K-THEORY OF BERNOULLI SHIFTS OF FINITE GROUPS ON UHF-ALGEBRAS

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ABSTRACT. We show that the Bernoulli shift and the trivial action of a finite group G on a UHF-algebra of infinite type are KK^G-equivalent and that the Bernoulli shift absorbs the trivial action up to conjugacy. As an application, we compute the K-theory of crossed products by approximately inner flips on classifiable C^*-algebras.

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1. INTRODUCTION

In topological dynamics, a very fertile class of examples is given by Bernoulli shifts, that is, by the shift action of a group G on the product $X^G := \prod_G X$ of G-many copies of a given compact space X. When the space X is moreover totally disconnected, the K-theory of the crossed product $C\left(X^G\right) \rtimes_r G$ can be computed in many cases [CEL13]. These computations and the techniques appearing in them are not only of intrinsic interest, but they make possible the computation of the K-theory of C^*-algebras associated to large classes of (inverse) semigroups, wreath products, and many more examples [CEL13, Li19, Li22]. The simplest non-commutative analogue of a totally disconnected space is a UHF-algebra, that is, a (possibly infinite) tensor product of matrix algebras. The non-commutative version of the Bernoulli shift is the shift action of a group G on the tensor product $A^{\otimes G} := \bigotimes_{g\in G} A$ for a given unital C^*-algebra A. Our main result computes

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the K-theory of the associated crossed product in the case that G is finite and that A is a UHF-algebra:

Theorem A (Theorem 2.8)**.** *Let* G *be a finite group, let* Z *be a* G*-set and let* $M_{\mathfrak{n}}$ *be a UHF-algebra of infinite type. Then* $M_{\mathfrak{n}}$ *is* KK^G*-equivalent to* $M_{\mathfrak{n}}^{\otimes Z}$ *where we equip* $M_{\mathfrak{n}}$ *with the trivial* G*-action and* $M_{\mathfrak{n}}^{\otimes Z}$ *with the Bernoulli shift. In particular, we have*

$$K_*\left(M_{\mathfrak{n}}^{\otimes Z} \rtimes G\right) \cong K_*(C^*(G) \otimes M_{\mathfrak{n}}) \cong K_*(C^*(G))[1/\mathfrak{n}].$$

The proof of Theorem A relies on a representation theoretic argument about invertibility of a certain element in the representation ring $R_{\mathbb{C}}(G)$ after inverting sufficiently many primes (see Proposition 2.1). A byproduct of the proof is that the Bernoulli shift absorbs the trivial action not only in KK-theory, but up to conjugacy. We point out that this fact may alternatively be extracted from [HW08, Lemma 3.1].

Theorem B (Theorem 2.7)**.** *With the notation as in Theorem A, there is a* G*-equivariant isomorphism*

$$M_{\mathfrak{n}}^{\otimes Z} \cong M_{\mathfrak{n}} \otimes M_{\mathfrak{n}}^{\otimes Z}.$$

One immediate consequence of Theorem A and [Izu04, Theorem 3.13] is that the Bernoulli shift $G \curvearrowright M_{\mathfrak{n}}^{\otimes Z}$ as above does not have the Rokhlin property (see Corollary 2.10). Beyond finite group actions, Theorem A also has consequences for infinite groups satisfying the Baum–Connes conjecture with coefficients [BCH94].

Corollary C (Corollary 2.11)**.** *Let* G *be a discrete group satisfying the Baum–Connes conjecture with coefficients, let* Z *be a* G*-set, let* A *be a* G-C^**-algebra and let* $M_{\mathfrak{n}}$ *be a UHF-algebra. Assume that* Z *is infinite or that* $\mathfrak{n}$ *is of infinite type. Then the inclusion* $A \to A \otimes M_{\mathfrak{n}}$ *induces an isomorphism*

$$K_*\left(A \rtimes_r G\right)[1/\mathfrak{n}] \cong K_*\left(\left(A \otimes M_{\mathfrak{n}}^{\otimes Z}\right) \rtimes_r G\right).$$

In particular, the right hand side is a $\mathbb{Z}[1/\mathfrak{n}]$*-module.*

Corollary C will be used in the follow-up paper [CEKN22] together with S. Chakraborty and S. Echterhoff to compute the K-theory of many more general Bernoulli shifts. Another consequence of Theorem A is that the Bernoulli shift of an amenable group G on a strongly self-absorbing (in the sense of [TW07]) C^*-algebra $\mathcal{D}$ satisfying the UCT is KK^G-equivalent to the trivial G-action on $\mathcal{D}$ (see Corollary 2.12; for $\mathcal{D} = \mathcal{O}_\infty$, this is [Sza18, Corollary 6.9]).

In Section 3, we apply Theorem A and compute $K_*\left(B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right)$ whenever B is a C^*-algebra with approximately inner flip (in the sense of [ER78]) satisfying the assumptions of the Elliott classification programme[1]. Thanks to Tikuisis' classification of such C^*-algebras [Tik16], the computation reduces to the following special case (only the case $\mathfrak{m} = 1$ is relevant):

[1]We refer to [Win18] and the references therein for an overview of the Elliott programme.

Theorem D (Theorem 3.2)**.** *For supernatural numbers* $\mathfrak{n}$ *and* $\mathfrak{m}$ *of infinite type, we have*

$$K_*\left(\mathcal{F}_{\mathfrak{m},\mathfrak{n}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{r}}/\mathbb{Z}, & * = 0 \\ \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}, & * = 1 \end{cases}$$

where $\mathfrak{r}$ *is the greatest common divisor of* $\mathfrak{m}$ *and* $\mathfrak{n}$*.*

We refer to Section 3 for the definition of the notation appearing above. Our methods heavily build on Izumi's computation of K-theory of flip automorphisms on C^*-algebras with finitely generated K-theory [Izu19].

