Classifiability of crossed products by nonamenable groups

C*-algebras: Structure and Dynamics

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Kirchberg–Phillips classification

Theorem (Kirchberg, Phillips)

Let A and B be unital, simple, separable, nuclear, purely infinite C^*-algebras satisfying the UCT. Then any isomorphism

$$\alpha : K_*(A) \xrightarrow{\cong} K_*(B), \quad \alpha([1_A]_0) = [1_B]_0$$

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Question

Let $G \curvearrowright X$ be an action of a group on a compact space. When is $C(X) \rtimes G$ a UCT Kirchberg algebra?

Amenable actions

Definition (Anantharaman-Delaroche–Renault)

An action $G \curvearrowright X$ of a countable discrete group on a compact metric space is called *amenable* if there is a sequence

$$\{\mu_n : X \to \mathrm{Prob}(G)\}_{n\in\mathbb{N}}$$

of continuous maps such that

$$\lim_{n\to\infty} \sup_{x\in X} \|g.\mu_n(x) - \mu_n(gx)\|_1 = 0 \quad \forall g \in G.$$

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Theorem (Ananatharaman-Delaroche)

An action $G \curvearrowright X$ is amenable if and only if $C(X) \rtimes G$ is nuclear.

Properties of amenable actions

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X has a G-invariant measure if and only if $C(X) \rtimes G$ has a trace if and only if G is amenable.

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When is $C(X) \rtimes G$ purely infinite?

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Definition (Kerr)

$G \curvearrowright X$ has *dynamical comparison* if for every open subset $V \subseteq X$, there is an open cover $X = U_1 \cup \ldots \cup U_n$ and $g_1, \ldots, g_n \in G$ such that $(g_i U_i)_{i=1\ldots,n}$ are pairwise disjoint subsets of V.

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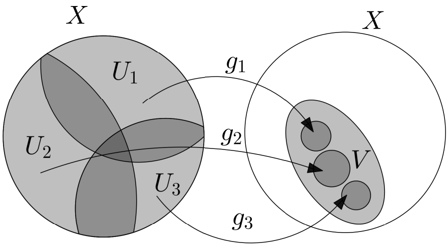
$$X \precsim V$$



From dynamical comparison to pure infiniteness

Theorem (Ma)

Let $G \curvearrowright X$ be a minimal, amenable, topologically free action of a nonamenable group. If $G \curvearrowright X$ has dynamical comparison, then $C(X) \rtimes G$ is purely infinite.

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In this case, $C(X) \rtimes G$ is a UCT Kirchberg algebra.

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When does $G \curvearrowright X$ have dynamical comparison?

Paradoxical towers

Definition (GGKN)

Let $n \in \mathbb{N}$. A countable group G has *n-paradoxical towers* if for every finite subset $D \subseteq G$, there are $A_1, \ldots, A_n \subseteq G$ and $g_1, \ldots, g_n \in G$ such that

a) The sets $\{dA_i\}_{d \in D, i=1,\ldots,n}$ are pairwise disjoint

b) $G = \cup_{i=1}^{n} g_i A_i$

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Theorem (GGKN)

The following nonamenable groups have paradoxical towers:

- *(Acylindrically) hyperbolic groups ($F_n, \ldots$)*
- *Lattices in many Lie groups ($SL_n(\mathbb{Z}), \ldots$)*
- *Groups with "nice boundary actions" ($BS(n,m), \ldots$)*
- $\ldots$

Dynamical comparison is automatic!

Theorem (GGKN)

Let $G = H \times K$ where H is a group with paradoxical towers and K is any countable group. Then every minimal, amenable action $G \curvearrowright X$ on a compact metric space has dynamical comparison.

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Assume moreover that the action $G \curvearrowright X$ is topologically free. Then $C(X) \rtimes G$ is a UCT Kirchberg algebra.

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Theorem (GGKN)

Let G be a group containing F_2 and let $G \curvearrowright X$ be a minimal, amenable, topologically free action on a compact metric space. Then $C(X) \rtimes G$ is properly infinite.

The proof for $G = F_2$

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Suppose that for every finite set $D \subseteq F_2$, there is an open set $V \subseteq X$ such that

- *The sets $\{dV\}_{d \in D}$ are pairwise disjoint.*
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Then $G \curvearrowright X$ has dynamical comparison.

Proof.

