The Baum-Connes assembly map(s)

Young topologist meeting 2021

Julian Kranz

University of Münster

July 12, 2021

The conjecture

Conjecture (Baum-Connes)

For every countable discrete group G, the assembly map

$$\mu : K_*^G(\mathcal{E}_{\mathrm{Fin}} G) \to K_*(C_r^*(G))$$

is an isomorphism.

The conjecture

Conjecture (Baum-Connes)

For every countable discrete group G, the assembly map

$$\mu : K_*^G(\mathcal{E}_{\mathsf{Fin}} G) \to K_*(C_r^*(G))$$

is an isomorphism.

Why should we care?

The Baum-Connes conjecture implies

- the strong Novikov conjecture
- the stable Gromov-Lawson-Rosenberg conjecture (but we need $\mathbb{R}$-coefficients)
- the Kadison-Kaplansky conjecture
- ...

The conjecture

Conjecture (Baum-Connes with coefficients)

For every G-C^*-algebra A, the assembly map

$$\mu : K_*^G(\mathcal{E}_{\mathrm{Fin}} G, A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

The conjecture

Conjecture (Baum-Connes with coefficients)

For every G-C^*-algebra A, the assembly map

$$\mu : K_*^G(\mathcal{E}_{\mathrm{Fin}} G, A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Question

- What is a G-C^*-algebra?
- What is $K_*^G(\mathcal{E}_{\mathrm{Fin}} G, A)$?
- What is $K_*(A \rtimes_r G)$?
- What does μ do?

The conjecture

Conjecture (Baum-Connes with coefficients)

For every G-C^*-algebra A, the assembly map

$$\mu : K_*^G(\mathcal{E}_{\mathrm{Fin}} G, A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Question

- What is a G-C^*-algebra?
- What is $K_*^G(\mathcal{E}_{\mathrm{Fin}} G, A)$?
- What is $K_*(A \rtimes_r G)$?
- What does μ do?

$\rightsquigarrow$ Let's do the abstract approach!

C^*-algebras

Definition (concrete)

A C^*-algebra is a $\|\cdot\|$-closed $*$-subalgebra $A \subseteq \mathcal{L}(H)$ where H is a complex Hilbert space.

C^*-algebras

Definition (concrete)

A C^*-algebra is a $\|\cdot\|$-closed $*$-subalgebra $A \subseteq \mathcal{L}(H)$ where H is a complex Hilbert space.

Definition (abstract)

A C^*-algebra is a complex Banachalgebra A with an involution $* : A \to A$ satisfying some axioms.

C^*-algebras

Definition (concrete)

A C^*-algebra is a $\|\cdot\|$-closed $*$-subalgebra $A \subseteq \mathcal{L}(H)$ where H is a complex Hilbert space.

Definition (abstract)

A C^*-algebra is a complex Banachalgebra A with an involution $*: A \to A$ satisfying some axioms.

Example

- compact operators $\mathcal{K}(H) \subseteq \mathcal{L}(H)$
- $C_0(X) := \{f : X \to \mathbb{C}, \text{ cts. \& vanishing at } \infty\}$
- $C_r^*(G) := \overline{\mathbb{C}G}^{\|\cdot\|} \subseteq \mathcal{L}(\ell^2(G))$

The maximal crossed product

Definition (maximal crossed product)

Let A be a (unital) C^*-algebra and $\alpha : G \to \mathrm{Aut}(A)$ an action.
$A \rtimes G$ is the universal C^*-algebra

$$C^* \left(A, G \;\middle|\; g \cdot h = gh, \quad g^* = g^{-1}, \quad \alpha_g(a) = gag^* \right).$$

The maximal crossed product

Definition (maximal crossed product)

Let A be a (unital) C^*-algebra and $\alpha : G \to \mathrm{Aut}(A)$ an action.
$A \rtimes G$ is the universal C^*-algebra

$$C^* \left(A, G \;\middle|\; g \cdot h = gh, \quad g^* = g^{-1}, \quad \alpha_g(a) = gag^* \right).$$

Remark

- $A \rtimes G$ is the 'homotopy quotient' of A by G where

  homotopy = conjugation by unitaries

- $A \rtimes G$ is the max. completion of the twisted group ring $A[G]_\alpha$.