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2. KK-THEORY OF BERNOULLI SHIFTS

For a finite group G, denote by $R_{\mathbb{C}}(G)$ its representation ring, given by the free abelian group generated by all irreducible complex representations of G with the tensor product as multiplication. The character of a finite-dimensional complex representation $\pi\colon G \to GL(V_\pi)$ is denoted by

$$\chi_\pi\colon G \to \mathbb{C}, \quad \chi_\pi(g) \coloneqq \mathrm{tr}\left(V_\pi \xrightarrow{\pi(g)} V_\pi\right),$$

where tr denotes the (non-normalized) trace. Recall that the map

$$R_{\mathbb{C}}(G) \to \mathbb{C}_{\text{class}}(G), \quad \pi \mapsto \chi_\pi$$

is an injective ring homomorphism with values in the algebra $\mathbb{C}_{\text{class}}(G)$ of conjugation invariant functions on G with pointwise multiplication. There is a natural isomorphism $R_{\mathbb{C}}(G) \cong KK^G(\mathbb{C},\mathbb{C})$. We refer to [Ser77] for an introduction to representation theory of finite groups and to [Kas88] for the definition of equivariant KK-theory.

Proposition 2.1. *Let* G *be a finite group, let* $k \geq 1$ *and let* Z *be a finite* G*-set. Denote by* $\pi_k\colon G \to GL\left(\ell^2\left(\{1,\dots,k\}^Z\right)\right)$ *the permutation representation associated to the* G*-set* $\{1,\dots,k\}^Z$*. Then the following hold.*

1. *There exist* $\alpha \in R_{\mathbb{C}}(G)$ *and* $r \geq 1$ *such that* $[\pi_k]^r = k\alpha$*.*
2. *There exist* $\beta \in R_{\mathbb{C}}(G)$ *and* $l \geq 1$ *such that* $[\pi_k] \cdot \beta = k^l$*.*

Proof. By considering the standard basis in $\ell^2\left(\{1,\dots,k\}^Z\right)$, it is easy to see that the trace of $\pi_k(g)$ for $g \in G$ is given by the number of g-fixed points in $\{1,\dots,k\}^Z$. In other words, the character of π_k is given by

$$\chi_{\pi_k}(g) = k^{|Z/\langle g\rangle|}.$$

We therefore have

$$\prod_{g\in G}\left(\chi_{\pi_k}-k^{|Z/\langle g\rangle|}\right)=0 \text{ in } \mathbb{C}_{\text{class}}(G).$$

Since the map $\pi\mapsto\chi_\pi$ is injective, we also have

$$\prod_{g\in G}\left([\pi_k]-k^{|Z/\langle g\rangle|}\right)=0 \text{ in } R_{\mathbb{C}}(G).$$

In particular, there are polynomials $p, q\in\mathbb{Z}[t]$ satisfying

$$[\pi_k]^{|G|}=kp([\pi_k]),\quad [\pi_k]\cdot q([\pi_k])=\prod_{g\in G}k^{|Z/\langle g\rangle|},$$

which proves the proposition. □

Definition 2.2. Let Z be a set and let $(A_z)_{z\in Z}$ be a collection of unital C^*-algebras. The infinite tensor product $\bigotimes_{z\in Z}A_z$ is defined as

$$\bigotimes_{z\in Z}A_z := \varinjlim_{F}\bigotimes_{z\in F}A_z,$$

where the inductive limit is taken over all finite subsets $F\subseteq Z$ with respect to the connecting maps $a\mapsto a\otimes 1$. Given a discrete group G, a unital C^*-algebra A and a G-set Z, the *Bernoulli shift* of G on $A^{\otimes Z}:=\bigotimes_Z A$ is the G-action induced by permuting the tensor factors according to the G-action on Z.

Countability of G and of a G-set Z and separability of A will be assumed *only if the statement involves* KK*-theory*. For example, we do not assume these when the statement is on K-theory of involved C^*-algebras.

Definition 2.3. A *supernatural number* is a formal product $\mathfrak{n}=\prod_p p^{n_p}$ where p runs over all primes and $n_p\in\{0,\dots,\infty\}$. The *UHF-algebra* associated to $\mathfrak{n}$ is the infinite tensor product

$$M_{\mathfrak{n}} := \bigotimes_p M_{p^{n_p}},$$

with $M_{p^\infty}:=M_p^{\otimes\mathbb{N}}$. We call $\mathfrak{n}$ or $M_{\mathfrak{n}}$ of *infinite type* if $n_p\in\{0,\infty\}$ for all p. We say that $\mathfrak{n}=\prod p^{n_p}$ divides $\mathfrak{m}=\prod p^{m_p}$ if $n_p\le m_p$ for all p.

Remark 2.4. Note that the above definition includes natural numbers and matrix algebras as a special case.

Definition 2.5. If M is an abelian group, we denote by $M[1/\mathfrak{n}]$ the inductive limit of the system

$$M\xrightarrow{\cdot p_1}M\xrightarrow{\cdot p_2}M\xrightarrow{\cdot p_3}\cdots$$

where $(p_1,p_2,\dots)$ contains each prime dividing $\mathfrak{n}$ infinitely many times.

