Amenability and weak containment for étale groupoids

Quantum Groups Seminar 2021

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Amenable groups

Theorem (Hulanicki)

Let G be a discrete group. TFAE:

a) G is amenable.

b) $C_r^*(G)$ is nuclear.

c) $C^*(G) = C_r^*(G)$ (weak containment property)

Question (Weak containment problem)

Given some dynamical input $\mathcal{X}$, how are the following related?

a) $\mathcal{X}$ is amenable.

b) $C_r^*(\mathcal{X})$ is nuclear.

c) $C^*(\mathcal{X}) = C_r^*(\mathcal{X})$.

Examples for dynamical input

- $G \curvearrowright X$ group action on a space $\rightsquigarrow C_0(X) \rtimes_{(r)} G$
  (Matsumura, Buss-Echterhoff-Willett)
- $G \curvearrowright X$ partial group action on a space $\rightsquigarrow C_0(X) \rtimes_{(r)} G$
  (Buss-Ferraro-Sehnem)
- $G \curvearrowright A$ (partial) group action on a C^*-algebra $\rightsquigarrow A \rtimes_{(r)} G$
  (Buss-Echterhoff-Willett, Buss-Ferraro-Sehnem)
- G groupoid $\rightsquigarrow C^*_{(r)}(G)$
  (Anantharaman–Delaroche, Willett, K.)
- $S \curvearrowright X$ partial action by inverse semigroup $\rightsquigarrow C_0(X) \rtimes_{(r)} S$
  (K.)

Amenability for group actions

Theorem (Anantharaman–Delaroche-Renault)

Let $G \curvearrowright X$ be a partial action of a discrete group on a loc. cpt. space. Consider the following conditions:

a) $G \curvearrowright X$ *is amenable*

b) $C_0(X) \rtimes_r G$ *is nuclear*

c) $C_0(X) \rtimes G = C_0(X) \rtimes_r G$

Then $a) \Leftrightarrow b) \Rightarrow c)$.

Question (Open)

Do we also have $c) \Rightarrow b)$?

The case of exact groups

Theorem (Matsumura)

*Let $G \curvearrowright X$ be an action of an **exact** discrete group on a compact space X. Then $C_0(X) \rtimes G = C_0(X) \rtimes_r G$ if and only if $G \curvearrowright X$ is amenable.*

Theorem (Buss-Echterhoff-Willett)

*Let $G \curvearrowright X$ be an action of an **exact** locally compact group on a loc. cpt. space X. Then $C_0(X) \rtimes G = C_0(X) \rtimes_r G$ if and only if $G \curvearrowright X$ is amenable.*

Theorem (Buss-Ferraro-Sehnem)

*Let $G \curvearrowright X$ be a **partial** action of an **exact** discrete group on a loc. cpt. space. Then $C_0(X) \rtimes G = C_0(X) \rtimes_r G$ if and only if $G \curvearrowright X$ is amenable.*

What about groupoids?

Groupoids

Definition

A topological groupoid consists of

a) An arrow space G
b) A closed unit space $G^{(0)} \subseteq G$
c) Range and source maps $r, s : G \to X$
d) A composition map $\bullet : G \times_{r,X,s} G \to G$
e) An inversion map $()^{-1} : G \to G$

satisfying a bunch of axioms.

Definition (equivalent)

A topological groupoid is a topological category where every morphism is invertible. (arrows=morphisms, units=objects)

Convention

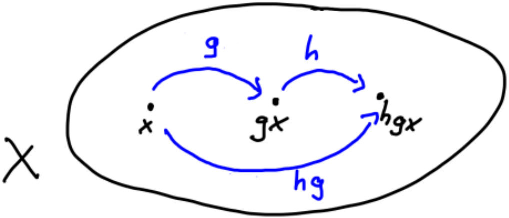
All groupoids are locally compact Hausdorff.

Example: Transformation groupoids

Definition

Let $G \curvearrowright X$ be an action. Then $X \rtimes G := X \times G$ is a groupoid with

- Unit space X
- Composition $(x, g) \cdot (y, h) := (y, gh)$
- Range map $r(x, g) := gx$
- Source map $s(x, g) = x$



Étale groupoids

Definition

A bisection of a groupoid G is a subset $U \subseteq G$ such that $r|_U : U \to r(U)$ and $s|_U : U \to s(U)$ are homeomorphisms.

Definition

A groupoid G is called étale, if its topology has a basis of open bisections.

Remark

If G is étale, then the *fibers*

$$G^x := \{g \in G : r(g) = x\}, \quad G_x := \{g \in G : s(g) = x\}$$

are discrete. Therefore, étale groupoids are the analogue of discrete groups.

