DYNAMICAL ENTROPY IN BANACH SPACES

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Abstract. We introduce a version of Voiculescu-Brown approximation entropy for isometric automorphisms of Banach spaces and develop within this framework the connection between dynamics and the local theory of Banach spaces as discovered by Glasner and Weiss. Our fundamental result concerning this contractive approximation entropy, or CA entropy, characterizes the occurrence of positive values both geometrically and topologically. This leads to various applications; for example, we obtain a geometric description of the topological Pinsker factor and show that a C^*-algebra is type I if and only if every multiplier inner *-automorphism has zero CA entropy. We also examine the behaviour of CA entropy under various product constructions and determine its value in many examples, including isometric automorphisms of ℓ_p for $1 \leq p \leq \infty$ and noncommutative tensor product shifts.

1. Introduction

In [26] E. Glasner and B. Weiss showed that if a homeomorphism from a compact metric space K to itself has zero topological entropy, then so does the induced homeomorphism on the space of probability measures on K with the weak* topology. One of the two proofs they gave of this striking result established a remarkable connection between topological dynamics and the local theory of Banach spaces. The key geometric fact is the exponential dependence of k on n given an approximately isometric embedding of ℓ_1^n into ℓ_∞^k, which is a consequence of the work of T. Figiel, J. Lindenstrauss, and V. D. Milman on almost Hilbertian sections of unit balls in Banach spaces [20].

The first author showed in [36] that Glasner and Weiss's geometric approach can be conceptually simplified from a functional-analytic viewpoint using Voiculescu-Brown entropy and also more generally applied to show that if a *-automorphism of a separable exact C^*-algebra has zero Voiculescu-Brown entropy then the induced homeomorphism on the unit ball of the dual has zero topological entropy. In this case the crucial Banach space fact is the exponential dependence of k on n given an approximately isometric embedding of ℓ_1^n into the matrix C^*-algebra M_k [36, Lemma 3.1], which can be deduced from the work of N. Tomczak-Jaegermann on the Rademacher type 2 constants of Schatten p-classes [52].

In the present paper we pursue this connection between dynamics and Banach space geometry within a general Banach space framework via the introduction of an analogue of Voiculescu-Brown entropy which we call contractive approximation entropy, or simply CA entropy. This dynamical invariant has the advantage of

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being defined for any isometric automorphism of a Banach space and often exhibits greater tractability as a result of its more basic structural context. With entropy a basic problem is to determine whether or not it is positive (in which case the dynamics can be thought of as being chaotic, or nondeterministic), and in the case of CA entropy we are able to characterize the occurrence of positive values both geometrically and topologically by expanding upon the arguments of Glasner and Weiss. A large part of the paper will involve applications of these characterizations. In particular, we give a description of the Pinsker algebra in topological dynamics in terms of dynamically generated sets equivalent to the standard basis of ℓ_1, and prove that a C^*-algebra is type I if and only if every multiplier inner *-automorphism has zero CA entropy. The latter result was conjectured by N. P. Brown for Voiculescu-Brown entropy [8] but seems to be out of reach in that case. For C^*-dynamics, the drawback of CA entropy in comparison to Voiculescu-Brown entropy is that, being oblivious to matricial structure, it can be much cruder as a numerical invariant, as we illustrate by computing the value for the tensor product shift on the UHF algebra $M_p^{\otimes\mathbb{Z}}$ to be infinity. Nevertheless, the condition of simplicity on a C^*-algebra does not necessarily rule out the existence of *-automorphisms with finite nonzero CA entropy, as we have discovered with the type 2^∞ Bunce-Deddens algebra.

We begin in Section 2 by defining CA entropy and recording some basic properties. In Section 3 we demonstrate that the topological entropy of a homeomorphism of compact Hausdorff space agrees with the CA entropy of the induced *-automorphism of the C^*-algebra of functions over the space, and then proceed to establish our geometric and topological characterizations of positive CA entropy (Theorem 3.5). Specifically, we show that positive CA entropy is equivalent to positivity of the topological entropy of the induced homeomorphism on the unit ball of the dual as well as to the existence of an element the restriction of whose orbit to a positive density subset of iterates is equivalent to the standard basis of ℓ_1. Several immediate corollaries ensue, such as the invariance of positive CA entropy under isomorphic conjugacy and the vanishing of CA entropy for every isometric automorphism of a Banach space with separable dual. We round out Section 3 by exhibiting $C(\mathbb{T})$ as an example of a Banach space which contains ℓ_1 isometrically but admits no isometric automorphism with positive CA entropy.

Section 4 briefly compares CA entropy with its matricial analogue for completely isometric automorphisms of exact operator spaces, which we call completely contractive approximation entropy, or CCA entropy. This was introduced in the C^*-algebraic setting by C. Pop and R. Smith, who showed that it coincides with Voiculescu-Brown entropy in that case [48]. Positive CA entropy implies positive CCA entropy, but the converse is false. However, we have been unable to resolve the problem of the converse for *-automorphisms of exact C^*-algebras. As our computation for the noncommutative shift in Section 9 demonstrates, it is also possible for the CA entropy to be strictly larger than the CCA entropy.

In Section 5 we obtain a geometric description of the topological Pinsker algebra, which is the C^*-algebraic manifestation of the Pinsker factor (i.e., the largest zero entropy factor) in topological dynamics [4]. It turns out that the elements of the topological Pinsker algebra are precisely those which do not generate a subspace canonically isomorphic to ℓ_1 along a positive density set of iterates. This yields geometric characterizations of positive and completely positive topological entropy, and implies that tame and HNS $\mathbb{Z}$-systems [22, 24] have zero entropy.

Section 6 contains our entropic characterization of type I C^*-algebras. For this we establish two lemmas which demonstrate that the property of having zero CA entropy behaves well with respect to Banach space quotients and continuous fields of Banach spaces.

Section 7 examines the behaviour of CA entropy under various product constructions. Subadditivity holds for the injective tensor product, but not necessarily for other tensor product norms, as we illustrate with the shift on a spin system. Adapting an argument of C. Pop and R. Smith based on Imai-Takai duality [48], we prove that, under taking a C^*-algebraic crossed product by the action of an amenable locally compact group G, zero CA entropy is preserved on a group element g such that $\mathrm{Ad}\, g$ generates a compact subgroup of $\mathrm{Aut}(G)$, and if the original C^*-algebra is commutative then every value of CA entropy is preserved in the same situation. The latter fact has the consequence that there exist simple C^*-algebras, for instance the type 2^∞ Bunce-Deddens algebra, which admit inner *-automorphisms with any prescribed value of CA entropy. Finally we deduce in Section 7 that zero entropy is preserved under taking reduced free products of commutative probability spaces by applying a result of N. P. Brown, K. Dykema, and D. Shlyakhtenko [12].

In Section 8 we study the prevalence of zero and infinite CA entropy in C^*-algebras. We find that, for many C^*-algebras which are subject to classification theory (more specifically, those which are tensorially stable with respect to the Jiang-Su algebra $\mathcal{Z}$), the collection of *-automorphisms with infinite CA entropy is point-norm dense, while in the special cases of UHF algebras, the Cuntz algebra $\mathcal{O}_2$, and the Jiang-Su algebra the collection of *-automorphisms with zero CA entropy is a point-norm dense G_δ set, giving a noncommutative version of a result of Glasner and Weiss on homeomorphisms of the Cantor set [25].

In the last three sections we apply a combinatorial argument (Lemma 9.1) to compute the CA entropy for some canonical examples. In Section 9 we prove that the tensor product shift on the UHF algebra $M_p^{\otimes\mathbb{Z}}$ has infinite CA entropy, in contrast to the value of $\log p$ for the Voiculescu-Brown entropy. This raises the question of whether there exist *-automorphisms of simple AF algebras with finite nonzero CA entropy. In Section 10 we show that the CA entropy of an isometric automorphism of ℓ_∞ is either zero or infinity depending on whether or not there is a finite bound on the cardinality of the orbits of the associated permutation of $\mathbb{Z}$. Finally, in Section 11 we show that the CA entropy of an isometric automorphism of ℓ_1 is either zero or infinity depending on whether or not there is an infinite orbit in the associated permutation of $\mathbb{Z}$. As a consequence we deduce that CA entropy is not an isomorphic conjugacy invariant.

All Banach spaces will be over the complex numbers, unless there is an indication to the contrary, such as the tag $\mathbb{R}$ when referring to the real version of a standard Banach space. We point out however that the relevant results in Sections 2, 3, and 10 are also valid over the real numbers, although in other situations differences between the real and complex cases can arise, as Remark 11.2 illustrates. For terminology related to Banach spaces see [41, 34]. The spaces ℓ_p for $1 \leq p \leq \infty$ will be indexed over $\mathbb{Z}$ so that we may conveniently speak of the shift automorphism as obtained from the shift $k \mapsto k+1$ on $\mathbb{Z}$. Given a Banach space X and $r > 0$ the ball $\{x \in X : \|x\| \leq r\}$ will be denoted $B_r(X)$. All C^*-tensor products will be minimal and written using an unadorned $\otimes$. The set of self-adjoint elements of an operator system X will be denoted $X_{\rm sa}$. We write M_d for the C^*-algebra of $d \times d$ complex matrices.

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2. Contractive approximation entropy

Let X and Y be Banach spaces and $\gamma : X \to Y$ a bounded linear map. Denote by $\mathcal{P}_{\rm f}(X)$ the collection of finite subsets of X. For each $\Omega \in \mathcal{P}_{\rm f}(X)$ and $\delta > 0$ we denote by $\mathrm{CA}(\gamma, \Omega, \delta)$ the collection of triples (ϕ, ψ, d) where d is a positive integer and $\phi : X \to \ell_\infty^d$ and $\psi : \ell_\infty^d \to Y$ are contractive linear maps such that

$$\|\psi \circ \phi(x) - \gamma(x)\| < \delta$$

for all $x \in \Omega$. By a *CA embedding* of a Banach space X we mean an isometric linear map ι from X to a Banach space Y such that $\mathrm{CA}(\iota, \Omega, \delta)$ is nonempty for every $\Omega \in \mathcal{P}_{\rm f}(X)$ and $\delta > 0$. Every Banach space admits a CA embedding; for example, the canonical map $X \to C(B_1(X^*))$ defined via evaluation is a CA embedding, as a standard partition of unity argument shows.

Let $\iota : X \to Y$ be a CA embedding. For each $\Omega \in \mathcal{P}_{\rm f}(X)$ and $\delta > 0$ we set

$$\mathrm{rc}(\Omega, \delta) = \inf\{d : (\phi, \psi, d) \in \mathrm{CA}(\iota, \Omega, \delta)\}.$$

We claim that this quantity is independent of the CA embedding, as our notation indicates. Indeed suppose $\iota_0 : X \to Y_0$ is another CA embedding and $(\phi, \psi, d) \in \mathrm{CA}(\iota, \Omega, \delta)$. Take an $\varepsilon > 0$ such that

$$\|\psi \circ \phi(x) - \iota(x)\| < \delta - \varepsilon$$

for all $x \in \Omega$, and choose a $(\phi_0, \psi_0, d_0) \in \mathrm{CA}(\iota_0, \Omega, \varepsilon)$. By the injectivity of $\ell_\infty^{d_0}$ we can extend $\phi \circ \iota^{-1} : \iota(X) \to \ell_\infty^{d_0}$ to a contractive linear map $\rho : Y \to \ell_\infty^{d_0}$. An application of the triangle inequality then shows that $(\phi, \psi_0 \circ \rho \circ \psi, d) \in \mathrm{CA}(\iota_0, \Omega, \delta)$, from which the claim follows.

We denote by $\mathrm{IA}(X)$ the collection of all isometric automorphisms of X. For $\alpha \in \mathrm{IA}(X)$ we set

$$\begin{aligned} \mathrm{hc}(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rc}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta), \\ \mathrm{hc}(\alpha, \Omega) &= \sup_{\delta>0} \mathrm{hc}(\alpha, \Omega, \delta), \\ \mathrm{hc}(\alpha) &= \sup_{\Omega\in\mathcal{P}_{\mathrm{f}}(X)} \mathrm{hc}(\alpha, \Omega) \end{aligned}$$

and refer to the last quantity as the *contractive approximation entropy* or *CA entropy* of α.

From the fact that $\mathrm{rc}(\Omega, \delta)$ does not depend on the CA embedding we see that CA entropy is an invariant with respect to conjugacy by isometric isomorphisms. It is not, however, an invariant with respect to conjugacy by arbitrary isomorphisms (see Remark 11.3). Nevertheless, it turns out that zero CA entropy is an isomorphic conjugacy invariant, as we will see in Section 3.

Proposition 2.1. Let α be an isometric automorphism of a Banach space X.

(i) If $Y \subseteq X$ is an α-invariant closed subspace then $\mathrm{hc}(\alpha|_Y) \leq \mathrm{hc}(\alpha)$ (monotonicity).
(ii) If $\{\Omega_\lambda\}_{\lambda\in\Lambda}$ is an increasing net in $\mathcal{P}_{\mathrm{f}}(X)$ such that $\bigcup_{\lambda\in\Lambda}\bigcup_{n\in\mathbb{Z}} \alpha^n(\Omega_\lambda)$ is total in X then $\mathrm{hc}(\alpha) = \sup_\lambda \mathrm{hc}(\alpha, \Omega_\lambda)$,
(iii) For every $k \in \mathbb{Z}$ we have $\mathrm{hc}(\alpha^k) = |k|\mathrm{hc}(\alpha)$.

Proof. Monotonicity follows from the fact that the restriction of a CA embedding to a closed subspace is a CA embedding, while for (ii) and (iii) we can proceed as in the proofs of Propositions 1.3 and 3.4, respectively, in [53]. □

Proposition 2.2. Let $X_1, \dots, X_r$ be Banach spaces with respective isometric automorphisms $\alpha_1, \dots, \alpha_r$. Then for the isometric automorphism $\alpha_1 \oplus \cdots \oplus \alpha_r$ of the ℓ_∞-direct sum $(X_1 \oplus \cdots \oplus X_r)_\infty$ we have

$$\mathrm{hc}(\alpha_1 \oplus \cdots \oplus \alpha_r) = \max_{1\leq i\leq r} \mathrm{hc}(\alpha_i).$$

Proof. The inequality $\mathrm{hc}(\alpha_1 \oplus \cdots \oplus \alpha_r) \geq \max_{1\leq i\leq r} \mathrm{hc}(\alpha_i)$ is a consequence of monotonicity (Proposition 2.1(i)), while the reverse inequality is readily seen using the fact that an ℓ_∞-direct sum of CA embeddings is a CA embedding. □

We also see by applying Proposition 2.2 in conjunction with Proposition 2.1(ii) that the CA entropy of a c_0-direct sum of isometric automorphisms is equal to the supremum of the CA entropies of the summands.

3. Topological and geometric characterizations of positive CA entropy

Let K be a compact Hausdorff space and $T : K \to K$ a homeomorphism. Recall that the *topological entropy* of T, denoted $\mathrm{h}_{\mathrm{top}}(T)$, is defined as the supremum over all finite open covers $\mathcal{W}$ of K of the quantities

$$\lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{W} \vee T^{-1}\mathcal{W} \vee \cdots \vee T^{-(n-1)}\mathcal{W}),$$

where $N(\cdot)$ denotes the smallest cardinality of a subcover [1] (see [13, 31, 54] for general references). The topological entropy of T may be equivalently expressed in terms of separated and spanning sets [5, 42] as follows.