Remark 2.6. If $\mathfrak{q} = \prod_p p^{n_p}$ with $n_p \geq 1$ for all p, then

$$M[1/\mathfrak{q}] \cong M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If $k \geq 1$ is a positive integer, then

$$\mathbb{Z}[1/k] \cong \left\{ \frac{m}{k^n} \middle| m \in \mathbb{Z}, n \in \mathbb{Z}_{>0} \right\} \subseteq \mathbb{Q}.$$

In general, the group $\mathbb{Z}[1/\mathfrak{n}]$ is different from the closely related group

$$\mathbb{Q}_{\mathfrak{n}} := \left\{ \frac{m}{k} \middle| m \in \mathbb{Z}, k \in \mathbb{Z}_{>0} \text{ divides } \mathfrak{n} \right\},$$

unless $\mathfrak{n}$ is of infinite type.

Theorem 2.7 (cf. [HW08, Lemma 3.1])**.** *Let G be a finite group, let $M_{\mathfrak{n}}$ be a UHF-algebra and let Z be a G-set. Assume that Z is infinite or that $\mathfrak{n}$ is of infinite type. Equip $M_{\mathfrak{n}}$ with the trivial G-action and $M_{\mathfrak{n}}^{\otimes Z}$ with the Bernoulli shift. Then there is an equivariant isomorphism*

$$M_{\mathfrak{n}}^{\otimes Z} \otimes M_{\mathfrak{n}} \cong M_{\mathfrak{n}}^{\otimes Z}.$$

If Z is infinite, and $m < \infty$, there is an equivariant isomorphism

$$M_m^{\otimes Z} \otimes M_{m^\infty} \cong M_m^{\otimes Z}.$$

Proof. Note that it suffices to prove the statement in the case that $M_{\mathfrak{n}} = M_{p^k}$ (or $M_m = M_{p^k}$) for a prime p and $k \in \{0, 1, \ldots, \infty\}$. As before, if Z is finite, we denote by π_p the permutation representation of G on $V_p := \ell^2(\{1, \ldots, p\}^Z)$, so that $M_p^{\otimes Z}$ is equivariantly isomorphic to $\mathrm{End}(V_p)$.

Assume first that $k = \infty$. We only need to prove the theorem for (any) one G-orbit of Z so we may assume that Z is finite. Let $\alpha \in R_{\mathbb{C}}(G)$ and $r \geq 1$ be as in Proposition 2.1 so that $[\pi_p]^r = p\alpha \in R_{\mathbb{C}}(G)$. Since $[\pi_p]$ is a non-negative linear combination of irreducible representations of G, α has to be the class of a finite-dimensional representation $\pi_\alpha \colon G \to GL(W_\alpha)$. In particular, we have an equivariant isomorphism $V_p^{\otimes r} \cong \mathbb{C}^p \otimes W_\alpha$. Passing to endomorphisms, we obtain an equivariant isomorphism

$$\left(M_p^{\otimes Z}\right)^{\otimes r} \cong M_p \otimes \mathrm{End}(W_\alpha)$$

with the trivial G-action on M_p. By taking the infinite tensor product we obtain an equivariant isomorphism

$$M_{p^\infty}^{\otimes Z} \cong M_{p^\infty} \otimes \mathrm{End}(W_\alpha)^{\otimes \mathbb{N}} \cong M_{p^\infty} \otimes M_{p^\infty} \otimes \mathrm{End}(W_\alpha)^{\otimes \mathbb{N}} \cong M_{p^\infty} \otimes M_{p^\infty}^{\otimes Z}.$$

Assume now that $k < \infty$ and that Z is infinite. Then Z contains infinitely many orbits of the same type. We may thus assume that Z is of the form $Z = \bigsqcup_{\mathbb{N}} G/H$ for some subgroup $H \subseteq G$. Then there is an equivariant isomorphism $M_{p^k}^{\otimes Z} \cong M_{p^\infty}^{\otimes G/H}$. This reduces the proof to the case considered above. □

Theorem 2.8. *Let G be a finite group, let Z be a countable G-set and let M_n be a UHF-algebra of infinite type. Then the canonical inclusions*

$$M_n \hookrightarrow M_n \otimes M_n^{\otimes Z} \hookleftarrow M_n^{\otimes Z}$$

are KK^G-equivalences, where M_n is endowed with the trivial action and where $M_n^{\otimes Z}$ is endowed with the Bernoulli shift. If Z is infinite, and $m < \infty$, the same conclusion holds for the inclusions

$$M_{m^\infty} \hookrightarrow M_{m^\infty} \otimes M_m^{\otimes Z} \hookleftarrow M_m^{\otimes Z}.$$

Proof. It follows from Theorem 2.7 and the fact that M_n is strongly self-absorbing (in the sense of [TW07]) that the map

$$M_n^{\otimes Z} \hookrightarrow M_n \otimes M_n^{\otimes Z}$$

is a KK^G-equivalence. Similarly, if Z is infinite and $m < \infty$, the map

$$M_m^{\otimes Z} \hookrightarrow M_{m^\infty} \otimes M_m^{\otimes Z}$$

is a KK^G-equivalence. We prove that the map

$$\mathrm{id}_{M_n} \otimes 1_{M_n^{\otimes Z}} : M_n \to M_n \otimes M_n^{\otimes Z} \tag{2.1}$$

is a KK^G-equivalence. Note that this map is the inductive limit of the maps

$$\mathrm{id}_{M_n} \otimes 1_{M_k^{\otimes Y}} : M_n \to M_n \otimes M_k^{\otimes Y} \tag{2.2}$$