Groupoid C^*-algebras

Definition

Let G be an étale groupoid. Equip $C_c(G)$ with the operations

$$f * g(\gamma) := \sum_{\eta\zeta=\gamma} f(\eta)g(\zeta), \quad f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

The *maximal groupoid C^*-algebra* $C^*(G)$ is the enveloping C^*-algebra of $C_c(G)$.
The *reduced groupoid C^*-algebra* $C_r^*(G)$ is the image of the left regular representation

$$\Lambda : C^*(G) \to \mathcal{L}_{C_0(G^{(0)})}(\ell^2(G)).$$

Is there a more algebraic description of $C^*(G)$?

Inverse semigroups

Definition

An *inverse semigroup* is a semigroup S such that for every $s \in S$, there is a unique s^* satisfying $ss^*s = s$ and $s^*ss^* = s^*$.

Example

Every inverse semigroup S can be realized as partial isometries on a Hilbert space.

Example

Let G be an étale groupoid. Then the set S of open bisections of G is an inverse semigroup.

Convention

All my inverse semigroups have a unit 1.

Partial actions

Definition

Let S be an inverse semigroup. A partial action $\theta : S \curvearrowright X$ is a collection of open subsets $\{D_s \subseteq X\}_{s \in S}$ and homeomorphisms $\{\theta_s : D_{s^*} \to D_s\}_{s \in S}$ satisfying

- $\theta_1 = \mathrm{id}_X$
- $\theta_{ts} \supseteq \theta_s \circ \theta_t$ for all $s, t \in S$.

Example

Let G be an étale groupoid and S its inverse semigroup of open bisections. There is a partial action $\theta : S \curvearrowright G^{(0)}$ given by

$$\theta_U : s(U) \xrightarrow{s^{-1}} U \xrightarrow{r} r(U), \quad U \in S.$$

Covariant representations

Definition (Sieben)

A *covariant representation* (π, v) of $(C_0(X), S)$ is a non-degenerate $*$-homomorphism $\pi : C_0(X) \to \mathcal{B}(H)$ and a representation $v : S \to \mathcal{B}(H)$ by partial isometries such that

- $\pi(C_0(D_s))H = v_{ss^*}H, \quad s \in S$
- $v_s a v_{s^*} = \theta_s(a), \quad s \in S, a \in D_{s^*}$

Crossed products by partial actions

Theorem (Sieben)

There is a universal covariant representation (ι, u) *such that for any covariant representation* $(\pi, v) : (C_0(X), S) \to \mathcal{B}(H)$ *and*

$$C_0(X) \rtimes S := \overline{\mathrm{span}}\{\iota(a)u_s : s \in S, a \in D_s\},$$

$\exists$ *unique* $*$*-homomorphism* $\pi \rtimes v : C_0(X) \rtimes S \to \mathcal{B}(H)$ *s.t.*

$$\pi(a)v_s = \pi \rtimes v(\iota(a)u_s), \quad s \in S, a \in D_s.$$

Theorem (Paterson)

$C^*(G) = C_0(G^{(0)}) \rtimes S$

Back to the original question

Question (Weak containment problem)

Given some dynamical input $\mathcal{X}$, how are the following related?

a) $\mathcal{X}$ is amenable.

b) $C_r^*(\mathcal{X})$ is nuclear.

c) $C^*(\mathcal{X}) = C_r^*(\mathcal{X})$.

The weak containment problem for groupoids

Theorem (Anantharaman–Delaroche-Renault)

Let G be an étale groupoid. Consider the following conditions:

a) *G is amenable*

b) *$C_r^*(G)$ is nuclear*

c) *$C^*(G) = C_r^*(G)$*

Then $a) \Leftrightarrow b) \Rightarrow c)$.

Example (Willett)

There is a non-amenable étale groupoid G with $C^*(G) = C_r^*(G)$.

Question

Do we get $c) \Rightarrow b)$ if we restrict to "exact" groupoids?

What is an exact groupoid?

Theorem (Matsumura)

Let $G \curvearrowright X$ be an action of an ***exact*** *discrete group on a compact space X. Then $C_0(X) \rtimes G = C_0(X) \rtimes_r G$ if and only if $G \curvearrowright X$ is amenable.*

Theorem (Ozawa)

A discrete group G is exact if and only if $\beta G \rtimes G$ is amenable.