Denote by $\mathcal{U}$ the unique uniformity compatible with the topology on K, i.e., the collection of all neighbourhoods of the diagonal in $K \times K$. Let $Q \subseteq K$ be a compact subset, and let $U \in \mathcal{U}$. A set $E \subseteq K$ is *(n, U)-separated (with respect to T)* if for every $s, t \in E$ with $s \neq t$ there exists a $0 \leq k \leq n-1$ such that $(T^k s, T^k t) \notin U$. A set $E \subseteq K$ is *(n, U)-spanning for Q (with respect to T)* if for every $s \in Q$ there is a $t \in E$ such that $(T^k s, T^k t) \in U$ for each $k = 0, \dots, n-1$. Denote by $\mathrm{sep}_n(T, Q, U)$ the largest cardinality of an (n, U)-separated subset of Q and by $\mathrm{spn}_n(T, Q, U)$ the smallest cardinality of an (n, U)-spanning set for Q. When $Q = K$ we simply write $\mathrm{sep}_n(T, U)$ and $\mathrm{spn}_n(T, U)$. We then have, by the same argument as in [5],

$$\sup_{U\in\mathcal{U}} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{sep}_n(T, Q, U) = \sup_{U\in\mathcal{U}} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{spn}_n(T, Q, U).$$

We also obtain the same quantity by taking either of the suprema over any given base for $\mathcal{U}$, the most important of which for our purposes is the collection consisting of the sets $U_{d,\varepsilon} = \{(s, t) \in K \times K : d(s, t) < \varepsilon\}$ where d is a continuous pseudometric on K and $\varepsilon > 0$. This quantity, which is invariant under conjugacy by homeomorphisms, we denote by $\mathrm{h}_{\mathrm{top}}(T, Q)$. When $Q = K$ it can be shown, as in [5], that we recover the topological entropy $\mathrm{h}_{\mathrm{top}}(T)$.

In the following proposition $\mathrm{ht}(\cdot)$ denotes Voiculescu-Brown entropy [53, 10].

Proposition 3.1. Let K be a compact Hausdorff space and $T : K \to K$ a homeomorphism. Let α_T be the *-automorphism of $C(K)$ given by $\alpha_T(f) = f \circ T$ for all $f \in C(K)$. Then $\mathrm{ht}(\alpha_T) = \mathrm{hc}(\alpha_T) = \mathrm{h}_{\mathrm{top}}(T)$.

Proof. The identity map from $C(K)$ to itself is a CA embedding, and the partition of unity argument in the proof of Proposition 4.8 in [53] demonstrates that $\mathrm{hc}(\alpha_T) \leq \mathrm{h}_{\mathrm{top}}(T)$. The inequality $\mathrm{ht}(\alpha_T) \leq \mathrm{hc}(\alpha_T)$ follows from the formulation of Voiculescu-Brown entropy in terms of completely contractive linear maps [48] and the observation that the identity on $C(K)$ is a nuclear embedding (see Section 4) along with the fact that a contractive linear map from an operator space into a commutative C^*-algebra is automatically completely contractive [45, Thm. 3.8]. It thus remains to show that $\mathrm{h}_{\mathrm{top}}(T) \leq \mathrm{ht}(\alpha_T)$. This holds when K is metrizable by Proposition 4.8 of [53], and we can reduce the general case to the metrizable one as follows.

Let $\delta > 0$. Then there is a neighbourhood U of the diagonal in $K \times K$ such that

$$\mathrm{h}_{\mathrm{top}}(T) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{sep}_n(T, U) + \delta.$$

We may assume that U is of the form $U_{d,\varepsilon}$ for some $\varepsilon > 0$ and pseudometric d of the form

$$d(s,t) = \sup_{f\in\Omega} |f(s) - f(t)|$$

for some $\Omega \in \mathcal{P}_f(C(K))$. Let A be the unital C^*-subalgebra of $C(K)$ generated by $\bigcup_{k\in\mathbb{Z}} \alpha_T^k(\Omega)$. Then A is separable and α_T-invariant. By separability we can enlarge Ω to a compact total subset Γ of the unit ball of A and define a metric

$$d'(s,t) = \sup_{f\in\Gamma} |f(s) - f(t)|$$

on the spectrum of A which is compatible with the weak* topology, and with respect to this metric the homeomorphism S of the spectrum of A induced from $\alpha_T|_A$ evidently satisfies $\mathrm{sep}_n(S, U_{d',\varepsilon}) \geq \mathrm{sep}_n(T, U_{d,\varepsilon})$ for all $n \in \mathbb{N}$. Knowing that the desired inequality holds in the metrizable case and applying monotonicity we thus obtain

$$\mathrm{h_{top}}(T) \leq \mathrm{h_{top}}(S) + \delta \leq \mathrm{ht}(\alpha_T|_A) + \delta \leq \mathrm{ht}(\alpha_T) + \delta,$$

completing the proof. □

Lemma 3.2. Let X be a Banach space. Let $\Omega = \{x_1, \dots, x_n\} \subseteq X$ and suppose that the linear map $\gamma : \ell_1^n \to X$ sending the ith standard basis element of ℓ_1^n to x_i for each $i = 1, \dots, n$ is an isomorphism. Let $\delta > 0$ be such that $\delta < \|\gamma^{-1}\|^{-1}$. Then

$$\log \mathrm{rc}(\Omega, \delta) \geq na\|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1} - \delta)^2$$

where $a > 0$ is a universal constant.

Proof. Let $\iota : X \to Y$ be a CA embedding, and suppose $(\phi, \psi, d) \in \mathrm{CA}(\iota, \Omega, \delta)$. For any linear combination $\sum c_i x_i$ of the elements $x_1, \dots, x_n$ we have

$$\begin{aligned}
\Big\| \sum c_i x_i \Big\| &\leq \Big\| \iota\Big(\sum c_i x_i\Big) - (\psi \circ \phi)\Big(\sum c_i x_i\Big) \Big\| \\
&\qquad\qquad + \Big\| (\psi \circ \phi)\Big(\sum c_i x_i\Big) \Big\| \\
&\leq \delta \sum |c_i| + \Big\| \phi\Big(\sum c_i x_i\Big) \Big\| \\
&\leq \delta \|\gamma^{-1}\| \Big\| \sum c_i x_i \Big\| + \Big\| \phi\Big(\sum c_i x_i\Big) \Big\|
\end{aligned}$$

and so $\big\|\phi\big(\sum c_i x_i\big)\big\| \geq (1 - \delta\|\gamma^{-1}\|)\big\|\sum c_i x_i\big\|$. Since ϕ is contractive, it follows that the composition $\phi \circ \gamma$ is a $\|\gamma\|(\|\gamma^{-1}\|^{-1} - \delta)^{-1}$-isomorphism onto its image in ℓ_∞^d. The desired conclusion now follows from the ℓ_∞^d version of Lemma 3.1 in [36], which can be deduced by the same kind of argument using (Rademacher) type 2 constants, or as an immediate consequence by viewing ℓ_∞^d as the diagonal in the matrix algebra M_d. □

We remark in passing that Lemma 3.2 shows that the CA entropy is infinite for the universal separable unital C^*-dynamical system, i.e., the shift on the infinite full free product $(C(\mathbb{T})^{*\mathbb{N}})^{*\mathbb{Z}}$, since the set of canonical unitary generators is isometrically isomorphic to the standard basis of ℓ_1 (cf. [47, Sect. 8]).

The following lemma is well known in Banach space theory and follows readily from Théorème 5 of [44], as indicated in the proof of Lemma 3.2 in [36].

Lemma 3.3. For every $\varepsilon > 0$ and $\lambda > 0$ there exist $d > 0$ and $\delta > 0$ such that the following holds for all $n \in \mathbb{N}$: if $S \subseteq B_1(\ell_\infty^n)$ is a symmetric convex set which contains an ε-separated set F of cardinality at least $e^{\lambda n}$ then there is a subset $I_n \subseteq \{1, 2, \dots, n\}$ with cardinality at least dn such that

$$B_\delta(\ell_\infty^{I_n}) \subseteq \pi_n(S),$$

where $\pi_n : \ell_\infty^n \to \ell_\infty^{I_n}$ is the canonical projection.

Before coming to the statement of the main result of this section we introduce and recall some terminology and notation. By saying that a set Δ in a Banach space is *equivalent* to the standard basis E of ℓ_1^I for some index set I we mean that there is a bijection $E \to \Delta$ which extends to an isomorphism $\gamma : \ell_1^I \to \overline{\mathrm{span}}\Delta$. We also say *$K$-equivalent* if $\|\gamma\|\|\gamma^{-1}\| \leq K$.

Definition 3.4. Let X be a Banach space and $\alpha \in \mathrm{IA}(X)$. Let $x \in X$. We say that a subset $I \subseteq \mathbb{Z}$ is an *ℓ_1-isomorphism set for x* if $\{\alpha^i(x) : i \in I\}$ is equivalent to the standard basis of ℓ_1^I.

Given an isometric automorphism α of a Banach space X, we denote by T_α the weak* homeomorphism of the unit ball $B_1(X^*)$ of the dual of X given by $T_\alpha(\omega) = \omega \circ \alpha$. Recall that the *upper density* of a set $I \subseteq \mathbb{Z}$ is defined as

$$\limsup_{n\to\infty} \frac{|I \cap \{-n, -n+1, \dots, n\}|}{2n+1},$$

and if these ratios converge then the limit is referred to as the *density* of I.

Theorem 3.5. Let X be a Banach space and $\alpha \in \mathrm{IA}(X)$. Let Z be a closed T_α-invariant subset of $B_1(X^*)$ such that the natural linear map $X \to C(Z)$ is an isomorphism from X to a (closed) linear subspace of $C(Z)$. Then the following are equivalent:

(1) $\mathrm{hc}(\alpha) > 0$,
(2) $\mathrm{h}_{\mathrm{top}}(T_\alpha) > 0$,
(3) $\mathrm{h}_{\mathrm{top}}(T_\alpha) = \infty$,
(4) there exist an $x \in X$, constants $K \geq 1$ and $d > 0$, a sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ tending to infinity, and sets $I_k \subseteq \{0, 1, \dots, n_k - 1\}$ of cardinality at least dn_k such that $\{\alpha^i(x) : i \in I_k\}$ is K-equivalent to the standard basis of $\ell_1^{I_k}$ for each $k \in \mathbb{N}$,
(5) there exists an $x \in X$ with an ℓ_1-isomorphism set of positive density,
(6) $\mathrm{h}_{\mathrm{top}}(T_\alpha|_Z) > 0$.

We may moreover take x in (4) and (5) to be in any given total subset Δ of X.

Proof. (1)⇒(2). The *-automorphism of $C(B_1(X^*))$ induced from T_α has positive CA entropy in view of the canonical equivariant isometric embedding $X \hookrightarrow C(B_1(X^*))$ and monotonicity. Thus $\mathrm{h}_{\mathrm{top}}(T_\alpha) > 0$ by Proposition 3.1.

(2)⇒(4). Without loss of generality we may assume that $\Delta \subseteq B_1(X)$. By assumption there exist a continuous pseudometric d on $B_1(X^*)$, $\varepsilon > 0$, $\lambda > 0$, a sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ tending to infinity, and an $(n_k, U_{d,\varepsilon})$-separated set $E_k \subseteq B_1(X^*)$ with cardinality at least $e^{\lambda n_k}$ for each $k \in \mathbb{N}$. By a simple approximation argument we may assume that there is a finite subset Ω of Δ such that

$$d(\sigma, \omega) = \sup_{x\in\Omega} |\sigma(x) - \omega(x)|$$

for all $\sigma, \omega \in B_1(X^*)$.

Define a linear map $\phi : X^* \to \ell_\infty^{\Omega\times n_k}$ by $(\phi(f))_{x,i} = f(\alpha^i(x))$, where the standard basis of $\ell_\infty^{\Omega\times n_k}$ is indexed by $\Omega \times \{0, \ldots, n_k - 1\}$. Then ϕ is a contraction and $\phi(E_k)$ is ε-separated. By Lemma 3.3 there exist $d > 0$ and $\delta > 0$ depending only on ε and λ such that for every $k \geq 1$ there is a set $J_k \subseteq \Omega \times \{0, \ldots, n_k - 1\}$ with

(i) $|J_k| \geq d|\Omega|n_k$, and
(ii) $\pi(\phi(B_1(X^*))) \supseteq B_\delta(\ell_\infty^{J_k})$, where $\pi : \ell_\infty^{\Omega\times n_k} \to \ell_\infty^{J_k}$ is the canonical projection.

Then for every $k \geq 1$ there exist some $x_k \in \Omega$ and a set $I_k \subseteq \{0, \ldots, n_k - 1\}$ such that $|I_k| \geq dn_k$ and $\{x_k\} \times I_k \subseteq J_k$. Consequently $\pi'(\phi(B_1(X^*))) \supseteq B_\delta(\ell_\infty^{I_k})$, where $\pi' : \ell_\infty^{\Omega\times n_k} \to \ell_\infty^{I_k}$ is the canonical projection. The dual $(\pi' \circ \phi)^*$ is an injection of $(\ell_\infty^{I_k})^* = \ell_1^{I_k}$ into X^{**} and the norm of the inverse of this injection is bounded above by δ^{-1}. Notice that $X \subseteq X^{**}$, and from our definition of ϕ it is clear that $(\pi' \circ \phi)^*$ sends the standard basis element of $\ell_1^{I_k}$ associated with $i \in I_k$ to $\alpha^i(x_k)$.

Since Ω is a finite set, there is an $x \in \Omega$ such that $x_k = x$ for infinitely many k. By taking a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ if necessary we may assume that $x_k = x$ for all $k \in \mathbb{N}$, and so we obtain (4).

(5)⇒(1). This follows from Lemma 3.2.

(4)⇒(3). Multiplying x by a scalar we may assume that $\|x\| = 1$. On $B_1(X^*)$ define the weak* continuous pseudometric

$$d(\sigma, \omega) = |\sigma(x) - \omega(x)|.$$

Denote $\mathrm{span}\{\alpha^i(x) : i \in I_k\}$ by V_k, and let γ_k denote the linear map from $\ell_1^{I_k}$ to V_k sending the standard basis element of $\ell_1^{I_k}$ associated with $i \in I_k$ to $\alpha^i(x)$. For each $f \in (\ell_1^{I_k})^*$ we have $(\gamma_k^{-1})^*(f) \in V_k^*$ and $\|(\gamma_k^{-1})^*(f)\| \leq K\|f\|$. By the Hahn-Banach theorem we may extend $(\gamma_k^{-1})^*(f)$ to an element in X^* of norm at most $K\|f\|$, which we will still denote by $(\gamma_k^{-1})^*(f)$. Let $0 < \varepsilon < (2K)^{-1}$, and let $M = \lfloor (2K\varepsilon)^{-1} \rfloor$ be the largest integer no greater than $(2K\varepsilon)^{-1}$. Let $\{g_i : i \in I_k\}$ be the standard basis of $(\ell_1^{I_k})^* = \ell_\infty^{I_k}$. For each $f \in \{1, \ldots, M\}^{I_k}$ set $\tilde{f} = \sum_{i\in I_k} 2f(i)\varepsilon g_i$. Then $f' := (\gamma_k^{-1})^*(\tilde{f})$ is in $B_1(X^*)$.

We claim that the set $\{f' : f \in \{1, \ldots, M\}^{I_k}\}$ is $(n_k, U_{d,\varepsilon})$-separated. Suppose $f, g \in \{1, \ldots, M\}^{I_k}$ and $f(i) < g(i)$ for some $i \in I_k$. Then

$$\begin{aligned} d(T_\alpha^i(f'), T_\alpha^i(g')) &= |(T_\alpha^i(f'))(x) - (T_\alpha^i(g'))(x)| \\ &= |f'(\alpha^i(x)) - g'(\alpha^i(x))| \\ &= 2(g(i) - f(i))\varepsilon \end{aligned}$$

$$> \varepsilon,$$

establishing our claim. Therefore $\mathrm{sep}_{n_k}(T_\alpha, \varepsilon) \geq M^{|I_k|} \geq M^{dn_k}$. It follows that $\mathrm{h_{top}}(T_\alpha) \geq d \log M$. Letting $\varepsilon \to 0$ we get $\mathrm{h_{top}}(T_\alpha) = \infty$.

(3)⇒(2). Trivial.

(4)⇒(5). Let Y_x be the collection of sets $I \subseteq \mathbb{Z}$ such that the linear map from ℓ_1^I to $\overline{\mathrm{span}}\{\alpha^i(x) : i \in I\}$ which sends the standard basis element of ℓ_1^I associated with $i \in I$ to $\alpha^i(x)$ is a K-isomorphism. If we identify subsets of $\mathbb{Z}$ with elements of $\{0,1\}^{\mathbb{Z}}$ via their characteristic functions then Y_x is a closed shift-invariant subset of $\{0,1\}^{\mathbb{Z}}$. It follows by the argument in the second paragraph of the proof of Theorem 3.2 in [26] that Y_x has an element J with density at least d. Clearly J is an ℓ_1-isomorphism set for x.