where k ranges over all positive integers that divide n and where Y ranges over all finite G-subsets of Z. It follows from the finiteness of G, the nuclearity of the involved algebras and [MN06, Proposition 2.6, Lemma 2.7] that the map in (2.1) is also the homotopy colimit (with respect to the triangulated structure of KK^G) of the maps in (2.2). Since a homotopy colimit of KK^G-equivalences is a KK^G-equivalence [2], it suffices to show that the maps appearing in (2.2) are KK^G-equivalences. Note that the maps in (2.2) can be identified with the elements $[\mathrm{id}_{M_n}] \otimes_{\mathbb{C}} [\pi_k] \in KK^G(M_n, M_n)$, where $\pi_k : G \to GL\left(\ell^2\left(\{1, \ldots, k\}^Y\right)\right)$ is the permutation representation and $[\pi_k]$ is its class in $KK^G(\mathbb{C}, \mathbb{C})$. By Proposition 2.1, there is an element $\beta \in KK^G(\mathbb{C}, \mathbb{C})$ and $l \geq 1$ such that $[\pi_k]\beta = k^l$. Thus $[\mathrm{id}_{M_n}] \otimes_{\mathbb{C}} [\pi_k]$ is invertible with inverse $\frac{1}{k^l}[\mathrm{id}_{M_n}] \otimes_{\mathbb{C}} \beta$. The same proof shows that, if Z is infinite and $m < \infty$, the map

$$\mathrm{id}_{M_{m^\infty}} \otimes 1_{M_m^{\otimes Z}} : M_{m^\infty} \to M_{m^\infty} \otimes M_m^{\otimes Z}$$

is a KK^G-equivalence. □

Remark 2.9. By [GL21, Theorem B] and [GHV22, Theorem B], a countable discrete group G is amenable if and only if for some (any) supernatural number $n \neq 1$ of infinite type, the Bernoulli shift on $M_n^{\otimes G}$ absorbs the trivial action on the Jiang-Su algebra $\mathcal{Z}$ up to cocycle conjugacy. In particular (since $M_n \cong M_n \otimes \mathcal{Z}$), the conclusion of Theorem 2.7 is false for

[2] This follows from the axioms of a triangulated category. The fact that homotopy colimits of maps are not unique does not cause a problem here.

non-amenable groups. On the other hand, Theorem 2.8 together with the Higson-Kasparov Theorem [HK01] (applied in the form of [MN06, Theorem 8.5]) implies that if G is an amenable group, then $M_{\mathfrak{n}}^{\otimes G}$ absorbs the trivial action on $M_{\mathfrak{n}}$ up to KK^G-equivalence. It is thus conceivable that a discrete group G is amenable if and only if the Bernoulli shift on $M_{\mathfrak{n}}^{\otimes G}$ absorbs the trivial action on $M_{\mathfrak{n}}$ up to cocycle conjugacy.

The following observation provides some evidence for this: Let G be an amenable group, $\mathfrak{n} \neq 1$ a supernatural number of infinite type, and A a G-C^*-algebra. By the remarks above, the unital embedding

$$\mathrm{id} \otimes 1 \colon (A \otimes M_{\mathfrak{n}}^{\otimes G}) \rtimes G \hookrightarrow (A \otimes M_{\mathfrak{n}}^{\otimes G}) \rtimes G \otimes M_{\mathfrak{n}}$$

is a KK-equivalence between $\mathcal{Z}$-stable C^*-algebras that induces an isomorphism on the trace spaces, in particular it induces an isomorphism on the Elliott invariants. If we additionally assume that $(A \otimes M_{\mathfrak{n}}^{\otimes G}) \rtimes G$ is simple, separable, nuclear, and satisfies the UCT (which happens in many cases of interest), then the classification of unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT [Phi00, EGLN15, TWW17, CET+21] implies that $(A \otimes M_{\mathfrak{n}}^{\otimes G}) \rtimes G \cong (A \otimes M_{\mathfrak{n}}) \rtimes G \otimes M_{\mathfrak{n}}$. This condition is certainly necessary for $M_{\mathfrak{n}}^{\otimes G}$ to absorb $M_{\mathfrak{n}}$ up to cocycle conjugacy.

Corollary 2.10. *Let $G \neq \{e\}$ be a finite group, let Z be a G-set and let $M_{\mathfrak{n}}$ be a UHF-algebra of infinite type. Then the Bernoulli shift of G on $M_{\mathfrak{n}}^{\otimes Z}$ does not have the Rokhlin property.*

Proof. Assume the contrary. Then [Izu04, Theorem 3.13] yields an isomorphism[3]

$$K_0\left(M_{\mathfrak{n}}^{\otimes Z} \rtimes G\right) \cong K_0\left(M_{\mathfrak{n}}^{\otimes Z}\right) = \mathbb{Z}[1/\mathfrak{n}].$$

On the other hand, Theorem 2.8 yields an isomorphism

$$K_0\left(M_{\mathfrak{n}}^{\otimes Z} \rtimes G\right) \cong K_0(C^*(G)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/\mathfrak{n}] \cong \mathbb{Z}[1/\mathfrak{n}]^{\oplus \hat{G}},$$

a contradiction. ∎

The following corollary will be used in the follow-up paper [CEKN22] with Sayan Chakraborty and Siegfried Echterhoff. We refer to [BCH94] for the formulation of the Baum–Connes conjecture with coefficients. Note that the Baum-Connes conjecture with coefficients holds for many groups, including a-T-menable groups [HK01] and hyperbolic groups [Laf12].

Corollary 2.11. *Let G be a countable discrete group satisfying the Baum–Connes conjecture with coefficients, let Z be a G-set, let A be a G-C^*-algebra and let $M_{\mathfrak{n}}$ be a UHF-algebra. Assume that Z is infinite or that $\mathfrak{n}$ is of infinite type. Then the inclusion $A \to A \otimes M_{\mathfrak{n}}$ induces an isomorphism*

$$K_*\left(A \rtimes_r G\right)[1/\mathfrak{n}] \cong K_*\left(\left(A \otimes M_{\mathfrak{n}}^{\otimes Z}\right) \rtimes_r G\right).$$

[3]Theorem 3.13 of [Izu04] is applicable by the combination of [Phi87, Proposition 7.1.3] and [Kis81, Theorem 3.1].

