$

In the case that $\mu_X(A)$ is not rational we can find, by the atomlessness of μ_X , an increasing sequence $A_1 \subseteq A_2 \subseteq \dots$ of subsets of A with rational μ_X -measure such that $A = \bigcup_{j=1}^{\infty} A_j$, in which case $\mu_X(A) = \lim_{j \rightarrow \infty} \mu_X(A_j)$ and hence, using the previous paragraph,

$$\mu(A \times B) = \lim_{j \rightarrow \infty} \mu(A_j \times B) = \lim_{j \rightarrow \infty} \mu_X(A_j) \mu_Y(B) = \mu_X(A) \mu_Y(B).$$

Since Borel rectangles generate the Borel σ -algebra of $X \times Y$, we conclude that μ is equal to the product measure $\mu_X \times \mu_Y$. $\square$

Lemma 3.2. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be continuous actions on compact metrizable spaces. Suppose that the product action $G \times H \curvearrowright X \times Y$ has the SBP. Then either $H \curvearrowright Y$ has the SBP or $\mathcal{M}_G(X)$ consists of atomless measures.*

Proof. Suppose that $H \curvearrowright Y$ does not have the SBP and there is a $\mu \in \mathcal{M}_G(X)$ which is not atomless. By Proposition 2.2 we can find distinct $y_1, y_2 \in Y$ such that for any open neighbourhood U of y_1 satisfying $y_2 \notin \overline{U}$, the boundary ∂U is not $\mathcal{M}_H(Y)$ -small. Also, we can find an $x \in X$ such that $\mu(\{x\}) > 0$.

Since $G \times H \curvearrowright X \times Y$ has the SBP, we can find an open neighbourhood O of (x, y_1) in $X \times Y$ such that $(x, y_2) \notin \overline{O}$ and ∂O is $\mathcal{M}_{G \times H}(X \times Y)$ -small. Denote by U the image of $O \cap (\{x\} \times Y)$ under the projection $X \times Y \rightarrow Y$. Then U is an open neighbourhood of y_1 , and $y_2 \notin \overline{U}$, and $\{x\} \times \partial U \subseteq \partial O$. By our choice of y_1 and y_2 , there exists a $\nu \in \mathcal{M}_H(Y)$ such that $\nu(\partial U) > 0$. Now $\mu \times \nu \in \mathcal{M}_{G \times H}(X \times Y)$, and

$$(\mu \times \nu)(\partial O) \geq (\mu \times \nu)(\{x\} \times \partial U) = \mu(\{x\}) \nu(\partial U) > 0,$$

a contradiction. Therefore either $H \curvearrowright Y$ has the SBP or $\mathcal{M}_G(X)$ consists of atomless measures. $\square$

Note that $\mathcal{M}_G(X)$ consists of atomless measures if and only if $G \curvearrowright X$ has no finite orbits.

Proof of Theorem A. (1) $\Rightarrow$ (2). Let $H \curvearrowright Y$ be a continuous action of a countable group on a compact metrizable space, and let (x_1, y_1) and (x_2, y_2) be two distinct points in $X \times Y$. To show (2), by Proposition 2.2 it suffices to find an open set O of $X \times Y$ such that $(x_1, y_1) \in O$, $(x_2, y_2) \notin \overline{O}$, and ∂O is $\mathcal{M}_{G \times H}(X \times Y)$ -small.

Consider first the case $x_1 \neq x_2$. Since $G \curvearrowright X$ has the SBP, we can find an open set $U \subseteq X$ such that $x_1 \in U$, $x_2 \notin \overline{U}$, and ∂U is $\mathcal{M}_G(X)$ -small. Set $O = U \times Y$. Then $(x_1, y_1) \in O$ and $(x_2, y_2) \notin \overline{O}$. Since $\partial O = \partial U \times Y$ and for every $\mu \in \mathcal{M}_{G \times H}(X \times Y)$ the μ -measure of this set is equal to the value of ∂U on the marginal measure, which is invariant under G , we know that ∂O is $\mathcal{M}_{G \times H}(X \times Y)$ -small.

Next consider the case $x_1 = x_2$. Fix a compatible metric ρ on Y such that $\rho(y_1, y_2) > 2$. For each $r \geq 0$ denote by S_r the set $\{y \in Y : \rho(y, y_1) = r\}$. By assumption the action $G \curvearrowright X$ does not have finite orbits, and so $\mathcal{M}_G(X)$ consists of atomless measures, which means that we can apply Proposition 2.6, with both A_0 and A_1 taken there to be the empty set, in order to obtain a continuous function $f : X \rightarrow [0, 1]$ such that $f^{-1}(t)$ is $\mathcal{M}_G(X)$ -small for all $t \in [0, 1]$. Put $g = f + 1$. Set

$$O = \{(x, y) \in X \times Y : \rho(y, y_1) < g(x)\}.$$

Then O is open in $X \times Y$, $(x_1, y_1) \in O$, and $(x_2, y_2) \notin \overline{O}$. Note that

$$\partial O \subseteq \{(x, y) \in X \times Y : \rho(y, y_1) = g(x)\}.$$

To show that ∂O is $\mathcal{M}_{G \times H}(X \times Y)$ -small, it suffices to show that $\mu(\partial O) = 0$ for every ergodic μ in $\mathcal{M}_{G \times H}(X \times Y)$. Indeed if we knew the latter and μ were an arbitrary measure in $\mathcal{M}_{G \times H}(X \times Y)$ then the ergodic disintegration $\int_Z \mu_z d\nu(z)$ of μ [7, Theorem 8.7] would yield $\mu(\partial O) = \int_Z \mu_z(\partial O) d\nu(z) = 0$.