(6)⇒(2). Trivial.

(5)⇒(6). By (5) we can find an $x \in X$ with an ℓ_1-isomorphism set of positive density. Then the image of x under the natural map $X \to C(Z)$ has the same ℓ_1 isomorphism set of positive density with respect to the induced automorphism α' of $C(Z)$. Since (5) implies (1) we get $\mathrm{hc}(\alpha') > 0$. Then (6) follows from Proposition 3.1. □

Remark 3.6. It is easily seen that for the subset Δ we need in fact only assume that $\bigcup_{j\in\mathbb{Z}} \alpha^j(\Delta)$ is total in X. Furthermore, if α is a *-automorphism of a unital commutative C^*-algebra then it suffices that $\bigcup_{j\in\mathbb{Z}} \alpha^j(\Delta)$ generate the C^*-algebra, since in the proof of (2)⇒(4) the sets E_k may be taken to lie in the pure state space (apply (2)⇒(1) and Proposition 3.1) and in this case pure states are multiplicative, so that we may choose the set Ω to lie in a given Δ of the desired type.

Remark 3.7. Combining Theorem 3.5 with Lemma 4.1 yields an alternate proof of Theorem 3.3 in [36].

Corollary 3.8. Let X_1 and X_2 be Banach spaces and let $\alpha_1 \in \mathrm{IA}(X_1)$ and $\alpha_2 \in \mathrm{IA}(X_2)$. Suppose there is an isomorphism $\gamma : X_1 \to X_2$ such that $\gamma \circ \alpha_1 = \alpha_2 \circ \gamma$. Then $\mathrm{hc}(\alpha_1) = 0$ if and only if $\mathrm{hc}(\alpha_2) = 0$.

Since a Banach space with separable dual contains no isomorphic copy of ℓ_1, we have:

Corollary 3.9. Every isometric automorphism of a Banach space with separable dual has zero CA entropy.

It follows, for example, that ℓ_p for $1 < p < \infty$, c_0, and the compact operators on a separable Hilbert space admit no isometric automorphisms with nonzero CA entropy. This is also true for the nonseparable versions of these spaces since the closed subspace dynamically generated by a finite subset lies within a copy of the corresponding separable version.

Since topological entropy is nonincreasing under passing to closed invariant sets, by Theorem 3.5 zero CA entropy is preserved under taking quotients:

Corollary 3.10. Let X be a Banach space, $Y \subseteq X$ a closed subspace, and $Q : X \to X/Y$ the quotient map. Let α be an isometric automorphism of X such that

$\alpha(Y) = Y$, and suppose that α has zero CA entropy. Then the induced isometric automorphism of X/Y has zero CA entropy.

The following corollary extends [26, Thm. A] to the nonmetrizable case.

Corollary 3.11. Let K be a compact Hausdorff space and $T : K \to K$ a homeomorphism. Let S_T be the induced homeomorphism of the space of probability measures on K with the weak* topology. Then $\mathrm{h_{top}}(T) = 0$ if and only if $\mathrm{h_{top}}(S_T) = 0$.

Corollary 3.12. Let K be a compact Hausdorff space and $T : K \to K$ a homeomorphism. Then $\mathrm{h_{top}}(T) = 0$ if and only if $\mathrm{h_{top}}(T') = 0$ for all metrizable factors T' of T.

Proof. The "only if" part is immediate as topological entropy is nonincreasing under taking factors. The "if" part follows from Theorem 3.5, Proposition 3.1, and the fact that there is a one-to-one correspondence between α_T-invariant unital separable C^*-subalgebras of $C(K)$ and metrizable factors of T, where α_T is the induced automorphism of $C(K)$. □

Corollary 3.13. Let I be a nonempty index set and for each $i \in I$ let α_i be an isometric automorphism of a Banach space X_i with $\mathrm{hc}(\alpha_i) = 0$. Let $1 \leq p < \infty$, and consider the isometric automorphism α of the ℓ_p-direct sum $\big(\bigoplus_{i\in I} X_i\big)_p$ given by $\alpha((x_i)_{i\in I}) = (\alpha_i(x_i))_{i\in I}$. Then $\mathrm{hc}(\alpha) = 0$.

Proof. By Proposition 2.1(ii) we may assume that I is finite. Then the formal identity map from $\big(\bigoplus_{i\in I} X_i\big)_p$ to the ℓ_∞-direct sum $\big(\bigoplus_{i\in I} X_i\big)_\infty$ is a linear contraction with bounded inverse (of norm $|I|^{1/p}$), and so we obtain the desired conclusion from Proposition 2.2 and Corollary 3.8. □

Corollary 3.14. Let (S, Σ, μ) be a σ-finite measure space and let α be an isometric automorphism of a Banach space X with $\mathrm{hc}(\alpha) = 0$. Let $1 \leq p < \infty$, and consider the isometric automorphism β of the space $L_p(S, \Sigma, \mu, X)$ given by $(\beta(f))(s) = \alpha(f(s))$. Then $\mathrm{hc}(\beta) = 0$.

Proof. Recall that $L_p(S, \Sigma, \mu, X)$ is the Banach space of all (equivalence classes of) strongly μ-measurable functions f from S into X such that the norm

$$\|f\|_p := \Big(\int_S \|f(s)\|^p \, d\mu(s) \Big)^{1/p}$$

is finite [55, Chapter V] [14, Chapter II]. The set of functions of the form $f = x\chi_E$, where $x \in X$ and χ_E is the characteristic function of some $E \in \Sigma$ with $\mu(E) < \infty$, is total in $L_p(S, \Sigma, \mu, X)$, and so the desired conclusion follows. □

In view of Theorem 3.5 we might ask whether there exists a Banach space which contains ℓ_1 isometrically but does not admit an isometric automorphism with positive CA entropy. We claim that $C(\mathbb{T})$ is such an example. By a theorem of Banach and Mazur $C(\mathbb{T})$ contains ℓ_1 isometrically (this result is usually stated for the Cantor set Δ or the unit interval, but we can embed $C(\Delta)$ into $C(\mathbb{T})$ isometrically by viewing Δ as a subset of $\mathbb{T}$ and extending functions linearly on the complement). It is also well known that $\mathbb{T}$ admits no homeomorphisms with positive topological entropy (see [54]). Thus we need only appeal to the following observation and proposition.

Let K be a compact Hausdorff space and α an isometric automorphism of $C(K)$. By the Banach-Stone theorem there are a unitary $w \in C(K)$ and a homeomorphism $T : K \to K$ such that $\alpha(f) = w(f \circ T)$ for all $f \in C(K)$.

Proposition 3.15. Let K, α, w, and T be as above. Then

$$\mathrm{hc}(\alpha) \leq \mathrm{h}_{\mathrm{top}}(T).$$

Also, if $\mathrm{hc}(\alpha) = 0$ then $\mathrm{h}_{\mathrm{top}}(T) = 0$.

Proof. We denote by E_K the weak* compact set of extreme points of $B_1(C(K)^*)$, whose elements are of the form $\lambda\delta_t$ where $\lambda \in \mathbb{T}$ and δ_t is the point mass at some $t \in K$. Let $\pi : E_K \to K$ be the continuous surjection $\lambda\delta_t \mapsto t$. Denoting by T' the homeomorphism of E_K given by $T'(\omega) = \omega \circ \alpha$, we then have a commutative diagram

$$\begin{array}{ccc} E_K & \xrightarrow{T'} & E_K \\ \pi \big\downarrow & & \big\downarrow \pi \\ K & \xrightarrow{T} & K. \end{array}$$

Given a continuous pseudometric d on K, we define a continuous pseudometric e on E_K by $e(\eta\delta_s, \lambda\delta_t) = \max(|\eta - \lambda|, d(s,t))$ for all $\eta, \lambda \in \mathbb{T}$ and $s, t \in K$. Then for a given $t \in K$ the map T' is isometric on $\pi^{-1}(t)$ with respect to e. Since the collection of pseudometrics e arising as above generate the weak* topology on E_K we infer that $\mathrm{h}_{\mathrm{top}}(T', \pi^{-1}(t)) = 0$. Applying Theorem 17 of [5] (see Lemma 6.1) and the fact that topological entropy does not increase under taking factors, we thus have

$$\mathrm{h}_{\mathrm{top}}(T) \leq \mathrm{h}_{\mathrm{top}}(T') \leq \mathrm{h}_{\mathrm{top}}(T) + \sup_{t \in K} \mathrm{h}_{\mathrm{top}}(T', \pi^{-1}(t)) = \mathrm{h}_{\mathrm{top}}(T)$$

so that $\mathrm{h}_{\mathrm{top}}(T') = \mathrm{h}_{\mathrm{top}}(T)$.

Let $\iota : C(K) \to C(E_K)$ be the isometric embedding given by $\iota(f)(\omega) = \omega(f)$ for all $f \in C(K)$ and $\omega \in E_K$, and β the *-automorphism of $C(E_K)$ given by $\beta(g)(\omega) = g(T'\omega)$ for all $g \in C(E_K)$ and $\omega \in E_K$. Then α may be viewed as the restriction of β to $\iota(C(K))$. By monotonicity, Proposition 3.1, and the previous paragraph we thus obtain

$$\mathrm{hc}(\alpha) \leq \mathrm{hc}(\beta) = \mathrm{h}_{\mathrm{top}}(T') = \mathrm{h}_{\mathrm{top}}(T).$$

The second assertion of the proposition follows from Theorem 3.5 in view of the fact that T' is a restriction of the induced homeomorphism T_α of $B_1(C(K)^*)$. ☐

4. Comparisons with matricial approximation entropies

We briefly examine here the relation between CA entropy and its matricial analogue for exact operator spaces, which we call CCA entropy (completely contractive approximation entropy). The latter was introduced for *-automorphisms of exact C^*-algebras in [48] in which case it was shown to coincide with Voiculescu-Brown entropy [48, Thm. 3.7]. For general references on operator spaces see [17, 47].

Let X and Y be operator spaces and $\gamma : X \to Y$ a bounded linear map. For each $\Omega \in \mathcal{P}_f(X)$ and $\delta > 0$ we denote by $\mathrm{CCA}(\gamma, \Omega, \delta)$ the collection of triples (ϕ, ψ, B) where B is a finite-dimensional C^*-algebra and $\phi : X \to B$ and $\psi : B \to Y$ are completely contractive linear maps such that

$$\|\psi \circ \phi(x) - \gamma(x)\| < \delta$$

for all $x \in \Omega$. By a *nuclear embedding* of an operator space X we mean a completely isometric linear map ι from X to an operator space Y such that $\mathrm{CCA}(\iota, \Omega, \delta)$ is nonempty for every $\Omega \in \mathcal{P}_f(X)$ and $\delta > 0$. An operator space is exact (in the sense of [17]) if and only if it admits a nuclear embedding [38, 16].

Let X be an exact operator space and $\iota : X \to Y$ a nuclear embedding. For each $\Omega \in \mathcal{P}_f(X)$ and $\delta > 0$ we set

$$\mathrm{rcc}(\Omega, \delta) = \inf\{\mathrm{rank}\, B : (\phi, \psi, B) \in \mathrm{CA}(\iota, \Omega, \delta)\}.$$

As in the contractive approximation setting this quantity is independent of the nuclear embedding, which can be seen in the same way using the operator injectivity of finite-dimensional C^*-algebras (i.e., Wittstock's extension theorem).

Let α be a completely isometric automorphism of X. We set

$$\begin{aligned}
\mathrm{hcc}(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcc}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta), \\
\mathrm{hcc}(\alpha, \Omega) &= \sup_{\delta>0} \mathrm{hcc}(\alpha, \Omega, \delta), \\
\mathrm{hcc}(\alpha) &= \sup_{\Omega\in\mathcal{P}_f(X)} \mathrm{hcc}(\alpha, \Omega)
\end{aligned}$$

and refer to the last quantity as the *completely contractive approximation entropy* or simply *CCA entropy* of α. As mentioned above, this coincides with Voiculescu-Brown entropy when X is an exact C^*-algebra [48, Thm. 3.7].

The following is the matricial version of Lemma 3.2, obtained by applying Lemma 3.1 of [36].

Lemma 4.1. Let X be an exact operator space. Let $x_1, \dots, x_n \in X$ and suppose that the linear map $\gamma : \ell_1^n \to X$ sending the ith standard basis element of ℓ_1^n to x_i for each $i = 1, \dots, n$ is an isomorphism. Let $\delta > 0$ be such that $\delta < \|\gamma^{-1}\|^{-1}$. Then

$$\log \mathrm{rcc}(\Omega, \delta) \geq na\|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1} - \delta)^2$$

where $a > 0$ is a universal constant.

Proposition 4.2. Let α be a completely isometric automorphism of an exact operator space X. Suppose that $\mathrm{hc}(\alpha) > 0$. Then $\mathrm{hcc}(\alpha) > 0$.

Proof. Apply Theorem 3.5 and Lemma 4.1. □

In general there is no inequality relating CA and CCA entropy. For instance, the tensor product shift on $M_p^{\otimes\mathbb{Z}}$ has infinite CA entropy (Theorem 9.3) but CCA entropy equal to $\log p$ [53, Prop. 4.7], while the following example shows that it is possible to have zero CA entropy in conjunction with positive CCA entropy.

Example 4.3. The CAR algebra A may be described as the universal unital C^*-algebra generated by self-adjoint unitaries $\{u_k\}_{k\in\mathbb{Z}}$ subject to the anticommutation relations

$$u_i u_j + u_j u_i = 0$$

for all $i, j \in \mathbb{Z}$ with $i \neq j$. We define a *-automorphism α of A by setting $\alpha(u_k) = u_{k+1}$ for all $k \in \mathbb{Z}$. This is an example of a bitstream C^*-dynamical system as studied in [27] (see also [49]). Let W be the operator space obtained as closure of the span of $\{u_k\}_{k\in\mathbb{Z}}$ in A. Then α restricts to a completely isometric automorphism of W, which we will denote by β.

Proposition 4.4. We have $\mathrm{hcc}(\beta) > 0$ while $\mathrm{hc}(\beta) = 0$.

Proof. For any $n \in \mathbb{N}$ the unitaries $u_1, \dots, u_n$ can be realized as tensor products of Pauli matrices (see [30, Sect. 2] and [47, Sect. 9.3]) from which it can be seen that the subset $\{u_1 \otimes u_1, \dots, u_n \otimes u_n\}$ of $A \otimes A$ is isometrically equivalent to the standard basis of ℓ_1^n over $\mathbb{R}$ and hence 2-equivalent to the standard basis of ℓ_1^n over $\mathbb{C}$. It follows by Lemma 4.1 that $\mathrm{hcc}(\alpha \otimes \alpha, \{u_1 \otimes u_1\}) > 0$. Since $\mathrm{hcc}(\alpha \otimes \alpha, \{u_1 \otimes u_1\}) \leq 2\,\mathrm{hcc}(\alpha, \{u_1\})$ by the local tensor product subadditivity of CCA entropy (cf. the proof of Proposition 3.10 in [53]), we obtain $\mathrm{hcc}(\beta) \geq \mathrm{hcc}(\beta, \{u_1\}) = \mathrm{hcc}(\alpha, \{u_1\}) > 0$.

Next we recall that the set $\{u_k\}_{k\in I}$ is equivalent to the standard basis of ℓ_2 (cf. [47, Sect. 9.3]). Indeed given a finite set $F \subseteq \mathbb{Z}$ and real numbers c_k for $k \in F$, the anticommutation relations between the u_i's yield $\big(\sum_{k\in F} c_k u_k\big)^2 = \sum_{k\in F} c_k^2 \cdot 1$ and hence

$$\Big\| \sum_{k\in F} c_k u_k \Big\|^2 = \Big\| \Big(\sum_{k\in F} c_k u_k \Big)^2 \Big\| = \sum_{k\in F} c_k^2,$$

from which it follows that $\{u_k\}_{k\in F}$ is isometrically equivalent over the real numbers to the standard basis of $(\ell_2^F)_{\mathbb{R}}$, and hence 2-equivalent over the complex numbers to the standard basis of ℓ_2^F. We conclude by Corollary 3.9 that $\mathrm{hc}(\beta) = 0$. $\square$

The above example is not a C^*-algebra automorphism, however, and so we ask the following question.