In particular, the right hand side is a $\mathbb{Z}[1/\mathfrak{n}]$*-module.*

Proof. By an inductive limit argument, we may assume Z is countable and A is separable. If G is finite, the statement follows from Theorem 2.8 considering the commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & A \otimes M_{\mathfrak{n}}^{\otimes Z} & \xrightarrow{\phi_1} & A \otimes M_{\mathfrak{n}^\infty} \otimes M_{\mathfrak{n}}^{\otimes Z} \\ & \searrow & & \nearrow_{\phi_2} & \\ & & A \otimes M_{\mathfrak{n}^\infty} & & \end{array} \tag{2.3}$$

where ϕ_1, ϕ_2 are KK^G-equivalences. Assume now that G is infinite. Consider the diagram (2.3). We know that (the restrictions of) ϕ_1, ϕ_2 are KK^H-equivalences for every finite subgroup $H \subseteq G$. Since G satisfies the Baum–Connes conjecture with coefficients, the results of [CEOO04] (see also [MN06]) imply that ϕ_1 and ϕ_2 induce isomorphisms of the K-theory groups of reduced crossed products by G. The statement follows from this by identifying $K_*((A \otimes M_{\mathfrak{n}^\infty}) \rtimes_r G) \cong K_*(A \rtimes_r G)[1/\mathfrak{n}]$. □

We end this section with an application to Bernoulli shifts on strongly self-absorbing C*-algebras. Recall that a separable, unital C*-algebra $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing [TW07] if there is an isomorphism $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the first factor inclusion $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$. Strongly self-absorbing C*-algebras are automatically simple, nuclear [TW07] and $\mathcal{Z}$-stable [Win11]. By the combination of [TW07, Proposition 5.1] and the classification of unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras in the UCT class [Phi00, EGLN15, TWW17, CET+21], a complete list of strongly self-absorbing C*-algebras satisfying the UCT is given by

$$\mathcal{Z},\ M_{\mathfrak{n}},\ \mathcal{O}_\infty,\ \mathcal{O}_\infty \otimes M_{\mathfrak{n}},\ \mathcal{O}_2, \tag{2.4}$$

where $\mathfrak{n} \neq 1$ is a supernatural number of infinite type. The following corollary is a generalization of [Sza18, Corollary 6.9]:

Corollary 2.12. *Let* $\mathcal{D}$ *be a strongly self-absorbing* C*-*algebra satisfying the UCT and let* G *be a discrete group having a* γ*-element equal to* 1[4]*. Then, for any* G*-set* Z*, the* G-C*-*algebra* $\mathcal{D}^{\otimes Z}$ *equipped with the Bernoulli shift is* KK^G*-equivalent to* $\mathcal{D}$ *equipped with the trivial* G*-action.*

For the proof, we need the following result of Izumi [Izu19] which we spell out here for later reference.

Theorem 2.13 ([Izu19, Theorem 2.1], see also [Sza18, Lemma 6.8])**.** *Let* A, B *be separable nuclear* C*-*algebras, let* H *be a finite group and let* Z *be a finite* H*-set. Then, there is a map from* $\mathrm{KK}(A, B)$ *to* $\mathrm{KK}^H(A^{\otimes Z}, B^{\otimes Z})$ *which in particular, sends a class of a* $*$*-homomorphism* ϕ *to the class of* $\phi^{\otimes Z}$*. Furthermore, this map*

[4]See [MN06, Section 7] for a definition of the γ-element. By the Higson–Kasparov theorem [HK01], this assumption is satisfied for all a-T-menable groups.

is compatible with the compositions and in particular sends a KK*-equivalence to a* KK^H*-equivalence. In particular, the Bernoulli shifts on* $A^{\otimes Z}$ *and* $B^{\otimes Z}$ *are* KK^H*-equivalent if* A *and* B *are* KK*-equivalent.*

Proof of Corollary 2.12. We claim that the unital embeddings

$$\mathcal{D} \hookrightarrow \mathcal{D} \otimes \mathcal{D}^{\otimes Z} \hookleftarrow \mathcal{D}^{\otimes Z} \tag{2.5}$$

are KK^G-equivalences. By the assumption on G, this amounts to showing that they are KK^H-equivalences for every finite subgroup $H \subseteq G$. (see [MN06, Theorem 7.3]). By the same homotopy co-limit argument as in the proof of Theorem 2.8, it is enough to show that the maps

$$\mathcal{D} \hookrightarrow \mathcal{D} \otimes \mathcal{D}^{\otimes Y} \hookleftarrow \mathcal{D}^{\otimes Y}$$

are KK^H-equivalences for all finite H-subsets Y of Z. Now Theorem 2.13 allows us to replace $\mathcal{D}$ by a KK-equivalent C^*-algebra. Thanks to the list (2.4), this reduces the problem to the cases $\mathcal{D} = \mathbb{C}$, $\mathcal{D} = 0$ and $\mathcal{D} = M_\mathfrak{n}$. The first two cases are trivial and the third one follows from Theorem 2.8. □