The reduced crossed product

Definition

$A \rtimes_r G$ is the image of the regular representation

$$\Lambda : A \rtimes G \to \mathcal{L}(\ell^2(G, A)).$$

The reduced crossed product

Definition

$A \rtimes_r G$ is the image of the regular representation

$$\Lambda : A \rtimes G \to \mathcal{L}(\ell^2(G, A)).$$

Remark

- $\Lambda : A \rtimes G \twoheadrightarrow A \rtimes_r G$ is injective when G is amenable.
- $C^*(G) = \mathbb{C} \rtimes G$
- $C^*_r(G) = \mathbb{C} \rtimes_r G$

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

- $kk^G(0) = 0$.

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

- $kk^G(0) = 0$.
- kk^G *is homotopy-invariant.*

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

- $kk^G(0) = 0$.
- kk^G *is homotopy-invariant.*
- kk^G *preserves split exact sequences.*

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

- $kk^G(0) = 0$.
- kk^G *is homotopy-invariant.*
- kk^G *preserves split exact sequences.*
- kk^G *inverts the* $A \otimes \mathcal{K}(H) \to A \otimes \mathcal{K}(H')$ *whenever* $H \hookrightarrow H'$ *is an inclusion of G-Hilbert spaces.*

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

- $kk^G(0) = 0$.
- kk^G *is homotopy-invariant.*
- kk^G *preserves split exact sequences.*
- kk^G *inverts the* $A \otimes \mathcal{K}(H) \to A \otimes \mathcal{K}(H')$ *whenever* $H \hookrightarrow H'$ *is an inclusion of* G*-Hilbert spaces.*
- kk^G *is universal with the above properties.*

Equivariant KK-theory

Theorem (Meyer, Bunke-Engel-Land)

Let $C^*_{G,sep}$ be the cat. of sep. G-C^*-algebras. There is a presentable stable ∞-category KK^G and a functor

$$kk^G : C^*_{G,sep} \to KK^G$$

with the following properties:

- $kk^G(0) = 0$.
- kk^G *is homotopy-invariant.*
- kk^G *preserves split exact sequences.*
- kk^G *inverts the* $A \otimes \mathcal{K}(H) \to A \otimes \mathcal{K}(H')$ *whenever* $H \hookrightarrow H'$ *is an inclusion of* G*-Hilbert spaces.*
- kk^G *is universal with the above properties.*

$\rightsquigarrow$ Think of a 'stable homotopy category'!

Some properties of KK^G

- We get a top. K-theory spectrum

$$K := KK(\mathbb{C}, -) : C^*_{sep} \to Sp$$

Some properties of KK^G

- We get a top. K-theory spectrum

$$K := KK(\mathbb{C}, -) : C^*_{sep} \to Sp$$

- We have functors

$$- \rtimes G : KK^G \to KK$$
$$- \rtimes_r G : KK^G \to KK$$
$$- \otimes A : KK^G \to KK^G$$
$$\mathrm{Res}^H_G : KK^G \to KK^H$$
$$\mathrm{Ind}^G_H : KK^H \to KK^G$$

where $H < G$ is a subgroup and $\mathrm{Ind}^G_H \dashv \mathrm{Res}^H_G$.

Localizing at finite subgroups

Definition (compactly contractible vs. compactly induced)

$$\mathcal{CC} := \left\{ A \in KK^G : \mathrm{Res}_G^H(A) \simeq 0 \quad \forall H < G \text{ finite} \right\}$$

$$\mathcal{CI} := \left\{ \mathrm{Ind}_H^G(B) : B \in KK^H, \quad H < G \text{ finite} \right\}$$

Localizing at finite subgroups

Definition (compactly contractible vs. compactly induced)

$$\mathcal{CC} := \left\{ A \in KK^G : \mathrm{Res}_G^H(A) \simeq 0 \quad \forall H < G \text{ finite} \right\}$$

$$\mathcal{CI} := \left\{ \mathrm{Ind}_H^G(B) : B \in KK^H, \quad H < G \text{ finite} \right\}$$

Let $\langle \mathcal{CI} \rangle$ be the smallest subcat. containing $\mathcal{CI}$ which is closed under equivalence, fiber sequences, suspension & countable sums.

Localizing at finite subgroups

Definition (compactly contractible vs. compactly induced)

$$\mathcal{CC} := \left\{ A \in KK^G : \mathrm{Res}_G^H(A) \simeq 0 \quad \forall H < G \text{ finite} \right\}$$

$$\mathcal{CI} := \left\{ \mathrm{Ind}_H^G(B) : B \in KK^H, \quad H < G \text{ finite} \right\}$$

Let $\langle \mathcal{CI} \rangle$ be the smallest subcat. containing $\mathcal{CI}$ which is closed under equivalence, fiber sequences, suspension & countable sums.