Question 4.5. Is there a *-automorphism of an exact C^*-algebra for which the Voiculescu-Brown entropy is strictly greater than the CA entropy?

5. A geometric description of the topological Pinsker algebra

Let $T : K \to K$ be a homeomorphism of a compact Hausdorff space. Since zero topological entropy is preserved under taking products and subsystems, by a standard argument (see Corollary 2.9(1) of [24]) T admits a largest factor with zero entropy, which we will refer to as the *topological Pinsker factor*, following [23]. The corresponding C^*-algebra will be called the *topological Pinsker algebra* and denoted $P_{K,T}$. It is an analogue of the Pinsker σ-algebra in ergodic theory.

In [4] F. Blanchard and Y. Lacroix constructed the topological Pinsker factor in the metrizable setting as the quotient system arising from the closed T-invariant equivalence relation on K generated by the collection of entropy pairs. Recall that an *entropy pair* is a pair $(s, t) \in K \times K$ with $s \neq t$ such that for every two-element open cover $\mathcal{U} = \{U, V\}$ with $s \in \mathrm{int}(K \setminus U)$ and $t \in \mathrm{int}(K \setminus V)$ the local topological entropy of T with respect to $\mathcal{U}$ is nonzero [2]. In [36] a description of $P_{K,T}$ for metrizable K was given in terms of local Voiculescu-Brown entropy. By applying the arguments from [36] and Section 3 we will obtain in Theorem 5.3 a geometric description of $P_{K,T}$ for general K.

For a function $f \in C(K)$ where K is a compact Hausdorff space, we denote by d_f the pseudometric on K given by

$$d_f(s,t) = |f(s) - f(t)|$$

for all $s, t \in K$. For notation relating to CCA entropy see Section 4 .

Proposition 5.1. Let K be a compact Hausdorff space and $T : K \to K$ a homeomorphism. Then for any $f \in C(K)$ the following are equivalent:

(1) $\mathrm{hcc}(\alpha_T, \{f\}) > 0$,
(2) $\mathrm{hc}(\alpha_T, \{f\}) > 0$,
(3) there exists an entropy pair $(s,t) \in K \times K$ with $f(s) \neq f(t)$,
(4) $h_{d_f}(T) > 0$,
(5) there exist $\lambda \geq 1$, $d > 0$, a sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ tending to infinity, and sets $I_k \subseteq \{0, 1, \ldots, n_k - 1\}$ of cardinality at least dn_k such that $\{\alpha_T^i(f) : i \in I_k\}$ is λ-equivalent to the standard basis of ℓ_1 for each $k \in \mathbb{N}$,
(6) f has an ℓ_1-isomorphism set of positive density.

Proof. By restricting to the α_T-invariant unital C^*-subalgebra of $C(K)$ generated by f we may assume that K is metrizable.

(1)⇒(2). Since the identity map on $C(K)$ is a nuclear embedding (see Section 4) we need only note that a contractive linear map from an operator space into a commutative C^*-algebra is automatically completely contractive [45, Thm. 3.8].

(2)⇒(3)⇒(4). These implications follow from the proofs of Theorem 4.3 and Lemma 4.2, respectively, in [36].

(4)⇒(5)⇒(6). Apply the same arguments as in the proofs of the respective implications (2)⇒(4)⇒(5) in Theorem 3.5.

(6)⇒(1). By assumption there exist a set $I \subseteq \mathbb{Z}$ of density greater than some $d > 0$ and an isomorphism $\gamma : \ell_1^I \to \overline{\mathrm{span}}\{\alpha_T^i(f) : i \in I\}$ sending the standard basis element of ℓ_1^I associated with $i \in I$ to $\alpha_T^i(f)$. Let $0 < \delta < \|\gamma^{-1}\|^{-1}$. By Lemma 4.1, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \log \mathrm{rcc}(\{\alpha_T^i(f) : i \in I \cap \{-n, -n+1, \ldots, n\}\}, \delta) \\ \geq a|I \cap \{-n, -n+1, \ldots, n\}| \|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1} - \delta)^2 \end{aligned}$$

for some universal constant $a > 0$. Since I has density greater than d and

$$\begin{aligned} \mathrm{rcc}(\{f, \alpha_T(f), \ldots, \alpha_T^{2n}(f)\}, \delta) \\ = \mathrm{rcc}(\{\alpha_T^{-n}(f), \alpha_T^{-n+1}(f), \ldots, \alpha_T^{n}(f)\}, \delta), \end{aligned}$$

we infer that

$$\log \mathrm{rcc}(\{f, \alpha_T(f), \ldots, \alpha_T^{2n}(f)\}, \delta) \geq d(2n+1)a\|\gamma\|^{-2}(\|\gamma^{-1}\|^{-1} - \delta)^2$$

for all sufficiently large $n \in \mathbb{N}$. Hence $\mathrm{hcc}(\alpha_T, \{f\}) > 0$. □

Corollary 5.2. Let A be a unital C^*-algebra and α an automorphism of A. Let α' be the automorphism of $C(S(A))$ given by $\alpha'(f)(\sigma) = f(\sigma \circ \alpha)$ for all $f \in C(S(A))$ and $\sigma \in S(A)$. Recall that there is an order isomorphism $x \mapsto \bar{x}$ from A to the affine function space $\mathrm{Aff}(S(A)) \subseteq C(S(A))$ given by $\bar{x}(\sigma) = \sigma(x)$ for all $x \in A$ and $\sigma \in S(A)$. Then for any $x \in A$ we have that $\mathrm{hc}(\alpha', \{\bar{x}\}) > 0$ implies $\mathrm{hc}(\alpha, \{x\}) > 0$.

Proof. The map $x \mapsto \bar{x}$ is a 2-isomorphism of Banach spaces which conjugates α to $\alpha'|_{\mathrm{Aff}(S(A))}$, and so we obtain the conclusion from the implication (2)⇒(6) in Proposition 5.1 and an appeal to Lemma 3.2. □

Theorem 5.3. Let $T : K \to K$ be a homeomorphism of a compact Hausdorff space. Then the topological Pinsker algebra $P_{K,T}$ is equal to the set of all $f \in C(K)$ which do not have an ℓ_1-isomorphism set of positive density.

Proof. This follows from Proposition 5.1 and Remark 3.6. □

Corollary 5.4. The homeomorphism T has positive topological entropy if and only if there is an $f \in C(K)$ with an ℓ_1-isomorphism set of positive density.

Corollary 5.5. The homeomorphism T has completely positive entropy (i.e., every nontrivial factor has positive topological entropy [3]) if and only if every nonconstant $f \in C(K)$ has an ℓ_1-isomorphism set of positive density.

Recently in [24] E. Glasner and M. Megrelishvili established a Bourgain-Fremlin-Talagrand dichotomy for metrizable topological dynamical systems according to which the enveloping semigroup either

(1) is a separable Rosenthal compactum (and hence has cardinality at most $2^{\aleph_0}$), or
(2) contains a homeomorphic copy of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ (and hence has cardinality $2^{2^{\aleph_0}}$).

A topological dynamical system (i.e., a compact space with an action of a topological group) is said to be *tame* if its enveloping semigroup is separable and Fréchet [22], which is equivalent to (1) in the context of the above dichotomy. In particular, the enveloping semigroup of a tame system has cardinality at most $2^{\aleph_0}$. Consider now a homeomorphism $T : K \to K$ of a compact Hausdorff space. If the system (K, T) is tame then it is $\mathbb{Z}$-*regular*, i.e., $C(K)$ does not contain a function f such that the orbit $\{f \circ T^n\}_{n\in\mathbb{Z}}$ admits an infinite subset equivalent to the standard basis of ℓ_1 (the $\mathbb{N}$-system version of this property is called *regularity* in [39]), for otherwise the enveloping semigroup would contain a homeomorphic copy of $\beta\mathbb{N}$ (see the proof of Corollary 5.4 in [39]). Thus Corollary 5.4 yields the following, which generalizes a result of Glasner [22], who proved it for (K, T) metrizable and minimal.

Corollary 5.6. A tame homeomorphism $T : K \to K$ of a compact Hausdorff space has zero topological entropy.

If the system (K, T) is HNS (hereditarily not sensitive) [24, Defn. 9.1] and K is metrizable then it is tame and hence has zero topological entropy by the above corollary. More generally we have:

Corollary 5.7. An HNS homeomorphism $T : K \to K$ of a compact Hausdorff space has zero topological entropy.

Proof. By Theorems 9.8 and 7.6 of [24] the system (K, T) is HNS if and only if every $f \in C(K)$ is in $\mathrm{Asp}(K)$, i.e., if and only if the pseudometric space $(K, \rho_{\mathbb{Z},f}|_K)$ is separable, where $\rho_{\mathbb{Z},f}$ is the pseudometric on $C(K)^*$ defined by

$$\rho_{\mathbb{Z},f}(\sigma, \omega) = \sup_{n \in \mathbb{Z}} |\sigma(f \circ T^n) - \omega(f \circ T^n)|.$$

By Lemma 1.5.3 of [19] the pseudometric space $(C(K)^*, \rho_{\mathbb{Z},f})$ is also separable. Thus the orbit $\{f \circ T^n\}_{n\in\mathbb{Z}}$ admits no infinite subset equivalent to the standard basis of ℓ_1, and we obtain the result by Corollary 5.4. □

Versions of Proposition 5.1 and Theorem 5.3 can be similarly established for topological sequence entropy [29, 32]. Recall that a system (K, T) consisting of a homeomorphism $T : K \to K$ of a compact Hausdorff space is said to be *null* if its topological sequence entropy is zero for all sequences. Nullness is preseved under taking products and subsystems, and so every system admits a largest null factor. In analogy with Theorem 5.3 we then have the following.

Theorem 5.8. Let $T : K \to K$ be a homeomorphism of a compact Hausdorff space. Then the largest null factor of the system (K, T), when viewed as a dynamically invariant C^*-subalgebra of $C(K)$, is equal to the set of all $f \in C(K)$ satisfying the property that for every $\lambda \geq 1$ there exists an $m \in \mathbb{N}$ such that if Ω is a subset of $\{f \circ T^n\}_{n\in\mathbb{N}}$ which is λ-equivalent to the standard basis of ℓ_1^Ω then $|\Omega| \leq m$. In particular, the system (K, T) is null if and only if the above property is satisfied by every $f \in C(K)$ (equivalently, by every f in a given $\Delta \subseteq C(K)$ such that $\bigcup_{j\in\mathbb{Z}} \alpha^j(\Delta)$ generates $C(K)$ as a C^*-algebra).

Corollary 5.9. A null homeomorphism $T : K \to K$ of a compact Hausdorff space is $\mathbb{Z}$-regular.

We also have the following analogue for nullness of [26, Thm. A] (cf. Corollary 3.11).

Theorem 5.10. A homeomorphism $T : K \to K$ of a compact Hausdorff space is null if and only if the induced weak* homeomorphism of the space $M(K)$ of probability measures on K is null.

Proof. The image of $C(K)$ in $C(M(K))$ under the equivariant map given by evaluation generates $C(M(K))$ as a C^*-algebra, and so we can apply Theorem 5.8 to obtain the nontrivial direction. □

We point out that, for minimal distal metrizable systems, nullness, $\mathbb{Z}$-regularity, and equicontinuity are equivalent. The equivalence of nullness and equicontinuity is established in [32] (see Corollaries 2.1(2) and 4.2 therein), while the equivalence of $\mathbb{Z}$-regularity and equicontinuity follows from Corollary 1.8 of [22] and the fact that tameness and $\mathbb{Z}$-regularity coincide in the metrizable case. The equivalence of nullness and $\mathbb{Z}$-regularity can also be extracted from [32]: if the system is not null then the proofs of Corollaries 4.2 and 4.1 and Lemma 3.1 in [32] show that there is a two-element open cover satisfying the property in the statement of Proposition 2.3 in

[32], yielding the existence of a real-valued continuous function on the space whose forward orbit contains a subsequence isometrically equivalent to the standard basis of ℓ_1 over $\mathbb{R}$, so that the system is not $\mathbb{Z}$-regular.

6. A dynamical characterization of type I C^*-algebras

By [43, 8] a separable unital C^*-algebra is type I if and only if every inner *-automorphism has zero Connes-Narnhofer-Thirring entropy with respect to each invariant state. The question thus arises of whether there is a topological version of this result, and indeed N. P. Brown conjectured in [8] that the analogous assertion for Voiculescu-Brown entropy also holds. One major difficulty is that the behaviour of zero Voiculescu-Brown entropy with respect to taking extensions is not well understood. For CA entropy, however, we can show that zero values persist under taking extensions by reducing the problem to a topological-dynamical one by means of Theorem 3.5. The topological-dynamical ingredient that we require is provided by the following lemma, which, using the notions of separated and spanning sets described in Section 3, can be established in the same way as its specialization to the metric setting [5, Thm. 17].

Lemma 6.1. Let K, J be compact Hausdorff spaces and $T : K \to K$, $S : J \to J$ homeomorphisms. Let $\pi : K \to J$ be a continuous surjective map such that $\pi \circ T = S \circ \pi$. Then

$$\mathrm{h_{top}}(T) \leq \mathrm{h_{top}}(S) + \sup_{s \in J} \mathrm{h_{top}}(T, \pi^{-1}(s)).$$

Lemma 6.2. Let X be a Banach space, $Y \subseteq X$ a closed subspace, and $Q : X \to X/Y$ the quotient map. Let α be an isometric automorphism of X such that $\alpha(Y) = Y$, and denote by $\bar{\alpha}$ the induced isometric automorphism of X/Y. Then $\mathrm{hc}(\alpha) = 0$ if and only if $\mathrm{hc}(\alpha|_Y) = \mathrm{hc}(\bar{\alpha}) = 0$.

Proof. The "only if" part follows from monotonicity and Corollary 3.10. For the "if" part we first observe that we have a commutative diagram

$$\begin{array}{ccc} B_1(X^*) & \xrightarrow{T_\alpha} & B_1(X^*) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ B_1(Y^*) & \xrightarrow{T_{\alpha|Y}} & B_1(Y^*) \end{array}$$

where Φ is the weak* continuous surjective map given by restriction. Lemma 6.1 then yields

$$\mathrm{h_{top}}(T_\alpha) \leq \mathrm{h_{top}}(T_{\alpha|Y}) + \sup_{\sigma \in B_1(Y^*)} \mathrm{h_{top}}(T_\alpha, \Phi^{-1}(\sigma)).$$

By Theorem 3.5 an isometric automorphism of a Banach space has zero CA entropy if and only if the induced homeomorphism of the unit ball of the dual has zero topological entropy, and thus, since $\mathrm{hc}(\alpha|_Y) = 0$ by hypothesis, the proof will be complete once we show that $\mathrm{h_{top}}(T_\alpha, \Phi^{-1}(\sigma)) = 0$ for all $\sigma \in B_1(Y^*)$.