3. K-THEORY OF APPROXIMATELY INNER FLIPS

In this section we apply Theorem 2.8 to the K-theory of approximately inner flips. Recall that a C^*-algebra B is said to have approximately inner flip if the flip automorphism $B \otimes B \to B \otimes B,\ \ a \otimes b \mapsto b \otimes a$ is approximately inner, i.e. a point-norm limit of inner automorphisms. A C^*-algebra B with approximately inner flip must be simple, nuclear and have at most one trace [ER78]. An approximately inner flip necessarily induces the identity map on $K_*(B \otimes B)$ and this largely restricts the class of C^*-algebras B with approximately inner flip. Effros and Rosenberg [ER78] showed that if B is AF, then B must be stably isomorphic to a UHF-algebra. Tikuisis [Tik16] determined a complete list of classifiable C^*-algebras with approximately inner flip. We would like to thank Dominic Enders, André Schemaitat and Aaron Tikuisis for informing us about a corrigendum stated below:

Theorem 3.1. *(*[EST22], *Corrigendum to* [Tik16, Theorem 2.2]*) Let* B *be a separable, unital* C^**-algebra with strict comparison, in the UCT class, which is either infinite or quasi-diagonal. The following are equivalent.*

(1) B *has approximately inner flip;*
(2) B *is Morita equivalent to one of the following* C^**-algebras:*
   - $\mathbb{C}$;
   - $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$;
   - $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}} \otimes \mathcal{O}_\infty$;
   - $\mathcal{F}_{1,\mathfrak{m}}$.

Here $\mathfrak{m}, \mathfrak{n}$ *are supernatural numbers of infinite type such that* $\mathfrak{m}$ *divides* $\mathfrak{n}$*,* $\mathcal{O}_\infty$ *is the Cuntz algebra on infinitely many generators,* $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$ *is the simple, separable, unital,* $\mathcal{Z}$*-stable, quasi-diagonal* C^**-algebra in the UCT class with unique trace satisfying*

$$K_0(\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}) \cong \mathbb{Q}_\mathfrak{n},\ \ [1]_0 = 1,\ \ K_1(\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}) \cong \mathbb{Q}_\mathfrak{m}/\mathbb{Z},$$

and $\mathcal{F}_{\mathfrak{m},\mathfrak{n}}$ is the unique unital Kirchberg algebra in the UCT class satisfying

$$K_0(\mathcal{F}_{\mathfrak{m},\mathfrak{n}}) \cong \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, \quad [1]_0 = 0, \quad K_1(\mathcal{F}_{\mathfrak{m},\mathfrak{n}}) \cong \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}.$$

Our strategy to compute the groups $K_*(B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2)$ with B as in Theorem 3.1 builds on Izumi's remarkable computation of $K_*(A^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2)$ for all separable nuclear C^*-algebras A in the UCT class and with finitely generated K-theory [Izu19]. Izumi's starting point is his Theorem 2.13 above. This allows him to replace the appearing C^*-algebras by finite direct sums of building blocks of the form $\mathbb{C}$, $C_0(\mathbb{R})$, $\mathcal{O}_{n+1}$ and D_n, where $\mathcal{O}_{n+1}$ denotes the Cuntz algebra on $n+1$ generators and where D_n denotes the dimension drop algebra. For B one of these building blocks, Izumi explicitly computes $K_*\left(B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right)$.

We follow the same strategy here. Thanks to Theorems 3.1 and 2.13 above, the following theorem and its corollary completely determine the K-theory groups $K_*\left(B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right)$ whenever B is a UCT C^*-algebra that is KK-equivalent to a classifiable C^*-algebra with an approximately inner flip.

Theorem 3.2. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be supernatural numbers of infinite type. Let $F_{\mathfrak{m},\mathfrak{n}}$ be any C^*-algebra satisfying the UCT such that*

$$K_*(F_{\mathfrak{m},\mathfrak{n}}) \cong \begin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, & * = 0 \\ \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}, & * = 1 \end{cases}.$$

Then we have

$$K_*\left(F_{\mathfrak{m},\mathfrak{n}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{r}}/\mathbb{Z}, & * = 0 \\ \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}, & * = 1 \end{cases},$$

where $\mathfrak{r}$ is the greatest common divisor of $\mathfrak{m}$ and $\mathfrak{n}$.

Corollary 3.3. *For any supernatural numbers $\mathfrak{m}$ and $\mathfrak{n}$ of infinite type, we have*

$$K_*\left(\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_{\mathfrak{n}} \oplus \mathbb{Q}_{\mathfrak{n}}, & * = 0 \\ \mathbb{Q}_{\mathfrak{m}\mathfrak{n}}/\mathbb{Q}_{\mathfrak{n}} \oplus \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, & * = 1 \end{cases}.$$

Proof. The algebra $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$ is KK-equivalent to $M_{\mathfrak{n}} \oplus \mathcal{F}_{1,\mathfrak{m}}$. Thus, using Theorem 2.13, we see that $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$ is KK-equivalent to

$$(M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2) \oplus (\mathcal{F}_{1,\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2) \oplus (M_{\mathfrak{n}} \otimes \mathcal{F}_{1,\mathfrak{m}}).$$

Note that $K_0(M_{\mathfrak{n}} \otimes \mathcal{F}_{1,\mathfrak{m}}) \cong 0$ and $K_1(M_{\mathfrak{n}} \otimes \mathcal{F}_{1,\mathfrak{m}}) \cong \mathbb{Q}_{\mathfrak{n}} \otimes_{\mathbb{Z}} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \cong \mathbb{Q}_{\mathfrak{m}\mathfrak{n}}/\mathbb{Q}_{\mathfrak{n}}$. The assertion now follows from Theorem 3.2. □

We break up the proof of Theorem 3.2 into two lemmas. We denote by $[e_0], [e_1], [1] \in K_0(C^*(\mathbb{Z}/2))$ the classes of the trivial representation, the sign representation and the unit of $C^*(\mathbb{Z}/2)$. We will abuse notation and write KK-elements as arrows between C^*-algebras, well-aware that they might not be induced by $*$-homomorphisms.