Theorem (Meyer-Nest)

$(\mathcal{CC}, \langle \mathcal{CI} \rangle)$ is complementary, i.e. $KK^G(\langle \mathcal{CI} \rangle, \mathcal{CC}) \simeq 0$ and for every $A \in KK^G$ we have a fiber sequence

$$\underbrace{\tilde{A}}_{\in \langle \mathcal{CI} \rangle} \xrightarrow{D} A \to \underbrace{N}_{\in \mathcal{CC}}$$

The assembly map

Theorem (Meyer-Nest)

For every $A \in KK^G$ we have a fiber sequence

$$\underbrace{\tilde{A}}_{\in\langle\mathcal{CI}\rangle} \xrightarrow{D} A \to \underbrace{N}_{\in\mathcal{CC}}$$

The assembly map

Theorem (Meyer-Nest)

For every $A \in KK^G$ we have a fiber sequence

$$\underbrace{\tilde{A}}_{\in\langle\mathcal{CI}\rangle} \xrightarrow{D} A \to \underbrace{N}_{\in\mathcal{CC}}$$

Definition (Meyer-Nest)

The assembly map is the map

$$K(\tilde{A} \rtimes_r G) \to K(A \rtimes_r G)$$

induced by D.

Identification with the Baum-Connes assembly map

Theorem (Meyer-Nest)

The indicated maps in the following diagram are isomorphisms

$$\begin{array}{ccc}
K_*^G(\mathcal{E}_{\mathrm{Fin}}G, \tilde{A}) & \xrightarrow{\cong} & K_*(\tilde{A} \rtimes_r G) \\
\downarrow \cong & & \downarrow MN \\
K_*^G(\mathcal{E}_{\mathrm{Fin}}G, A) & \xrightarrow{BC} & K_*(A \rtimes_r G)
\end{array}$$

The Davis-Lück picture

Fact (Green's imprimitivity theorem)

For every subgroup $H < G$, we have KK^G-equivalences

$$(\mathrm{Ind}_H^G A) \rtimes_r G \sim_{KK} A \rtimes_r H$$

The Davis-Lück picture

Fact (Green's imprimitivity theorem)

For every subgroup $H < G$, we have KK^G-equivalences

$$(\mathrm{Ind}_H^G A) \rtimes_r G \sim_{KK} A \rtimes_r H$$

$\rightsquigarrow$ Instead of building $\tilde{A}$ from induced algebras and then taking crossed products, directly take crossed products by subgroups.

The Davis-Lück picture

Fact (Green's imprimitivity theorem)

For every subgroup $H < G$, we have KK^G-equivalences

$$(\mathrm{Ind}_H^G A) \rtimes_r G \sim_{KK} A \rtimes_r H$$

$\rightsquigarrow$ Instead of building $\tilde{A}$ from induced algebras and then taking crossed products, directly take crossed products by subgroups.

Guess

The assembly map is equivalent to the map

$$\operatorname*{colim}_{H<G \text{ finite}} K(A \rtimes_r H) \to K(A \rtimes_r G)$$

The Davis-Lück picture

Fact (Green's imprimitivity theorem)

For every subgroup $H < G$, we have KK^G-equivalences

$$(\mathrm{Ind}_H^G A) \rtimes_r G \sim_{KK} A \rtimes_r H$$

$\rightsquigarrow$ Instead of building $\tilde{A}$ from induced algebras and then taking crossed products, directly take crossed products by subgroups.

Guess

The assembly map is equivalent to the map

$$\operatorname*{colim}_{H<G \text{ finite}} K(A \rtimes_r H) \to K(A \rtimes_r G)$$

$\rightsquigarrow$ almost right, but we need to work with orbits G/H instead of subgroups $H < G$.

Notation

Definition (Orbit category)

- $\mathrm{Or}(G)$ is the category of homogeneous G-spaces G/H.