So let $\sigma \in B_1(Y^*)$. Pick an $\omega \in \Phi^{-1}(\sigma)$ and define the weak* continuous map $\Psi : B_1((X/Y)^*) \to X^*$ by $\Psi(\rho) = 2\rho\circ Q + \omega$ for all $\rho \in B_1((X/Y)^*)$. Let $\{x_1, x_2, \dots, x_r\}$ be a finite subset of $B_1(X)$ and define on X^* the weak* continuous pseudometric

$$d(\eta, \tau) = \sup_{1\leq j\leq r} |\eta(x_j) - \tau(x_j)|.$$

Note that $d(c\eta, c\tau) = cd(\eta, \tau)$ and $d(\eta + \rho, \tau + \rho) = d(\eta, \tau)$ for all $\eta, \tau, \rho \in X^*$ and $c > 0$. We also define on $(X/Y)^*$ the weak* continuous pseudometric

$$\bar{d}(\eta, \tau) = \sup_{1\leq j\leq r} |\eta(Q(x_j)) - \tau(Q(x_j))|.$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$, and let $E \subseteq B_1((X/Y)^*)$ be an $(n, U_{\bar{d},\varepsilon/4})$-spanning set with respect to $T_{\bar{\alpha}}$. Now suppose $\eta \in \Phi^{-1}(\sigma)$. Since Q is a quotient map there is a $\rho \in (X/Y)^*$ such that $\rho \circ Q = (\eta - \omega)/2$ and $\|\rho\| = \|\eta - \omega\|/2 \leq 1$. We can then find a $\tau \in E$ such that

$$\bar{d}(T^k_{\bar{\alpha}}(\rho), T^k_{\bar{\alpha}}(\tau)) < \varepsilon/4$$

for each $k = 0, \dots, n - 1$. Then for each $k = 0, \dots, n - 1$ we have

$$\begin{aligned} d(T^k_\alpha(\eta), T^k_\alpha(\Psi(\tau))) &= d(T^k_\alpha(2\rho \circ Q + \omega), T^k_\alpha(2\tau \circ Q + \omega)) \\ &= 2d(T^k_\alpha(\rho \circ Q), T^k_\alpha(\tau \circ Q)) \\ &= 2\bar{d}(T^k_{\bar{\alpha}}(\rho), T^k_{\bar{\alpha}}(\tau)) \\ &< 2(\varepsilon/4) = \varepsilon/2. \end{aligned}$$

For every $\tau \in E$ we pick, if possible, an $\eta_\tau \in \Phi^{-1}(\sigma)$ such that

$$d(T^k_\alpha(\eta_\tau), T^k_\alpha(\Psi(\tau))) < \varepsilon/2$$

for each $k = 0, \dots, n-1$. For those $\tau \in E$ for which this is not possible we set $\eta_\tau = 0$. It is then easily checked that the set $F = \{\eta_\tau : \tau \in E\}$, which has cardinality at most that of E, is $(n, U_{d,\varepsilon})$-spanning with respect to T_α. Hence $\mathrm{spn}_n(T_\alpha, \Phi^{-1}(\sigma), U_{d,\varepsilon}) \leq \mathrm{spn}_n(T_{\bar{\alpha}}, U_{\bar{d},\varepsilon/4})$. Since $\mathrm{h_{top}}(T_{\bar{\alpha}}) = \mathrm{hc}(\bar{\alpha}) = 0$ by Theorem 3.5 and our hypothesis, we conclude that $\mathrm{h_{top}}(T_\alpha, \Phi^{-1}(\sigma)) = 0$, as desired. □

We also need to know that zero CA entropy is well behaved with respect to continuous fields over locally compact Hausdorff spaces.

Lemma 6.3. Let Z be a locally compact Hausdorff space, and let $(X_z)_{z\in Z}$ be a continuous field of Banach spaces over Z. Let X be the Banach space of continuous sections of $(X_z)_{z\in Z}$ vanishing at infinity. Let α be an isometric automorphism of X which arises from $\alpha_z \in \mathrm{IA}(X_z)$ for $z \in Z$. Then $\mathrm{hc}(\alpha) = 0$ if and only if $\mathrm{hc}(\alpha_z) = 0$ for all $z \in Z$.

Proof. The "only if" part follows from Corollary 3.10. Suppose then that $\mathrm{hc}(\alpha_z) = 0$ for all $z \in Z$, and let us show that $\mathrm{hc}(\alpha) = 0$. We first reduce the problem to the case Z is compact. For each compact subset $W \subseteq Z$ let X_W be the Banach space of continuous sections of the restriction field $(X_w)_{w\in W}$ of Banach spaces over W and let $Q_W : X \to X_W$ be the quotient map. The isometric automorphisms $(\alpha_w)_{w\in W}$ give us an isometric automorphism α_W of X_W. If $\mathrm{hc}(\alpha) > 0$ then by Theorem 3.5

we can find an $x \in X$ with an ℓ_1-isomorphism set of positive density. Using the fact that x vanishes at infinity, we can find a compact subset $W \subseteq Z$ such that $Q_W(x)$ has the same ℓ_1-isomorphism set of positive density under α_W. Then $\mathrm{hc}(\alpha_W) > 0$ by Theorem 3.5. Replacing Z by W, we may thus assume that Z is compact.

Next we reduce the problem to the case in which X contains an element x such that $Q_z(x)$ has norm 1 and is fixed under α_z for every $z \in Z$, where Q_z is the quotient map $X \to X_z$. Set Y_z to be the ℓ_∞-direct sum $X_z \oplus \mathbb{C}$ for each $z \in Z$. Then $(Y_z)_{z\in Z}$ is in a natural way a continuous field of Banach spaces over Z. The global section space Y of this field is the ℓ_∞-direct sum $X \oplus C(Z)$. The automorphisms α and α_z naturally extend to isometric automorphisms of Y and Y_z fixing $C(Z)$ and $\mathbb{C}$, respectively. By Lemma 6.2 we have $\mathrm{hc}(\gamma_z) = 0$ for every $z \in Z$, where γ_z is the extension of α_z. Note the constant function $1 \in C(T) \subseteq Y$ satisfies the above requirement. Replacing X by Y, we may assume that X contains such an element x.

For each $z \in Z$ denote by S_z the subset of $B_1(X_z^*)$ consisting of linear functionals σ satisfying $\sigma(Q_z(x)) \geq 1/2$. Clearly the sets $Q_z^*(S_z)$ for $z \in Z$ are pairwise disjoint. Let $S = \bigcup_{z\in Z} Q_z^*(S_z)$, and let $\psi : S \to Z$ be the map which sends $\sigma_z \in Q_z^*(S_z)$ to $z \in Z$. Denote by T_α (resp. T_{α_z}) the homeomorphism of $B_1(X^*)$ (resp. $B_1(X_z^*)$) induced by α (resp. α_z). One checks easily that ψ is surjective and continuous, and that S is a T_α-invariant closed subset of $B_1(X^*)$. By Theorem 3.5 we have $\mathrm{h_{top}}(T_{\alpha_z}|_{S_z}) = 0$. Applying Lemma 6.1 we get

$$\mathrm{h_{top}}(T_\alpha|_S) = \sup_{z\in Z} \mathrm{h_{top}}(T_\alpha, \psi^{-1}(z)) = \sup_{z\in Z} \mathrm{h_{top}}(T_{\alpha_z}|_{S_z}) = 0.$$

Now given any unit vectors v and w in a Banach space V and $r \in [0,1]$, there exists a $\sigma \in B_1(V^*)$ with $\sigma(v) \geq r$ and $|\sigma(w)| \geq (1-r)/3$ (indeed if σ is an element of $B_1(V^*)$ such that $\sigma(v) = 1$ and $|\sigma(w)| < (1-r)/3$, then choose a $\tau \in B_1(V^*)$ with $\tau(v) \geq 0$ and $|\tau(w)| = 1$ and replace σ with $(\sigma + (1-r)\tau)/\|\sigma + (1-r)\tau\|$). It follows that the natural linear map $X \to C(S)$ is an isomorphism from X to a closed linear subspace of $C(S)$. We thus conclude by Theorem 3.5 that $\mathrm{hc}(\alpha) = 0$. □

Given a C^*-algebra A we denote by $M(A)$ its multiplier algebra and by $\mathrm{Prim}(A)$ its primitive ideal space.

Theorem 6.4. Let A be a C^*-algebra. Then the following are equivalent:

(1) A is type I,
(2) $\mathrm{hc}(\alpha) = 0$ for every $\alpha \in \mathrm{Aut}(A)$ with trivial induced action on $\mathrm{Prim}(A)$,
(3) $\mathrm{hc}(\mathrm{Ad}\, u) = 0$ for every unitary $u \in M(A)$,
(4) $\mathrm{hc}(\mathrm{Ad}\, u) < \infty$ for every unitary $u \in M(A)$.

Proof. (1)⇒(2). By [46, Thm. 6.2.11] there is a composition series $(I_\rho)_{0\leq\rho\leq\mu}$ (i.e., an increasing family of closed two-sided ideals of A indexed by ordinals with $I_\mu = A$ and I_ρ equal the norm closure of $\bigcup_{\rho'<\rho} I_{\rho'}$ for any limit ordinal $\rho \leq \mu$) such that each quotient $I_{\rho+1}/I_\rho$ for $0 \leq \rho < \mu$ is a continuous trace C^*-algebra. By Lemma 6.2 and Proposition 2.1 we may therefore assume that A is a continuous trace C^*-algebra. Thus $\hat{A}$ is a locally compact Hausdorff space and A is the C^*-algebra of continuous sections vanishing at infinity of a continuous field of C^*-algebras over $\hat{A}$

with each fibre A_x equal to the compact operators $\mathcal{K}(\mathcal{H}_x)$ for some Hilbert space $\mathcal{H}_x$. By Corollary 3.9 (see the comment following it) every isometric automorphism of $\mathcal{K}(\mathcal{H}_x)$ has zero CA entropy, and so by Lemma 6.3 we conclude that $\mathrm{hc}(\alpha) = 0$.

(2)⇒(3)⇒(4). Trivial.

(4)⇒(1). Here we can apply the construction in the proof of Theorem 1.2 in [8]. Suppose that A is not type I. Denote by $\tilde{A}$ the unitization of A. Let γ be the tensor product shift on the CAR algebra $M_{2^\infty} = M_2^{\otimes\mathbb{Z}}$. Set $\beta = \gamma^{\otimes\mathbb{N}} \in \mathrm{Aut}((M_{2^\infty})^{\otimes\mathbb{N}})$. As in the proof of Theorem 1.2 in [8], we can find a unital C^*-subalgebra $B \subseteq \tilde{A}$ such that there are a unitary $u \in \tilde{A}$ and a surjective *-homomorphism $\pi : B \to (M_{2^\infty})^{\otimes\mathbb{N}}$ with $\pi\circ\mathrm{Ad}\, u = \beta\circ\pi$. For each $j \in \mathbb{N}$ let x_j be the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ viewed as an element of the zeroeth copy of M_2 in the jth copy of M_{2^∞} in $(M_{2^\infty})^{\otimes\mathbb{N}}$, and pick a $b_j \in B$ such that $\|b_j\| = 1$ and $\pi(b_j) = x_j$. Let $m \in \mathbb{N}$ and set $\Omega_m = \{b_j : j = 1, \dots, m\}$. For each $b \in \Omega_m$ write $b = \lambda_b 1 + a_b$ where $\lambda_b \in \mathbb{C}$ and $a_b \in A$, and note that $\|a_b\| \leq 2$ (otherwise a_b would be invertible).

Denote by φ the contractive linear map from $\ell_1^{\{1,\dots,m\}\times\mathbb{Z}}$ to $\overline{\mathrm{span}} \bigcup_{k\in\mathbb{Z}} \mathrm{Ad}\, u^k(\Omega_m)$ which sends standard basis elements to elements in $\bigcup_{k\in\mathbb{Z}} \mathrm{Ad}\, u^k(\Omega_m)$ respecting the indexing in the obvious way. It is easily checked that φ is a 2-isomorphism. Set $\Omega'_m = \{a_b : b \in \Omega_m\}$ and let φ' be the bounded linear map from $\ell_1^{\{1,\dots,m\}\times\mathbb{Z}}$ to $\overline{\mathrm{span}} \bigcup_{k\in\mathbb{Z}} \mathrm{Ad}\, u^k(\Omega'_m)$ which corresponds to φ on the standard basis elements via the association of $\mathrm{Ad}\, u^k(a_b)$ with $\mathrm{Ad}\, u^k(b)$. We will argue that φ' has an inverse of norm at most 2. Suppose f is a norm one element of $\ell_1^{\{1,\dots,m\}\times\{1,\dots,n\}}$. Then $g = f \oplus (-f) \in \ell_1^{\{1,\dots,m\}\times\{1,\dots,2n\}}$ has norm 2, and so $\|\varphi(g)\| \geq 1$. Since $\mathrm{Ad}\, u$ is unital we have

$$\varphi(g) = \varphi(f) - u^n\varphi(f)u^{-n} = \varphi'(f) - u^n\varphi'(f)u^{-n} = \varphi'(g)$$

and hence at least one of $\|\varphi'(f)\|$ and $\|u^n\varphi'(f)u^{-n}\|$ is greater than or equal to 1/2. Therefore $\|\varphi'(f)\| \geq 1/2$, as desired.

Having shown that φ' is an isomorphism (in fact a 4-isomorphism since $\|\varphi'\| \leq 2$), it follows from Lemma 3.2 that for a given $0 < \delta < 1/2$ we have $\mathrm{hc}(\mathrm{Ad}\, u, \Omega'_m, \delta) \geq am$ for some $a > 0$ which does not depend on m, and consequently $\mathrm{hc}(\mathrm{Ad}\, u) = \infty$. □

7. Tensor products, crossed products, and free products

We have tensor product subadditivity with respect to the injective tensor product:

Proposition 7.1. Let X_1 and X_2 be Banach spaces and let $\alpha_1 \in \mathrm{IA}(X_1)$ and $\alpha_2 \in \mathrm{IA}(X_2)$. Then the isometric automorphism $\alpha_1\check{\otimes}\alpha_2$ of $X_1\check{\otimes}X_2$ satisfies

$$\mathrm{hc}(\alpha_1\check{\otimes}\alpha_2) \leq \mathrm{hc}(\alpha_1) + \mathrm{hc}(\alpha_2).$$

Proof. Let $\iota_1 : X_1 \to Y_1$ and $\iota_2 : X_2 \to Y_2$ be CA embeddings. Then $\iota_1\check{\otimes}\iota_2 : X_1\check{\otimes}X_2 \to Y_1\check{\otimes}Y_2$ is a CA embedding, and if for given finite subsets $\Omega_1 \subseteq B_1(X_1)$ and $\Omega_2 \subseteq B_1(X_2)$ and $\delta > 0$ we have $(\phi_i, \psi_i, d_i) \in \mathrm{CA}(\iota_i, \Omega_i, \delta)$ for $i = 1, 2$ then since $\ell_\infty^{d_1}\check{\otimes}\ell_\infty^{d_2} = \ell_\infty^{d_1 d_2}$ it is readily checked that

$$(\phi_1\check{\otimes}\phi_2, \psi_1\check{\otimes}\psi_2, d_1 d_2) \in \mathrm{CA}(\iota_1\check{\otimes}\iota_2, \Omega_1 \otimes \Omega_2, 2\delta),$$

from which the result follows. □

The subadditivity of Proposition 7.1 can fail for other tensor product norms, in particular for the minimal operator space tensor product, as the following example demonstrates. Let β be the completely isometric automorphism of the operator subspace W of the CAR algebra as described in Example 4.3. The CAR algebra is *-isomorphic to the infinite tensor product UHF algebra $M_2^{\otimes\mathbb{Z}}$, and so W is exact. In Proposition 4.4 it was shown that $\mathrm{hc}(\beta) = 0$. On the other hand, applying monotonicity and using Corollary 5.5 as in the first part of the proof of Proposition 4.4 (to which we refer for notation) we have

$$\mathrm{hc}(\beta \otimes \beta) \geq \mathrm{hc}(\alpha \otimes \alpha, \{u_0 \otimes u_0\}) > 0.$$

We turn next to C^*-crossed products. All crossed products considered here will be reduced and so for economy we won't bother to tag the product symbol to indicate this, contrary to usual practice. Note however that the full and reduced crossed products coincide if the acting group is amenable, as will ultimately be the case here.

We will establish in Theorem 7.4 below an analogue of Theorem 5.3 of [48], whose proof based on Imai-Takai duality we will adapt. In our case we cannot establish equality beyond the zero entropy case unless the action is on a commutative C^*-algebra, since tensor product subadditivity only holds with respect to the injective tensor product.

The following lemma and proposition are the analogues of Lemma 2.2 and Theorem 5.2, respectively, in [48].

Lemma 7.2. Let X_1 and X_2 be Banach spaces and $\alpha_1 \in \mathrm{IA}(X_1)$ and $\alpha_2 \in \mathrm{IA}(X_2)$. Suppose there exists a net

$$X_1 \xrightarrow{S_\lambda} X_2 \xrightarrow{T_\lambda} X_1$$

of contractive linear maps such that $T_\lambda \circ S_\lambda$ converges to id_{X_1} in the point-norm topology and $S_\lambda \circ \alpha_1 = \alpha_2 \circ S_\lambda$ and $T_\lambda \circ \alpha_2 = \alpha_1 \circ T_\lambda$ for all λ. Then $\mathrm{hc}(\alpha_1) \leq \mathrm{hc}(\alpha_2)$.