Lemma 3.4. *We have*

$$K_*\left(F_{m,1}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_m/\mathbb{Z}, & * = 0 \\ 0, & * = 1 \end{cases}.$$

Proof. By the Kirchberg–Phillips classification theorem [Phi00], there is a unital $*$-homomorphism

$$M_m \otimes \mathcal{O}_\infty \to \mathcal{F}_{m,1}$$

such that the composition

$$\phi\colon M_m \xrightarrow{\mathrm{id} \otimes 1} M_m \otimes \mathcal{O}_\infty \to \mathcal{F}_{m,1}$$

induces the canonical quotient map $\mathbb{Q}_m \to \mathbb{Q}_m/\mathbb{Z}$ on K_0. Denote by

$$M_\phi = \{(a, f) \in M_m \oplus (C[0,1] \otimes \mathcal{F}_{m,1}) \mid \phi(a) = f(0)\}$$

the mapping cylinder of ϕ and by $C_\phi \coloneqq \ker(\mathrm{ev}_1\colon M_\phi \to \mathcal{F}_{m,1})$ the mapping cone of ϕ. Note that the inclusion $M_m \hookrightarrow M_\phi$ is a KK-equivalence. The short exact sequence

$$0 \longrightarrow C_\phi \longrightarrow M_\phi \longrightarrow \mathcal{F}_{m,1} \longrightarrow 0, \tag{3.1}$$

induces an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q}_m \longrightarrow \mathbb{Q}_m/\mathbb{Z} \longrightarrow 0 \tag{3.2}$$

on K_0 and 0 on K_1. We streamline the notations and re-write the exact sequence (3.1) as

$$0 \longrightarrow I \longrightarrow B \longrightarrow A_0 \longrightarrow 0.$$

The only properties that we will use are the induced sequence (3.2) on K_0 and that the map $I \to B$ can be identified with the unital inclusion $\mathbb{C} \hookrightarrow M_m$ in KK-theory. From now on, we follow the beautiful computations of [Izu19, Theorem 3.4]. Writing $I_1 \coloneqq I \otimes B + B \otimes I \subseteq B^{\otimes \mathbb{Z}/2}$, we have the following short exact sequences of $\mathbb{Z}/2$-C^*-algebras

$$0 \longrightarrow I_1 \longrightarrow B^{\otimes \mathbb{Z}/2} \longrightarrow A_0^{\otimes \mathbb{Z}/2} \longrightarrow 0\ , \tag{3.3}$$

$$0 \longrightarrow I^{\otimes \mathbb{Z}/2} \longrightarrow I_1 \longrightarrow (I \otimes A_0) \oplus (A_0 \otimes I) \longrightarrow 0\ . \tag{3.4}$$

Taking crossed-products and applying K-theory for (3.4) produces the 6-term exact sequence

$$\begin{array}{ccccc} \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & K_0(I_1 \rtimes \mathbb{Z}/2) & \longrightarrow & \mathbb{Q}_m/\mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(I_1 \rtimes \mathbb{Z}/2) & \longleftarrow & 0 \end{array} . \tag{3.5}$$

Here the generators of $\mathbb{Z} \oplus \mathbb{Z}$ are the image of $[1]$ and $[e_1]$ via the KK-equivalence $\mathbb{C}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to I^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$ obtained from the KK-equivalence

$\mathbb{C} \to I$ as in Theorem 2.13, and $\mathbb{Q}_m/\mathbb{Z}$ is identified with $K_0(I \otimes A_0)$. The canonical map from the exact sequence

$$\underbrace{K_0(I \otimes I)}_{\cong \mathbb{Z}} \oplus \mathbb{Z} \to \underbrace{K_0(I \otimes B)}_{\cong \mathbb{Q}_m} \oplus \mathbb{Z} \to \underbrace{K_0(I \otimes A_0)}_{\cong \mathbb{Q}_m/\mathbb{Z}} \tag{3.6}$$

to the top row of (3.5) is an isomorphism (since it clearly is on the left and right hand terms). From this we see that $K_0(I_1 \rtimes \mathbb{Z}/2) \cong \mathbb{Q}_m \oplus \mathbb{Z}$, with the generator of $\mathbb{Z}$ being the image of $[e_1]$ via the KK-element $\mathbb{C}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to I^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to I_1 \rtimes \mathbb{Z}/2$ and with $\mathbb{Q}_m$ being the image of $K_0(M_m)$ via the KK-element $M_m \to B \to I \otimes B \to I_1$.

Taking crossed products and applying K-theory for (3.3) yields the 6-term exact sequence

$$\begin{array}{ccccc} \mathbb{Q}_m \oplus \mathbb{Z} & \longrightarrow & \mathbb{Q}_m \oplus \mathbb{Q}_m & \longrightarrow & K_0\left(A_0^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \\ \uparrow & & & & \downarrow \\ K_1\left(A_0^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) & \longleftarrow & 0 & \longleftarrow & 0 \end{array} \tag{3.7}$$

where the generators $\mathbb{Q}_m \oplus \mathbb{Q}_m$ over $\mathbb{Q}_m$ are the images of $[1]$ and $[e_1]$ by the KK-equivalence $M_m^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$. Here we have used Theorem 2.8. Thus the first arrow in the top row of (3.7) is the natural inclusion. We get $K_1\left(A_0^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong 0$ and $K_0\left(A_0^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \mathbb{Q}_m/\mathbb{Z}$, generated by the image of $\mathbb{Q}_m[e_1]$ in $K_0\left(M_m^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right)$. □