Notation

Definition (Orbit category)

- $\mathrm{Or}(G)$ is the category of homogeneous G-spaces G/H.
- $\mathrm{Or}_{\mathrm{Fin}}(G) := \{G/H \quad | \quad H \text{ finite}\} \subseteq \mathrm{Or}(G)$

Notation

Definition (Orbit category)

- $\mathrm{Or}(G)$ is the category of homogeneous G-spaces G/H.
- $\mathrm{Or}_{\mathrm{Fin}}(G) := \{G/H \mid H \text{ finite}\} \subseteq \mathrm{Or}(G)$

Definition

Let $\overline{G/H}$ be the groupoid with objects G/H and morphisms

$$\mathrm{Hom}_{\overline{G/H}}(x, y) := \{g \in G \mid gx = y\}.$$

Notation

Definition (Orbit category)

- $\mathrm{Or}(G)$ is the category of homogeneous G-spaces G/H.
- $\mathrm{Or}_{\mathrm{Fin}}(G) := \{G/H \mid H \text{ finite}\} \subseteq \mathrm{Or}(G)$

Definition

Let $\overline{G/H}$ be the groupoid with objects G/H and morphisms

$$\mathrm{Hom}_{\overline{G/H}}(x, y) := \{g \in G \mid gx = y\}.$$

Remark

- $\exists$ functor $\overline{G/H} \to G$
- $\Rightarrow$ every G-C^*-algebra is also a $\overline{G/H}$-C^*-algebra.

The functor on the orbit category

Fact

- $\exists$ *crossed product* *C^*-categories* $A \rtimes \overline{G/H}$ *and* $A \rtimes_r \overline{G/H}$.

The functor on the orbit category

Fact

- $\exists$ *crossed product C*-categories* $A \rtimes \overline{G/H}$ *and* $A \rtimes_r \overline{G/H}$.
- K*-theory also works for* C^**-categories.*

The functor on the orbit category

Fact

- $\exists$ *crossed product* *C^*-categories* $A \rtimes \overline{G/H}$ *and* $A \rtimes_r \overline{G/H}$.
- *K-theory also works for C^*-categories.*
- *We have* $A \rtimes_r \overline{G/H} \sim_{KK} A \rtimes_r H$.

The functor on the orbit category

Fact

- $\exists$ *crossed product* *C^*-categories* $A \rtimes \overline{G/H}$ *and* $A \rtimes_r \overline{G/H}$.
- *K-theory also works for C^*-categories.*
- *We have* $A \rtimes_r \overline{G/H} \sim_{KK} A \rtimes_r H$.

Corollary

We get a functor

$$\boldsymbol{K}_A^G : \mathrm{Or}(G) \to Sp, \quad G/H \mapsto K(A \rtimes_r \overline{G/H})$$

The functor on the orbit category

Fact

- $\exists$ *crossed product C*-categories* $A \rtimes \overline{G/H}$ *and* $A \rtimes_r \overline{G/H}$.
- *K-theory also works for C^*-categories.*
- *We have* $A \rtimes_r \overline{G/H} \sim_{KK} A \rtimes_r H$.

Corollary

We get a functor

$$\boldsymbol{K}_A^G : \mathrm{Or}(G) \to Sp, \quad G/H \mapsto K(A \rtimes_r \overline{G/H})$$

Definition (Davis-Lück)

The assembly map is the map

$$\operatorname*{colim}_{G/H \in \mathrm{Or}_{\mathrm{Fin}}(G)} K(A \rtimes_r \overline{G/H}) \to K(A \rtimes_r G)$$

Identification with the Meyer-Nest assembly map

Theorem (K.)

The indicated maps in the following diagram are isomorphisms.

$$\begin{array}{ccc}
\operatorname*{colim}_{G/H\in \mathrm{Or}_{\mathrm{Fin}}(G)} K(\tilde{A} \rtimes_r \overline{G/H}) & \xrightarrow{\cong} & K(\tilde{A} \rtimes_r G) \\
\downarrow{\cong} & & \downarrow{MN} \\
\operatorname*{colim}_{G/H\in \mathrm{Or}_{\mathrm{Fin}}(G)} K(A \rtimes_r \overline{G/H}) & \xrightarrow{DL} & K(A \rtimes_r G)
\end{array}$$

References

- P. Baum, A. Connes, N. Higson. *Classifying spaces for proper actions and K-theory of group C^*-algebras.* (1994)
- J. Davis, W. Lück. *Spaces over a category and isomorphism conjectures in K- and L-theory.* (1998)
- R. Meyer, R. Nest. *The Baum-Connes conjecture via localization of categories.* (2004)
- J. Kranz. *An identification of the Baum-Connes and Davis-Lück assembly maps.* (2020)
- U. Bunke, A. Engel, M. Land. *A stable ∞-category for equivariant KK-theory.* (2021)
- U. Bunke, A. Engel, M. Land. *Paschke duality and assembly maps.* (2021)