Proof. Let $\iota_1 : X_1 \to Y_1$ and $\iota_2 : X_2 \to Y_2$ be CA embeddings. We may assume that Y_1 is injective by taking ι_1 to be, for example, the composition of the map $X_1 \to C(B_1(X^*))$ defined via evaluation with the canonical embedding of $C(B_1(X^*))$ into its second dual. Let $\Omega \in \mathcal{P}_{\mathrm{f}}(X_1)$ and $\delta > 0$. Pick a λ such that $\|T_\lambda \circ S_\lambda(x) - x\| < \delta$ for all $x \in \Omega$. Let $n \in \mathbb{N}$ and suppose $(\phi, \psi, d) \in \mathrm{CA}(\iota_2, S_\lambda(\Omega) \cup \alpha_2(S_\lambda(\Omega)) \cup \cdots \cup \alpha_2^{n-1}(S_\lambda(\Omega)), \delta)$. By the injectivity of Y_1 we can extend $T_\lambda \circ \iota_2^{-1}|_{\iota_2(X_2)}$ to a contractive linear map $\gamma : Y_2 \to Y_1$. From our assumption we have $S_\lambda \circ \alpha_1^k(x) = \alpha_2^k(S_\lambda(x))$ and $T_\lambda \circ \alpha_2^k(S_\lambda(x)) = \alpha_1^k \circ T_\lambda(S_\lambda(x))$ for all $x \in \Omega$ and $k \in \mathbb{Z}$, and so by an estimate using the triangle inequality we have

$$(\phi \circ S_\lambda, \tau \circ \gamma \circ \psi, d) \in \mathrm{CA}(\iota_1, \Omega \cup \alpha_1\Omega \cup \cdots \cup \alpha_1^{n-1}\Omega, 2\delta).$$

We infer that $\mathrm{hc}(\alpha_1, \Omega, 2\delta) \leq \mathrm{hc}(\alpha_2, S_\lambda(\Omega), \delta)$, from which we conclude that $\mathrm{hc}(\alpha_1) \leq \mathrm{hc}(\alpha_2)$. □

Proposition 7.3. Let A be a C^*-algebra, G a locally compact group, and α a strongly continuous action of G on A by *-automorphisms. Let $g \in G$, and suppose that e has a basis of neighbourhoods N satisfying $gNg^{-1} = N$. Then

$$\mathrm{hc}(\alpha_g) \leq \mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}).$$

Proof. In the proof of Theorem 5.2 in [48] it is shown that the pair

$$(A, \alpha_g), (A \rtimes_\alpha G, \mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G})$$

has the covariant completely contractive factorization property (see Section 2 of [48]), and so we can apply Lemma 7.2 to obtain the result. □

For a locally compact group G we consider $\mathrm{Aut}(G)$ with its standard topological group structure, for which a neighbourhood basis of id_G is formed by the sets $\{\alpha \in \mathrm{Aut}(G) : \alpha(x) \in Vx \text{ and } \alpha^{-1}(x) \in Vx \text{ for all } x \in K\}$ where $K \subseteq G$ is compact and V is a neighbourhood of e in G.

Theorem 7.4. Let A be a C^*-algebra, G an amenable locally compact group, and α a strongly continuous action of G on A by *-automorphisms. Let $g \in G$, and suppose that the closure of the subgroup of $\mathrm{Aut}(G)$ generated by $\mathrm{Ad}\,g$ is compact. Then $\mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) = 0$ if and only if $\mathrm{hc}(\alpha_g) = 0$. If A is furthermore assumed to be commutative then $\mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) = \mathrm{hc}(\alpha_g)$.

Proof. By [48, Lemma 5.1] and Proposition 7.3 it suffices to prove that $\mathrm{hc}(\alpha_g) = 0$ implies $\mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) = 0$, and, in the case that A is commutative, that $\mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) \leq \mathrm{hc}(\alpha_g)$.

It is shown in the proof of Theorem 5.3 in [48] that the pair

$$(A \rtimes_\alpha G, \mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}), (A \rtimes_\alpha G \rtimes_{\hat{\alpha}} G, \mathrm{Ad}\,(\lambda_g \otimes l_g r_g)|_{A\rtimes_\alpha G\rtimes_{\hat{\alpha}} G})$$

has the covariant completely contractive factorization property (see Section 2 of [48]) and hence by Lemma 7.2 we have

$$\mathrm{hc}(\alpha_g) \leq \mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) \leq \mathrm{hc}(\mathrm{Ad}\,(\lambda_g \otimes l_g r_g)|_{(A\rtimes_\alpha G)\rtimes_{\hat{\alpha}} G}).$$

By Theorem 4.1 of [48] we have

$$\mathrm{Ad}\,(\lambda_g \otimes l_g r_g)|_{A\rtimes_\alpha G\rtimes_{\hat{\alpha}} G} = \alpha_g \otimes (\mathrm{Ad}\,l_g r_g|_{\mathcal{K}(L^2(G))}).$$

Since the minimal C^*-tensor product coincides with the injective tensor product if one of the factors is commutative, it follows by Proposition 7.1 that if A is commutative then

$$\mathrm{hc}(\mathrm{Ad}\,(\lambda_g \otimes l_g r_g)|_{A\rtimes_\alpha G\rtimes_{\hat{\alpha}} G}) \leq \mathrm{hc}(\alpha_g) + \mathrm{hc}(\mathrm{Ad}\,l_g r_g|_{\mathcal{K}(L^2(G))})$$

and hence $\mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) = \mathrm{hc}(\alpha_g)$ since every *-automorphism of the compact operators has zero CA entropy by Corollary 3.9.

Assuming now that A is not commutative, we suppose that $\mathrm{hc}(\alpha_g) = 0$. For economy we set $\beta_g = \mathrm{Ad}\,l_g r_g|_{\mathcal{K}(L^2(G))}$. Let $x \in A \otimes \mathcal{K}(L^2(G))$. To obtain the desired equality $\mathrm{hc}(\mathrm{Ad}\,\lambda_g|_{A\rtimes_\alpha G}) = 0$, it suffices by the general observations above and Theorem 3.5 to show that x admits no ℓ_1-isomorphism set of positive density with respect to the *-automorphism $\alpha_g \otimes \beta_g$. Since the span of rank one projections is dense in $\mathcal{K}(L^2(G))$, we may assume that x is of the form $\sum_{k=1}^r a_k \otimes p_k$ where $p_1, \dots, p_r$ are rank one projections in $\mathcal{B}(L^2(G))$. Denoting by p_ξ the orthogonal projection onto the subspace spanned by a given vector $\xi \in L^2(G)$, it is readily verified that $\beta_g(p_\xi) = p_{\xi\circ \mathrm{Ad}\, g^{-1}}$ for all $\xi \in L^2(G)$ using the unimodularity of g [48, Lemma 5.1] and that the function from $L^2(G) \setminus \{0\}$ to $\mathcal{B}(L^2(G))$ given by $\xi \mapsto p_\xi$ is norm continuous. Thus, since for a given $\xi \in L^2(G)$ the function from $\mathrm{Aut}(G)$ to $L^2(G)$ defined by $\gamma \mapsto \xi \circ \gamma^{-1}$ is continuous by Proposition IV.5.2 of [7], we infer in view of our assumption on g that the set

$$\Theta = \big\{\beta_g^n(p_k) : k = 1, \dots, r \text{ and } n \in \mathbb{Z}\big\}$$

has compact closure in $\mathcal{K}(L^2(G))$. Now suppose $I \subseteq \mathbb{Z}$ is a positive density subset and let $\varepsilon > 0$. Since Θ has compact closure there is a finite subset $F \subseteq \Theta$ which is δ-dense in Θ for $\delta = \varepsilon\big(2r \max_{1\leq k\leq r} \|a_k\|\big)^{-1}$. Using the fact that every finite partition of a positive density subset of $\mathbb{Z}$ contains at least one member of positive upper density, we can apply a diagonal argument across k to find a subset $J \subseteq I$ of positive upper density and $q_1, \dots, q_r \in F$ such that

$$\Big(\max_{1\leq k\leq r} \|a_k\|\Big)\|\beta_g^n(p_k) - q_k\| \leq \frac{\varepsilon}{2r}$$

for every $k = 1, \dots, r$ and $n \in J$. By Proposition 2.2 the ℓ_∞-direct sum of r copies of α_g has zero CA entropy, and so by Theorem 3.5 there exist a finite subset $E \subseteq J$ and a norm one element $(c_n)_{n\in E}$ of ℓ_1^E such that

$$\sup_{1\leq k\leq r} \Big\| \sum_{n\in E} c_n \alpha_g^n(a_k) \Big\| \leq \frac{\varepsilon}{2r}.$$

We then have, for each $k = 1, \dots, r$,

$$\begin{aligned}
&\Big\| \sum_{n\in E} c_n (\alpha_g \otimes \beta_g)^n (a_k \otimes p_k) \Big\| \\
&\qquad \leq \Big\| \Big(\sum_{n\in E} c_n \alpha_g^n(a_k)\Big) \otimes q_k \Big\| \\
&\qquad\qquad + \Big\| \sum_{n\in E} c_n \alpha_g^n(a_k) \otimes (\beta_g^n(p_k) - q_k) \Big\| \\
&\qquad \leq \Big\| \sum_{n\in E} c_n \alpha_g^n(a_k) \Big\| + \sum_{n\in E} |c_n| \|a_k\| \|\beta_g^n(p_k) - q_k\| \\
&\qquad \leq \frac{\varepsilon}{2r} + \frac{\varepsilon}{2r} = \frac{\varepsilon}{r}
\end{aligned}$$

and hence

$$\begin{aligned}
\Big\| \sum_{n\in E} c_n (\alpha_g \otimes \beta_g)^n (x) \Big\| &\leq \sum_{k=1}^{r} \Big\| \sum_{n\in E} c_n (\alpha_g \otimes \beta_g)^n (a_k \otimes p_k) \Big\| \\
&\leq r\frac{\varepsilon}{r} = \varepsilon.
\end{aligned}$$

Since ε was arbitrary we conclude that I is not an ℓ_1-isomorphism set for x, completing the proof. □

One consequence of Theorem 7.4 is the existence of a simple separable unital nuclear C^*-algebra on which every possible value for CA entropy is realized by an inner *-automorphism. Indeed we argue that this happens for the Bunce-Deddens algebra B of type 2^∞ (see Example 3.2.11 of [50]) as follows. By the classification theorem of [18] B can be expressed as the crossed product associated to the dyadic odometer, since these two C^*-algebras are real rank zero A$\mathbb{T}$-algebras with the same Elliott invariant (see Section 3.2 of [50]). Now given any $r \in [0, \infty]$, by [6] there is a minimal homeomorphism of the Cantor set which is strong orbit equivalent to the dyadic odometer and has topological entropy equal to r, and by Theorem 2.1 of [21] the crossed product associated to this homeomorphism is *-isomorphic to B. Applying Theorem 7.4 we thus obtain an inner *-automorphism of B with CA entropy equal to r.

We close this section with a result on reduced free product *-automorphisms. For notation and terminology see Section 2 of [12].

Proposition 7.5. Let D be a finite-dimensional C^*-algebra. Let (A, E) and (B, F) be D-probability spaces with A and B commutative, and suppose that the GNS representations of E and F are faithful. Let α and β be *-automorphisms of (A, E) and (B, F), respectively, such that $\alpha|_D = \beta|_D$ and $\mathrm{hc}(\alpha) = \mathrm{hc}(\beta) = 0$. Then

$$\mathrm{hc}(\alpha * \beta) = 0.$$

Proof. For a *-automorphism of a unital commutative C^*-algebra the CA entropy and Voiculescu-Brown entropy agree by Proposition 3.1. We can thus apply Theorem 5.7 of [12] to obtain $\mathrm{ht}(\alpha * \beta) = 0$, and this implies that $\mathrm{hc}(\alpha * \beta) = 0$ by Theorem 3.5 and [36, Thm. 3.3] (see Proposition 4.2). □

8. On the prevalence of zero and infinite CA entropy in C^*-algebras

Let A be a unital C^*-algebra A. We denote by $\mathcal{U}(A)$ the unitary group of A and by $\mathcal{U}_0(A)$ the subgroup of $\mathcal{U}(A)$ consisting of those unitaries which are homotopic to 1. We denote by $\mathrm{Aut}(A)$ the group of *-automorphisms of A, by $\mathrm{Inn}(A)$ the subgroup of inner *-automorphisms, and by $\mathrm{Inn}_0(A)$ the subgroup of inner *-automorphisms that can be expressed as $\mathrm{Ad}\, u$ for some $u \in \mathcal{U}_0(A)$. Unless stated otherwise, the topology on $\mathrm{Aut}(A)$ will be the point-norm one, i.e., the topology which has as a base sets of the form $\{\beta \in \mathrm{Aut}(A) : \|\beta(a) - \alpha(a)\| < \varepsilon \text{ for all } a \in \Omega\}$ for some $\varepsilon > 0$ and finite set $\Omega \subseteq \mathrm{Aut}(A)$. In particular, $\overline{\mathrm{Inn}}(A)$ and $\overline{\mathrm{Inn}_0}(A)$ refer to point-norm closures. For separable A the space $\mathrm{Aut}(A)$ is Polish.

Proposition 8.1. Let A be a unital C^*-algebra and $u \in A$ a unitary with finite spectrum. Then $\mathrm{hc}(\mathrm{Ad}\, u) = 0$.

Proof. By the functional calculus there exist pairwise orthogonal projections $p_1, \dots, p_n \in A$ with sum 1 and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of unit modulus such that $u = \lambda_1 p_1 + \cdots + \lambda_n p_n$. For $1 \leq i, j \leq n$ set $A_{ij} = p_i A p_j$. Then A decomposes as the direct sum of the A_{ij}'s, and $\mathrm{Ad}\,(u)$ acts on A_{ij} by multiplication by λ_i / λ_j. Thus if Ω is a finite subset of the union of the A_{ij}'s then $\mathrm{hc}(\mathrm{Ad}\, u, \Omega) = 0$. The result now follows from Proposition 2.1(ii). □

Proposition 8.1 shows in particular that, within the set of inner *-automorphisms of a von Neumann algebra, those with zero CA entropy are norm dense, since any unitary can be approximated in norm by one with finite spectrum using the Borel functional calculus.

Recall that a C^*-algebra is said to be *subhomogeneous* if it is *-isomorphic to a C^*-subalgebra of $M_n(C_0(X))$ for some $n \in \mathbb{N}$ and locally compact Hausdorff space X. By the structure theory for von Neumann algebras, a von Neumann algebra is subhomogeneous if and only if it can be written as a finite direct sum of matrix algebras over commutative von Neumann algebras.

Proposition 8.2. For a von Neumann algebra M the following are equivalent:

(1) M is not subhomogeneous,
(2) $\mathrm{hc}(\alpha) > 0$ for some $\alpha \in \mathrm{Inn}(M)$,
(3) The set of $\alpha \in \mathrm{Inn}(M)$ such that $\mathrm{hc}(\alpha) = \infty$ is norm dense in $\mathrm{Inn}(M)$.