Let $\mu \in \mathcal{M}_{G \times H}(X \times Y)$ be ergodic. By Lemma 3.1 we have $\mu = \mu_X \times \mu_Y$ for some $\mu_X \in \mathcal{M}_G(X)$ and $\mu_Y \in \mathcal{M}_H(Y)$. Denote by W the set of all $r \geq 0$ such that $\mu_Y(S_r) > 0$. This set is countable since the sets S_r for $r \geq 0$ are pairwise disjoint, and so $\mu_X(g^{-1}(W)) = 0$. For each $x \in X$, denote by $(\partial O)_x$ the image of $\partial O \cap (\{x\} \times Y)$ under the projection $X \times Y \rightarrow Y$. Then

$$\partial O = \bigcup_{x \in X} (\{x\} \times (\partial O)_x) \subseteq \bigcup_{x \in X} (\{x\} \times S_{g(x)}).$$

By Fubini's theorem,

$$\mu(\partial O) = \int_X \mu_Y((\partial O)_x) d\mu_X(x) = \int_{g^{-1}(W)} \mu_Y((\partial O)_x) d\mu_X(x) = 0.$$

(2) $\Rightarrow$ (1). Take a continuous action $H \curvearrowright Y$ of a countable group on a compact metrizable space such that the action does not have the SBP and $\mathcal{M}_H(Y)$ does not consist entirely of atomless measures. For example, one may take the shift action $\mathbb{Z} \curvearrowright [0, 1]^{\mathbb{Z}}$, which has nonzero mean dimension [22, Proposition 3.3] and hence fails to have the SBP [22, Theorem 5.4]. By (2) the product action $G \times H \curvearrowright X \times Y$ has the SBP. It follows from Lemma 3.2 that (1) holds. $\square$

4. SBP PAIRS

Let $G \curvearrowright X$ be a continuous action of a countable group on a compact metrizable space. Set $\Delta(X) = \{(x, x) : x \in X\} \subseteq X^2$.

Motivated by Proposition 2.2, we make the following definition.

Definition 4.1. We say that a pair $(x_1, x_2) \in X^2 \setminus \Delta(X)$ is an *SBP pair* if there is no open neighbourhood U of x_1 such that $x_2 \notin \bar{U}$ and ∂U is $\mathcal{M}_G(X)$ -small. Denote by $\text{SBP}_2(X)$ the set of all SBP pairs of X .

For disjoint closed subsets A_1 and A_2 of X , we say that A_1 and A_2 are *separated by closed $\mathcal{M}_G(X)$ -small sets* if there is a partition $\{U_1, Y, U_2\}$ of X such that U_j is an open set with $A_j \subseteq U_j$ for $j = 1, 2$ and Y is $\mathcal{M}_G(X)$ -small. The following is straightforward.

Lemma 4.2. *Let A_1 and A_2 be disjoint closed subsets of X that are not separated by closed $\mathcal{M}_G(X)$ -small sets. Let $A_{1,1}$ and $A_{1,2}$ be closed subsets of A_1 such that $A_1 = A_{1,1} \cup A_{1,2}$. Then either $A_{1,2}$ and A_2 are not separated by closed $\mathcal{M}_G(X)$ -small sets or $A_{1,2}$ and A_2 are not separated by closed $\mathcal{M}_G(X)$ -small sets.*

Proposition 4.3. *The following hold.*

- (1) $G \curvearrowright X$ does not have the SBP if and only if $\text{SBP}_2(X) \neq \emptyset$.
- (2) $\text{SBP}_2(X) \cup \Delta(X)$ is closed and G -invariant.
- (3) If A_1 and A_2 are disjoint closed subsets of X such that A_1 and A_2 are not separated by closed $\mathcal{M}_G(X)$ -small sets, then $(A_1 \times A_2) \cap \text{SBP}_2(X) \neq \emptyset$.
- (4) Let $\pi : X \rightarrow Y$ be a factor map. Then $\pi \times \pi(\text{SBP}_2(X)) \subseteq \text{SBP}_2(Y) \cup \Delta(Y)$.

Proof. The first statement is part of Proposition 2.2, while (2) and (4) are obvious and (3) follows from Lemma 4.2. $\square$

When G is sofic and Σ is a sofic approximation sequence for G , García-Ramos and Gutman defined a pair $(x_1, x_2) \in X^2 \setminus \Delta(X)$ to be a sofic mean dimension pair if for any disjoint closed neighbourhoods A_1 and A_2 of x_1 and x_2 , respectively, one has $\text{mdim}_\Sigma(\mathcal{U}) > 0$ for $\mathcal{U} = \{X \setminus A_1, X \setminus A_2\}$ [4, Definition 4.1]. When G is amenable, one can also define mean dimension pairs using $\text{mdim}(\mathcal{U})$, i.e., by averaging over Følner sets instead of a sofic approximation sequence. When G is amenable and infinite, using [18, Lemma 3.5] and the proof of [18, Lemma 3.7] one can see that these two definitions are equivalent.

In the case that G is amenable and infinite, Lindenstrauss and Weiss showed that if $G \curvearrowright X$ has the SBP then it has mean dimension zero [22, Theorem 5.4]. Their argument also works here.

Theorem 4.4. *Suppose that G is amenable and infinite. Then every mean dimension pair is an SBP pair.*

Question 4.5. For free minimal actions of $\mathbb{Z}^d$, must SBP pairs be mean dimension pairs?

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