Definition (Anantharaman-Delaroche)

An étale groupoid G is *strongly amenable at infinity*, if $\beta_r G \rtimes G$ is amenable where

$$\beta_r G = \mathrm{Spec}\left(\left\{ f \in C_b(G) : \forall \varepsilon > 0 \exists K \subseteq_{cpt} G^{(0)} : \|f|_{G \setminus r^{-1}(K)}\|_\infty < \varepsilon \right\}\right)$$

is the *fiberwise Stone-Čech compactification* of G.

Matsumura's Theorem for groupoids

Theorem (K.)

Let G be an étale groupoid which is strongly amenable at infinity. Suppose that $C^*(G) = C^*_r(G)$. Then $C^*_r(G)$ is nuclear. In particular, G is amenable.

Strategy:

Factor the inclusion $C^*_r(G) \hookrightarrow C^*_r(G)^{**}$ as a composition

$$C^*_r(G) \hookrightarrow \underbrace{C^*_r(\beta_r G \rtimes G)}_{\text{nuclear}} \xrightarrow{\text{cpc}} C^*_r(G)^{**}.$$

This implies that $C^*_r(G)$ itself is nuclear.

Why do we need partial actions?

Recall: Let G be an étale groupoid with inverse semigroup S.

- $C^*(G) = C_0(G^{(0)}) \rtimes S$
- $C^*(\beta_r G \rtimes G) = C_0(\beta_r G) \rtimes S$

Lemma

The partial action $S \curvearrowright C_0(G^{(0)})$ *extends to a partial action* $S \curvearrowright C_0(G^{(0)})^{**}$. *We have canonical* $*$*-homomorphisms*

$$C_0(G^{(0)}) \rtimes S \to C_0(G^{(0)})^{**} \rtimes S \to (C_0(G^{(0)}) \rtimes S)^{**}$$

whose composition is the natural inclusion.

The Haagerup standard form

Theorem

There is a faithful normal covariant representation

$$(\pi, v) : (C_0(G^{(0)})^{**}, S) \to \mathcal{B}(H)$$

such that

$$\pi(C_0(G^{(0)}))' = \pi(C_0(G^{(0)})^{**}).$$

Proof of the main theorem

Theorem (K.)

Let G be étale and strongly amenable at infinity such that $C^*(G) = C^*_r(G)$. Then $C^*_r(G)$ is nuclear.

Proof

Let $(\pi, v) : (C_0(G^{(0)})^{**}, S) \to \mathcal{B}(H)$ be the Haagerup standard form. By Arveson's theorem, we find a c.p.p map ϕ as follows:

$$\begin{array}{ccc} C_0(\beta_r G) \rtimes S & & \\ \uparrow & \overset{\phi}{\searrow} & \\ C_0(G^{(0)}) \rtimes S & \xrightarrow{\pi \rtimes v} & \mathcal{B}(H) \end{array}$$

We used that reduced crossed products preserve inclusions.

$$\begin{array}{ccc} C_0(\beta_r G) \rtimes S & & \\ \hookuparrow & \overset{\phi}{\searrow} & \\ C_0(G^{(0)}) \rtimes S & \xrightarrow{\pi \rtimes v} & \mathcal{B}(H) \end{array}$$

Using multiplicative domain arguments, we show that

- $\phi(C_0(\beta_r G)) \subseteq \pi(C_0(G^{(0)}))' \cong C_0(G^{(0)})^{**}$
- $\psi := \phi|_{C_0(\beta_r G)}$ is S-equivariant.

We get a factorization

$$C_0(G^{(0)}) \rtimes S \hookrightarrow C_0(\beta_r G) \rtimes S \xrightarrow{\psi \rtimes S} C_0(G^{(0)})^{**} \rtimes S \to (C_0(G^{(0)}) \rtimes S)^{**}$$

$$\begin{array}{ccc}
C_0(\beta_r G) \rtimes S & & \\
\uparrow & \overset{\phi}{\searrow} & \\
C_0(G^{(0)}) \rtimes S & \xrightarrow{\pi \rtimes v} & \mathcal{B}(H)
\end{array}$$

Using multiplicative domain arguments, we show that

- $\phi(C_0(\beta_r G)) \subseteq \pi(C_0(G^{(0)}))' \cong C_0(G^{(0)})^{**}$
- $\psi := \phi|_{C_0(\beta_r G)}$ is S-equivariant.

We get a factorization

$$C_r^*(G) \hookrightarrow \underbrace{C_r^*(\beta_r G \rtimes G)}_{\text{nuclear}} \xrightarrow{\psi \rtimes S} C_0(G^{(0)})^{**} \rtimes S \to C_r^*(G)^{**}$$

Thus $C_r^*(G)$ is nuclear. $\square$

Thank you for your attention!