Let $U \subseteq X$ open. By minimality, $D^{-1}U = X$ for finite $D \subseteq G$. Then $X \precsim V \precsim U$. □

Lemma ("strong" paradoxical towers)

Let $D \subseteq F_2$ *be finite. Then there are* $A_1, A_2, A_3 \subseteq F_2$ *and* $g_1, g_2, g_3 \in F_2$ *such that*

- *The sets* $(dA_i)_{d\in D, i=1,\dots,n}$ *are pairwise disjoint.*
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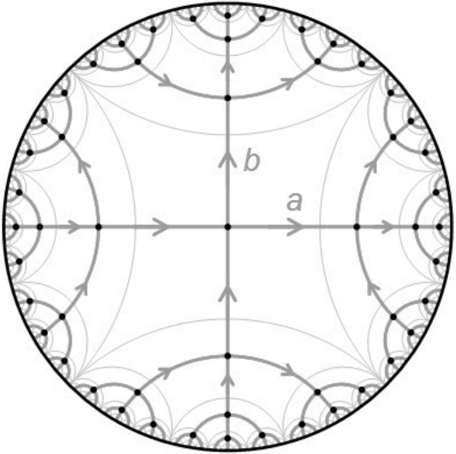

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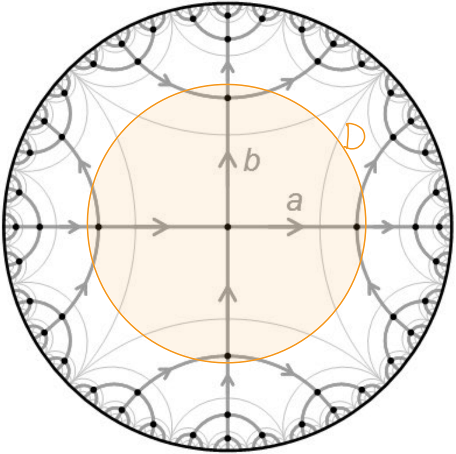

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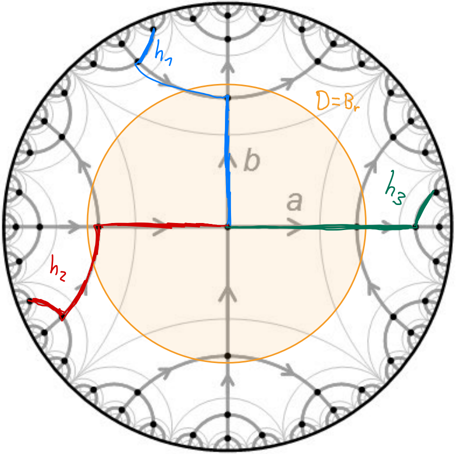

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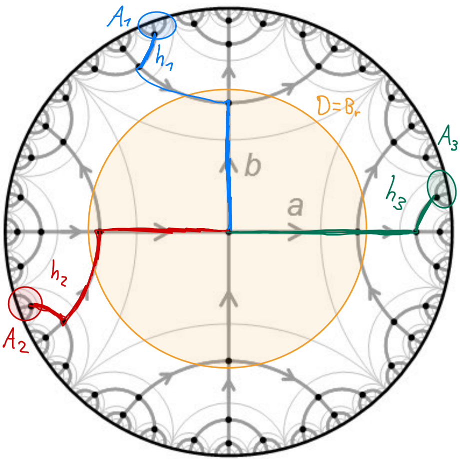

Thm: $F_2 \curvearrowright X$ has dynamical comparison

Let $0 < \varepsilon < \frac{1}{12}$. Let $\mu : X \to \mathrm{Prob}(G)$ such that

$$\sup_{x \in X} \|g.\mu(x) - \mu(gx)\|_1 < \varepsilon, \quad \forall g \in D^{-1} \cup \{g_1, g_2, g_3\}.$$

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Define

$$V_i := \left\{ x \in X \mid \mu(x)(A_i) > \frac{1}{2} + \varepsilon \right\} \subseteq X, \quad i = 1, 2, 3.$$

and $V = V_1 \cup V_2 \cup V_3$.

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Since $(dA_i)_{d \in D, i=1,2,3}$ are pairwise disjoint, $(dV_i)_{d \in D, i=1,2,3}$ are pairwise disjoint $\Rightarrow (dV)_{d \in D}$ are pairwise disjoint.

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Thank you very much!