Proof. The implications (3)⇒(1) and (1)⇔(2) follow from Theorem 6.4. Suppose then that (1) holds. We will obtain (3) upon showing that, given a unitary $u \in M$ and an $\varepsilon > 0$, there is a unitary $v \in M$ with $\|v - u\| < \varepsilon$ and $\mathrm{hc}(\mathrm{Ad}\, v) = \infty$. By the Borel functional calculus we can find a set Γ of pairwise orthogonal projections with sum 1 in the von Neumann subalgebra of M generated by u with the property that for each $p \in \Gamma$ there is a $\lambda \in \mathbb{C}$ of unit modulus such that $\|pup - \lambda p\| < \varepsilon$. By (1) there must be a $q \in \Gamma$ such that qMq is not subhomogeneous. Then qMq is not a type I C^*-algebra and so by Theorem 6.4 there is a unitary $w \in qMq$ with $\mathrm{hc}(\mathrm{Ad}\, w) = \infty$. Perturbing u via the Borel functional calculus if necessary, we may assume that $quq = \lambda q$ for some $\lambda \in \mathbb{C}$ of unit modulus. By taking a branch of the logarithm function we can apply the Borel functional calculus to obtain a unitary $z \in qMq$ such that $\|z - 1_{qMq}\| < \varepsilon$ and $z^n = w$ for some $n \in \mathbb{N}$. Put $v = u(z + 1 - q) \in M$. Then v is a unitary with

$$\|v - u\| \leq \|z - q\| < \varepsilon.$$

It remains to observe that $\mathrm{Ad}\, v$ restricts to $\mathrm{Ad}\, z$ on qMq so that by monotonicity and Proposition 2.1(iii) we obtain

$$\mathrm{hc}(\mathrm{Ad}\, v) \geq \mathrm{hc}(\mathrm{Ad}\, z) = \frac{1}{n}\mathrm{hc}(\mathrm{Ad}\, z^n) = \frac{1}{n}\mathrm{hc}(\mathrm{Ad}\, w) = \infty,$$

completing the proof. □

Since $\mathrm{Aut}(M) = \mathrm{Inn}(M)$ for a type I factor we obtain the following.

Corollary 8.3. Let M be an infinite-dimensional factor of type I. Then the set of $\alpha \in \mathrm{Aut}(M)$ with $\mathrm{hc}(\alpha) = 0$ (resp. $\mathrm{hc}(\alpha) = \infty$) is norm dense in $\mathrm{Aut}(M)$.

Let X be a separable Banach space. Choose a dense sequence $\{x_1, x_2, \dots\}$ in the unit ball of X and define on $B_1(X^*)$ the metric

$$d(\sigma, \omega) = \sum_{n=1}^{\infty} 2^{-n} |\sigma(x_n) - \omega(x_n)|,$$

which is compatible with the weak* topology. On the set of homeomorphisms of $B_1(X^*)$ we consider the metric

$$\rho(T,S) = \sup_{\sigma\in B_1(X^*)} d(S\sigma, T\sigma) + \sup_{\sigma\in B_1(X^*)} d(S^{-1}\sigma, T^{-1}\sigma).$$

Then on $\mathrm{IA}(X)$ the topology arising from ρ (via the identification of an element $\alpha \in \mathrm{IA}(X)$ with the induced homeomorphism T_α of $B_1(X^*)$) and the point-norm topology agree, as is readily checked. We can then apply Lemma 2.4 of [25] and Theorem 3.5 to obtain the following lemma.

Lemma 8.4. Let X be a separable Banach space. Then the elements α of $\mathrm{IA}(X)$ with $\mathrm{hc}(\alpha) = 0$ form a G_δ subset of $\mathrm{IA}(X)$ in the point-norm topology.

Proposition 8.5. Let A be a separable unital C^*-algebra with real rank zero. Then the set of *-automorphisms in $\overline{\mathrm{Inn}_0}(A)$ (resp. in $\mathrm{Inn}_0(A)$) with zero CA entropy is a dense (resp. norm dense) G_δ subset.

Proof. By [40] a unital C^*-algebra has real rank zero if and only if for every unitary $u \in \mathcal{U}_0(A)$ and $\varepsilon > 0$ there is a unitary $v \in A$ with finite spectrum such that $\|u - v\| < \varepsilon$. Thus by Proposition 8.1 and Lemma 8.4 we obtain the result. □

Proposition 8.6. Let A be a separable C^*-algebra which is an inductive limit of type I C^*-algebras. Then the set of *-automorphisms in $\overline{\mathrm{Inn}}(A)$ (resp. in $\mathrm{Inn}(A)$) with zero CA entropy is a dense (resp. norm dense) G_δ subset.

Proof. Apply Theorem 6.4 and Proposition 2.1(ii). □

The following is the noncommutative analogue of Corollary 2.5 in [25]. For the definition of the Jiang-Su C^*-algebra $\mathcal{Z}$ see [33].

Proposition 8.7. Let A be a UHF C^*-algebra, the Cuntz C^*-algebra $\mathcal{O}_2$, or the Jiang-Su C^*-algebra $\mathcal{Z}$. Then the *-automorphisms in $\mathrm{Aut}(A)$ with zero CA entropy form a dense G_δ subset.

Proof. If A is a UHF algebra or $\mathcal{Z}$ then it is an inductive limit of type I C^*-algebras and $\overline{\mathrm{Inn}}(A) = \mathrm{Aut}(A)$ (see [50, 33]), and so the conclusion follows from Proposition 8.6. In the case of $\mathcal{O}_2$ every unitary is homotopic to 1 and every *-automorphism is approximately inner (see [50]) whence $\overline{\mathrm{Inn}_0}(A) = \mathrm{Aut}(A)$, and thus since $\mathcal{O}_2$ has real rank zero we can apply Proposition 8.5. □

We will next establish an infinite entropy density result for *-automorphisms of C^*-algebras which are tensorially stable with respect to the Jiang-Su C^*-algebra $\mathcal{Z}$. We say briefly that a C^*-algebra A is *$\mathcal{Z}$-stable* if $A \otimes \mathcal{Z}$ is *-isomorphic to A (note that the C^*-tensor product is unique in this case by the nuclearity of $\mathcal{Z}$). The class of $\mathcal{Z}$-stable C^*-algebras is of importance in the classification theory for nuclear C^*-algebras and includes all Kirchberg algebras and all simple unital infinite-dimensional AH algebras of bounded dimension [33, Thm. 5] [50, Example 3.4.5] (for obstructions to $\mathcal{Z}$-stability see [28]). Recently A. Toms and W. Winter have informed us that they have established $\mathcal{Z}$-stability for all separable approximately divisible C^*-algebras.

Lemma 8.8. Let A be a $\mathcal{Z}$-stable C^*-algebra. For any $\Omega \in \mathcal{P}_{\mathrm{f}}(A)$ and $\varepsilon > 0$ there exists a *-isomorphism $\Phi : A \to A \otimes \mathcal{Z}$ such that $\|\Phi(x) - x \otimes 1_{\mathcal{Z}}\| < \varepsilon$ for all $x \in \Omega$.

Proof. Since A is $\mathcal{Z}$-stable, we may write A as $A \otimes \mathcal{Z}$, and by a simple approximation argument we may assume that Ω is contained in the algebraic tensor product of A and $\mathcal{Z}$. It suffices then to consider the case $A = \mathcal{Z}$. By [33, Thm. 4] we can find a *-isomorphism $\Psi : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$. Denote by ψ the embedding $\mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ sending $a \in \mathcal{Z}$ to $a \otimes 1_{\mathcal{Z}}$. Then $\psi \circ \Psi^{-1}$ is a unital *-endomorphism of $\mathcal{Z} \otimes \mathcal{Z}$ which is approximately inner by [33, Thm. 3]. Thus we can find a unitary $u \in \mathcal{Z} \otimes \mathcal{Z}$ such that $\|uyu^* - \psi(\Psi^{-1}(y))\| < \varepsilon$ for all $y \in \Psi(\Omega)$. Set $\Phi = \mathrm{Ad}\, u \circ \Psi$. Then $\|\Phi(x) - x \otimes 1_{\mathcal{Z}}\| < \varepsilon$ for all $x \in \Omega$. □

Proposition 8.9. Let A be a unital $\mathcal{Z}$-stable C^*-algebra. Then the set of *-automorphisms in $\mathrm{Aut}(A)$ (resp. in $\mathrm{Inn}(A)$) with infinite CA entropy is dense in $\mathrm{Aut}(A)$ (resp. in $\mathrm{Inn}(A)$).

Proof. Let $\alpha \in \mathrm{Aut}(A)$, and let $\Omega \in \mathcal{P}_{\mathrm{f}}(A)$ and $\varepsilon > 0$. By Lemma 8.8 we can find a *-isomorphism $\Phi : A \to A \otimes \mathcal{Z}$ such that $\|\Phi(x) - x \otimes 1_{\mathcal{Z}}\| < \varepsilon/2$ for all $x \in \Omega \cup \alpha(\Omega)$. Since $\mathcal{Z}$ is non-type I, by Theorem 6.4 it contains a unitary v with $\mathrm{hc}(\mathrm{Ad}\, v) = \infty$. Set $\beta = \Phi^{-1} \circ (\alpha \otimes \mathrm{Ad}\, v) \circ \Phi$. Note that when α is inner, so is β. Then $\mathrm{hc}(\beta) = \infty$ by monotonicity, and for all $x \in \Omega$ we have

$$\begin{aligned} \|\beta(x) - \alpha(x)\| &= \|(\alpha \otimes \mathrm{Ad}\, v)(\Phi(x)) - \Phi(\alpha(x))\| \\ &\leq \|(\alpha \otimes \mathrm{Ad}\, v)(\Phi(x)) - (\alpha \otimes \mathrm{Ad}\, v)(x \otimes 1_{\mathcal{Z}})\| \\ &\qquad + \|\alpha(x) \otimes 1_{\mathcal{Z}} - \Phi(\alpha(x))\| \\ &< \varepsilon, \end{aligned}$$

completing the proof. □

Question 8.10. Does there exist a non-type I C^*-algebra A such that the set of *-automorphisms in $\mathrm{Aut}(A)$ with infinite CA entropy is not dense in $\mathrm{Aut}(A)$?

Question 8.11. Let A be a unital nuclear C^*-algebra and $u \in A$ a unitary with $\mathrm{hc}(\mathrm{Ad}\, u) > 0$. Must the spectrum of u be the entire unit circle?

Note that the answer to Question 8.11 is no if, for example, A is a von Neumann algebra, for in this case by the Borel functional calculus there is a unitary $v \in A$ without full spectrum such that $v^2 = u$, and $\mathrm{hc}(\mathrm{Ad}\, v) > 0$ by Proposition 2.1(iii).

9. The shift on $M_p^{\otimes \mathbb{Z}}$

In this section we will show that for $p \geq 2$ the tensor product shift on the UHF C^*-algebra $M_p^{\otimes \mathbb{Z}}$ (obtained from the shift $k \mapsto k+1$ on the index set $\mathbb{Z}$) has infinite CA entropy. The key to obtaining arbitrarily large lower bounds is the following lemma, which will also be applied in Sections 10 and 11.

Lemma 9.1. Let X be a Banach space and $\alpha \in \mathrm{IA}(X)$. Let $\Omega \subseteq X$ be a finite set of unit vectors such that $\|x+y\| < 2$ for all $x, y \in \Omega$ with $x \neq y$ and $\left\| \sum_{k=0}^{n-1} \alpha^k(x_k) \right\| = n$ for all $n \in \mathbb{N}$ and $(x_k)_k \in \Omega^n$. Then $\mathrm{hc}(\alpha, \Omega) \geq \log |\Omega|$.

Proof. Let $0 < \delta < 2 - \max\{\|x+y\| : x, y \in \Omega \text{ and } x \neq y\}$ and $n \in \mathbb{N}$. Let $\iota : X \to Y$ be a CA embedding and let $(\varphi, \psi, d) \in \mathrm{CA}(\iota, \Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta^2)$. For every $x = (x_k)_k \in \Omega^n$ we can find a $1 \leq b_x \leq d$ such that $\big|\sum_{k=0}^{n-1} \varphi(\alpha^k(x_k))(b_x)\big| \geq n(1-\delta^2)$. Take a maximal subset $Q_n \subseteq \Omega^n$ with the property that $|\{k : x_k \neq y_k\}| \geq 3n\delta$ for all $x, y \in Q_n$ with $x \neq y$ (in which case $\big\|\sum_{k=0}^{n-1} \alpha^k(x_k + y_k)\big\| < n(2 - 3\delta^2)$). If x and y are any distinct elements of Q_n with $b_x = b_y$ then the arguments of $\sum_{k=0}^{n-1} \varphi(\alpha^k(x_k))(b_x)$ and $\sum_{k=0}^{n-1} \varphi(\alpha^k(y_k))(b_y)$ are separated by at least δ^2, and so we get the inequality $d \geq \delta^2 (2\pi)^{-1} |Q_n|$.

With a view to obtaining a lower bound for $|Q_n|$, for each $x \in Q_n$ define Θ_x to be the subset of Ω^n consisting of all y such that $|\{k : x_k \neq y_k\}| < 3n\delta$. Setting $m = |\Omega|$ and supposing that $\delta < 1/6$ and n is sufficiently large, we have the estimate

$$|\Theta_x| = \sum_{k=0}^{\lfloor 3n\delta \rfloor} (m-1)^k \binom{n}{k} \leq m^{3n\delta} \binom{n}{3n\delta}.$$

By Stirling's formula we can find an $M > 0$ such that for all $n \in \mathbb{N}$ and $0 < \delta < 1/6$ we have

$$\binom{n}{3n\delta} \leq \frac{M}{\sqrt{n\delta}} (1 - 3\delta)^{-n} \left(\frac{1 - 3\delta}{3\delta}\right)^{3n\delta}.$$

Using the maximality of Q_n, it follows in our case that

$$\begin{aligned} |Q_n| &\geq \frac{m^n}{\max_{x \in Q_n} |\Theta_x|} \\ &\geq m^{n(1-3\delta)} \frac{\sqrt{n\delta}}{M} (1 - 3\delta)^n \left(\frac{3\delta}{1 - 3\delta}\right)^{3n\delta}. \end{aligned}$$

Since the logarithm of this last expression divided by n tends to $\log m + C(\delta)$ as $n \to \infty$ where $\lim_{\delta \to 0^+} C(\delta) = 0$, we obtain the assertion of the lemma. ☐

For each $k \in \mathbb{N}$ we denote by u_k and v_k the self-adjoint matrices $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, respectively, viewed as elements of $M_p^{\otimes \mathbb{Z}}$ by first identifying M_2 with the upper left-hand 2×2 corner of M_p and then embedding M_p into $M_p^{\otimes \mathbb{Z}}$ as the kth tensor product factor. For each $n \in \mathbb{N}$ we set $Y_n = \mathrm{span}_{\mathbb{R}}\{u_k, v_k\}_{k=1}^n \subseteq (M_p^{\otimes \mathbb{Z}})_{\mathrm{sa}}$ and denote by W_n the ℓ_1-direct sum of n copies of ℓ_2^2 over the real numbers.

Lemma 9.2. For each $n \in \mathbb{N}$ the linear map from W_n to Y_n given by

$$((c_k, d_k))_{k=1}^n \mapsto \sum_{k=1}^n c_k u_k + d_k v_k$$

is an isometric isomorphism.

Proof. Let $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{R}$. For each $k = 1, \dots, n$ we note that $c_k u_k + d_k v_k$, as an element of M_2, has characteristic polynomial $x^2 - (c_k^2 + d_k^2)$ and hence admits a unit eigenvector whose corresponding vector state σ_k satisfies

$$\sigma_k(c_k u_k + d_k v_k) = (c_k^2 + d_k^2)^{1/2} = \|c_k u_k + d_k v_k\|.$$

Thus

$$\begin{aligned}\sum_{k=1}^{n}(c_k^2+d_k^2)^{1/2} &= \sum_{k=1}^{n}\|c_ku_k+d_kv_k\| \\ &\geq \Big\| \sum_{k=1}^{n} c_ku_k+d_kv_k \Big\| \\ &\geq \Big|(\sigma_1\otimes\cdots\otimes\sigma_n)\Big(\sum_{k=1}^{n} c_ku_k+d_kv_k\Big)\Big| \\ &= \Big|\sum_{k=1}^{n}\sigma_k(c_ku_k+d_kv_k)\Big| \\ &= \sum_{k=1}^{n}(c_k^2+d_k^2)^{1/2}\end{aligned}$$

so that $\big\|\sum_{k=1}^{n} c_ku_k+d_kv_k\big\| = \sum_{k=1}^{n}(c_k^2+d_k^2)^{1/2}$, which establishes the proposition. □

Theorem 9.3. The tensor product shift α on $M_p^{\otimes\mathbb{Z}}$ has infinite CA entropy.

Proof. By Lemma 9.2 every finite subset of the unit sphere of Y_1 satisfies the hypotheses of Lemma 9.1, and so we obtain the result. □

Theorem 9.3 prompts the following question. (Compare the discussion after Theorem 7.4.)

Question 9.4. Is there a *-automorphism of $M_p^{\otimes\mathbb{Z}}$ (or any other simple AF algebra) with finite nonzero CA entropy?

We also remark that we don't know whether there exists a *-automorphism of a simple purely infinite C^*-algebra (in particular, the Cuntz algebra $\mathcal{O}_2$) with finite nonzero CA entropy (cf. [12]).

10. Isometric automorphisms of ℓ_∞

By the Banach-Stone theorem, for every isometric automorphism α of ℓ_∞ there are a double-sided sequence $\lambda : \mathbb{Z}\to\mathbb{T}$ and permutation σ of $\mathbb{Z}$ such that $\alpha(x)(s) = \lambda(s)x(\sigma(s))$ for all $x\in\ell_\infty$ and $s\in\mathbb{Z}$ (cf. [41, Prop. 2.f.14]).

Proposition 10.1. For every isometric automorphism α of ℓ_∞ we have either $\mathrm{hc}(\alpha) = 0$ or $\mathrm{hc}(\alpha) = \infty$ depending on whether or not there is a finite bound on the cardinality of the orbits of the associated permutation of $\mathbb{Z}$.

Proof. Let α be an isometric automorphism of ℓ_∞ with associated $\mathbb{T}$-valued sequence λ and permutation σ of $\mathbb{Z}$, as above. Suppose first that there is a $d\in\mathbb{N}$ such that every orbit of σ is of cardinality at most d. To show that $\mathrm{hc}(\alpha) = 0$ we may assume that σ is the identity by replacing α with $\alpha^{d!}$ and applying Proposition 2.1(iii). We may then view α as the isometric automorphism of $C(\beta\mathbb{Z}) \cong \ell_\infty$ given by multiplication by the continuous $\mathbb{T}$-valued extension of λ to the Stone-Čech compactification $\beta\mathbb{Z}$, so that we can appeal to Proposition 3.15 to conclude that $\mathrm{hc}(\alpha) = 0$, as desired.

Suppose now that there is no finite bound on the cardinality of the orbits of σ. Let $m \in \mathbb{N}$. By relabelling the coordinates of ℓ_∞ if necessary we can associate to each $n \in \mathbb{N}$ and $f \in \{1, \dots, m\}^{\{0,\dots,n-1\}}$ an interval I_f in $\mathbb{Z}$ of the form $[s_f, s_f + n - 1]$ such that $\sigma(t) = t + 1$ for all $t \in I_f$, and we can arrange for the intervals I_f to be pairwise disjoint. Now let $1 \leq j \leq m$ and define the unit vector $x_j \in \ell_\infty$ as follows. For every $n \in \mathbb{N}$, $f \in \{1, \dots, m\}^{\{0,\dots,n-1\}}$, and $k = 0, \dots, n-1$ we set

$$x_j(s_f + k) = \begin{cases} \overline{\lambda(s_f)\lambda(s_f+1)\cdots\lambda(s_f+k-1)} & \text{if } f(k) = j, \\ 0 & \text{otherwise.} \end{cases}$$

At all remaining coordinates we take x_j to be zero. Then the set $\Omega = \{x_1, \dots, x_m\}$ satisfies the hypotheses of Lemma 9.1, from which we obtain $\mathrm{hc}(\alpha, \Omega) \geq \log m$. Since m is arbitrary we conclude that $\mathrm{hc}(\alpha) = \infty$, completing the proof. ☐

Remark 10.2. From Propositions 3.1 and 10.1 we see that the Stone-Čech compactification $\beta\mathbb{Z}$ (i.e., the spectrum of ℓ_∞) provides an example of a compact Hausdorff space which admits a homeomorphism with infinite topological entropy but no homeomorphism with finite nonzero topological entropy.

11. Isometric automorphisms of ℓ_1

Given an isometric automorphism α of ℓ_1 there are a double-sided sequence $\lambda : \mathbb{Z} \to \mathbb{T}$ and a permutation σ of $\mathbb{Z}$ such that $\alpha(x)(s) = \lambda(s)x(\sigma(s))$ for all $x \in \ell_1$ and $s \in \mathbb{Z}$ [41, Prop. 2.f.14].

Proposition 11.1. For every isometric automorphism α of ℓ_1 we have either $\mathrm{hc}(\alpha) = \infty$ or $\mathrm{hc}(\alpha) = 0$ depending on whether or not there is an infinite orbit in the associated permutation of $\mathbb{Z}$.

Proof. Suppose first that there is an infinite orbit in the permutation of $\mathbb{Z}$ associated to α. Let k be an integer contained in this infinite orbit and let e_k be the kth standard unit basis vector in ℓ_1. Then for any finite subset Ω of the unit sphere of $\mathrm{span}\{e_k\}$ we have $\mathrm{hc}(\alpha, \Omega) \geq \log |\Omega|$ by Lemma 9.1, whence $\mathrm{hc}(\alpha) = \infty$.

If on the other hand there is no infinite orbit in the associated permutation of $\mathbb{Z}$, then ℓ_1 contains a dense union of finite-dimensional α-invariant subspaces, so that $\mathrm{hc}(\alpha) = 0$ by Proposition 2.1(ii). ☐

Remark 11.2. The infinite value of CA entropy for the shift on ℓ_1 is a reflection of the fact that the unit sphere of the scalar field $\mathbb{C}$ is infinite. If we work instead over $\mathbb{R}$, whose unit sphere has two elements, then the CA entropy of the shift α' on $(\ell_1)_\mathbb{R}$ is $\log 2$. Indeed $(\ell_1^n)_\mathbb{R}$ embeds isometrically in $(\ell_\infty^{2^n})_\mathbb{R}$ so that $\mathrm{hc}(\alpha') \leq \log 2$, while from Lemma 9.1 we get the lower bound $\mathrm{hc}(\alpha') \geq \log 2$ by taking $\Omega = \{e, -e\}$, where e is any standard unit basis vector in $(\ell_1)_\mathbb{R}$.

Remark 11.3. Using Proposition 11.1 we can show that infinite CA entropy is not an isomorphic conjugacy invariant, in contrast to zero CA entropy. Consider the shift T on $\{1,-1\}^{\mathbb{Z}}$. Define the self-adjoint unitary function $f \in C(\{1,-1\}^{\mathbb{Z}})$ by

$$f((a_k)_k) = a_0$$

for all $(a_k)_k \in \{1,-1\}^{\mathbb{Z}}$. Then $\{f \circ T^k\}_{k\in\mathbb{Z}}$ is isometrically equivalent to the standard basis of ℓ_1 over $\mathbb{R}$, and hence equivalent to the standard basis of ℓ_1 over $\mathbb{C}$. Denoting by β_T the restriction of the induced *-automorphism of $C(\{1,-1\}^{\mathbb{Z}})$ to $\overline{\mathrm{span}}\{f \circ T^k\}_{k\in\mathbb{Z}}$, we obtain an isometric automorphism which is isomorphically conjugate to the shift on ℓ_1. While the latter has infinite CA entropy, however, we have $\mathrm{hc}(\beta_T) \leq \log 2$ by monotonicity and Proposition 3.1.

Example 11.4. Let σ be a permutation of $\mathbb{Z}$. Let σ_* be the corresponding free permutation *-automorphism of the full free group C^*-algebra $C^*(F_{\mathbb{Z}})$ sending u_j to $u_{\sigma(j)}$, where $\{u_j\}_{j\in\mathbb{Z}}$ is the set of canonical unitary generators of $C^*(F_{\mathbb{Z}})$. Then $\mathrm{hc}(\sigma_*) = \infty$ if and only if σ has an infinite orbit. As pointed out in [47, Sect. 8], $\{u_j\}_{j\in\mathbb{Z}}$ is isometrically equivalent to the standard basis of ℓ_1, and so the "if" part holds by Proposition 11.1 and monotonicity. For the "only if" part, supposing that σ has no infinite orbits we can take Δ in Theorem 3.5 to be the set of unitaries in $C^*(F_{\mathbb{Z}})$ corresponding to elements of $F_{\mathbb{Z}}$ to get $\mathrm{h}_{\mathrm{top}}(T_{\sigma_*}) = 0$, as desired. We remark that the corresponding *-automorphisms of the reduced free group C^*-algebra $C^*_{\mathrm{r}}(F_{\mathbb{Z}})$ all have zero CA entropy by [11, 15] and Proposition 4.2.

References

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.* **114** (1965), 309–319.

[2] F. Blanchard. A disjointness theorem involving topological entropy. *Bull. Soc. Math. France* **121** (1993), 465–478.

[3] F. Blanchard. Fully positive topological entropy and topological mixing. *Symbolic dynamics and its applications.* New Haven, CT, 1991, pp. 95–105.

[4] F. Blanchard and Y. Lacroix. Zero entropy factors of topological flows. *Proc. Amer. Math. Soc.* **119** (1993), 985–992.

[5] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** (1971), 401–414.

[6] M. Boyle and D. Handelman. Entropy versus orbit equivalence for minimal homeomorphisms. *Pacific J. Math.* **164** (1994), 1–13.

[7] J. Braconnier. Sur les groupes topologiques localement compacts. *J. Math. Pures Appl. (9)* **27** (1948), 1–85.

[8] N. P. Brown. Characterizing type I C^*-algebras via entropy. *C. R. Math. Acad. Sci. Paris* **339** (2004), 827–829.

[9] N. P. Brown. Topological entropy, embeddings and unitaries in nuclear quasidiagonal C^*-algebras. *Proc. Amer. Math. Soc.* **128** (2000), 2603–2609.

[10] N. P. Brown. Topological entropy in exact C^*-algebras. *Math. Ann.* **314** (1999), 347–367.

[11] N. P. Brown and M. Choda. Approximation entropies in crossed products with an application to free shifts. *Pacific J. Math.* **198** (2001), 331–346.

[12] N. P. Brown, K. Dykema, and D. Shlyakhtenko. Topological entropy of free product automorphisms. *Acta Math.* **189** (2002), 1–35.

[13] M. Denker, C. Grillenberger, and K. Sigmund. *Ergodic Theory on Compact Spaces.* Lecture Notes in Math., vol. 527. Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[14] J. Diestel and J. J. Uhl, Jr. *Vector Measures.* Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.

[15] K. Dykema. Topological entropy of some automorphisms of reduced amalgamated free product C^*-algebras. *Ergod. Th. Dynam. Sys.* **21** (2001), 1683–1693.

[16] E. G. Effros, N. Ozawa, and Z.-J. Ruan. On injectivity and nuclearity for operator spaces. *Duke Math. J.* **110** (2001), 489–521.

[17] E. G. Effros and Z.-J. Ruan. *Operator Spaces.* London Mathematical Society Monographs. New Series, 23. The Clarendon Press, Oxford University Press, New York, 2000.

[18] G. A. Elliott. On the classification of C^*-algebras of real rank zero. *J. Reine Angew. Math.* **443** (1993), 179–219.

[19] M. Fabian. *Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces.* Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1997.

[20] T. Figiel, J. Lindenstrauss, and V. D. Milman. The dimension of almost spherical sections of convex bodies. *Acta Math.* **139** (1977), 53–94.

[21] T. Giordano, I. F. Putnam, and C. F. Skau. Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.* **469** (1995), 51–111.

[22] E. Glasner. On tame enveloping semigroups. arXiv:math.DS/0406549 v1.

[23] E. Glasner. A simple characterization of the set of μ-entropy pairs and applications. *Israel J. Math.* **102** (1997), 13–27.

[24] E. Glasner and M. Megrelishvili. Linear representations of hereditarily non-sensitive dynamical systems. arXiv:math.DS/0406192 v2.

[25] E. Glasner and B. Weiss. The topological Rohlin property and topological entropy. *Amer. J. Math.* **123** (2001), 1055–1070.

[26] E. Glasner and B. Weiss. Quasi-factors of zero entropy systems. *J. Amer. Math. Soc.* **8** (1995), 665–686.

[27] V. Ya. Golodets and E. Størmer. Entropy of C^*-dynamical systems defined by bitstreams. *Ergod. Th. Dynam. Sys.* **18** (1998), 859–874.

[28] G. Gong, X. Jiang, and H. Su. Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras. *Canad. Math. Bull.* **43** (2000), 418–426.

[29] T. N. T. Goodman. Topological sequence entropy. *Proc. London Math. Soc.* **29** (1974), 331–350.

[30] L. Harris. Bounded symmetric homogeneous domains in infinite dimensional spaces. Lecture Notes in Math., vol. 364. Springer-Verlag, Berlin, 1974, pp. 13–40.

[31] B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems.* Encyclopedia of Mathematics and its Applications, vol. 54. Cambridge University Press, Cambridge, 1995.

[32] W. Huang, S. M. Li, S. Shao, and X. D. Ye. Null systems and sequence entropy pairs. *Ergod. Th. Dynam. Sys.* **23** (2003), 1505–1523.

[33] X. Jiang and H. Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.* **121** (1999), 359–413.

[34] W. B. Johnson and J. Lindenstrauss. Basic concepts in the geometry of Banach spaces. In: *Handbook of the geometry of Banach spaces, Vol. I*, 1–84. North-Holland, Amsterdam, 2001.

[35] M. Junge, N. Ozawa, and Z.-J. Ruan. On $\mathcal{OL}_\infty$ structures of nuclear C^*-algebras. *Math. Ann.* **325** (2003), 449–483.

[36] D. Kerr. Entropy and induced dynamics on state spaces. *Geom. Funct. Anal.* **14** (2004), 575–594.

[37] D. Kerr and H. Li. Positive topological entropy and ℓ_1. arXiv:math.DS/0303161 v1.

[38] E. Kirchberg. On subalgebras of the CAR-algebra. *J. Funct. Anal.* **129** (1995), 35–63.

[39] A. Köhler. Enveloping semigroups for flows. *Proc. Roy. Irish Acad. Sect. A* **95** (1995), 179–191.

[40] H. Lin. Exponential rank of C^*-algebras with real rank zero and the Brown-Pedersen conjectures. *J. Funct. Anal.* **114** (1993), 1–11.

[41] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces I. Sequence Spaces.* Springer-Verlag, Berlin-New York, 1977.

[42] M. Misiurewicz. A short proof of the variational principle for a $\mathbb{Z}_+^N$ action on a compact space. *International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975). Astérisque* **40** (1976), 147–157.

[43] S. Neshveyev and E. Størmer. Entropy of type I algebras. *Pacific J. Math.* **201** (2001), 421–428.

[44] A. Pajor. Volumes mixtes et sous-espaces l_1^n des espaces de Banach. *Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). Astérisque* **131** (1985), 401–411.

[45] V. I. Paulsen. *Completely Bounded Maps and Dilations.* Pitman Research Notes in Mathematics **146**, Longman, 1986.

[46] G. K. Pedersen. *C^*-algebras and their Automorphism Groups.* Academic Press Inc., London-New York, 1979.

[47] G. Pisier. *Introduction to Operator Space Theory.* London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003.

[48] C. Pop and R. R. Smith. Crossed products and entropy of automorphisms. *J. Funct. Anal.* **206** (2004), 210–232.

[49] R. T. Powers and G. L. Price. Binary shifts on the hyperfinite II_1 factor. In: *Representation theory of groups and algebras,* 453–464. *Contemp. Math.* **145**. AMS, Providence, RI, 1993.

[50] M. Rørdam. Classification of nuclear, simple C^*-algebras. In: *Classification of Nuclear C^*-algebras. Entropy in Operator Algebras*, pp. 1–145. Berlin: Springer, 2002.

[51] R. R. Smith. Completely contractive factorizations of C^*-algebras. *J. Funct. Anal.* **64** (1985), 330–337.

[52] N. Tomczak-Jaegermann. The moduli of smoothness and convexity and the Rademacher averages of trace classes $S_p\,(1 \leq p < \infty)$. *Studia Math.* **50** (1974), 163–182.

[53] D. V. Voiculescu. Dynamical approximation entropies and topological entropy in operator algebras. *Comm. Math. Phys.* **170** (1995), 249–281.

[54] P. Walters. *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics, 79. Springer-Verlag, New York, Berlin, 1982.

[55] K. Yosida. *Functional Analysis.* Reprint of the sixth (1980) edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

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