Lemma 3.5. *We have*

$$K_*\left(F_{1,m}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} 0, & * = 0 \\ \mathbb{Q}_m/\mathbb{Z} \oplus \mathbb{Q}_m/\mathbb{Z}, & * = 1 \end{cases}.$$

Proof. We write $A_0 := F_{m,1}$ as in the proof of Lemma 3.4 and use $A := C_0(\mathbb{R}) \otimes A_0$ as a model for $F_{1,m}$. Note that the flip action on $C_0(\mathbb{R})^{\otimes \mathbb{Z}/2}$ is conjugate to the action on $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ that is trivial on the first factor and reflects at the origin $0 \in \mathbb{R}$ on the second factor. We thus have

$$K_*\left(A^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong K_{*+1}\left(\left(C_0(\mathbb{R}) \otimes A_0^{\otimes \mathbb{Z}/2}\right) \rtimes \mathbb{Z}/2\right). \tag{3.8}$$

We consider the short exact sequence

$$0 \to (C_0(-\infty, 0) \oplus C_0(0, \infty)) \otimes A_0^{\otimes \mathbb{Z}/2} \to C_0(\mathbb{R}) \otimes A_0^{\otimes \mathbb{Z}/2} \to A_0^{\otimes \mathbb{Z}/2} \to 0 \tag{3.9}$$

of $\mathbb{Z}/2$-C^*-algebras. We have

$$\begin{aligned} K_*\left(\left((C_0(-\infty, 0) \oplus C_0(0, \infty)) \otimes A_0^{\otimes \mathbb{Z}/2}\right) \rtimes \mathbb{Z}/2\right) &\cong K_{*+1}\left(A_0^{\otimes \mathbb{Z}/2}\right) \\ &\cong \begin{cases} \mathbb{Q}_m/\mathbb{Z}, & * = 0 \\ 0, & * = 1 \end{cases} \end{aligned}$$

by the Künneth theorem (since $\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}) \cong \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}$). In view of this and Lemma 3.4, taking crossed products and applying K-theory for (3.9) produces the 6-term exact sequence

$$
\begin{array}{ccccc}
\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} & \longrightarrow & K_0\left(\left(C_0(\mathbb{R}) \otimes A_0^{\otimes \mathbb{Z}/2}\right) \rtimes \mathbb{Z}/2\right) & \longrightarrow & \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & K_1\left(\left(C_0(\mathbb{R}) \otimes A_0^{\otimes \mathbb{Z}/2}\right) \rtimes \mathbb{Z}/2\right) & \longleftarrow & 0
\end{array} \tag{3.10}
$$

By [Tik16, Lemma 1.1], the top row of (3.10) splits. Now the Lemma follows from (3.10) and (3.8). □

By Theorem 3.1, the list of the classifiable C^*-algebras with approximately inner flip is up to KK-equivalences given by

$$\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}, \ \mathcal{F}_{1,\mathfrak{m}}$$

for supernatural numbers $\mathfrak{m}$ and $\mathfrak{n}$ where $\mathfrak{m}$ divides $\mathfrak{n}$. Theorem 3.2 and Corollary 3.3 say that for any of these algebras A, we have an isomorphism

$$K_*\left(A^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong K_*(A \otimes C^*_r(\mathbb{Z}/2)).$$

The Künneth formula moreover implies $K_*\left(A^{\otimes \mathbb{Z}/2}\right) \cong K_*(A)$. This naturally raises the following question.

Question 3.6. *Let A be any C^*-algebra satisfying the UCT with approximately inner flip. Is the $\mathbb{Z}/2$-C^*-algebra $A^{\otimes \mathbb{Z}/2}$ equipped with the flip action $KK^{\mathbb{Z}/2}$-equivalent to A equipped with the trivial action?*

We note that in order to answer this question positively, it would suffice to answer it positively for $A = \mathcal{F}_{1,\mathfrak{m}}$. Indeed, $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$ is KK-equivalent to $M_{\mathfrak{n}} \oplus \mathcal{F}_{1,\mathfrak{m}}$ and $M_{\mathfrak{n}} \otimes \mathcal{F}_{1,\mathfrak{m}}$ is KK-equivalent to zero if $\mathfrak{m}$ divides $\mathfrak{n}$. In particular, the flip on $\mathcal{E}^{\otimes \mathbb{Z}/2}_{\mathfrak{n},1,\mathfrak{m}}$ is $KK^{\mathbb{Z}/2}$-equivalent to the sum of the flips on $M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \oplus \mathcal{F}^{\otimes \mathbb{Z}/2}_{1,\mathfrak{m}}$. Since Theorem 2.8 provides a positive answer to Question 3.6 for $A = M_{\mathfrak{n}}$, a positive answer for $A = \mathcal{F}_{1,\mathfrak{m}}$ would provide a positive answer for $A = \mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$.

Unfortunately, the methods used to establish the $KK^{\mathbb{Z}/2}$-equivalence between $M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2}$ and $M_{\mathfrak{n}}$ in Theorem 2.8 do not apply in the situation of Question 3.6. For once, there is no analogue of the representation theoretic argument in Proposition 2.1 for $\mathcal{F}_{1,\mathfrak{m}}$. Furthermore, the diagram

$$M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \to M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \otimes M_{\mathfrak{n}} \leftarrow M_{\mathfrak{n}}$$

of $KK^{\mathbb{Z}/2}$-equivalences in Theorem 2.8 does not have an analogue for $\mathcal{F}_{1,\mathfrak{m}}$ since the unit class $[1]_0 \in K_0(\mathcal{F}_{1,\mathfrak{m}}) \cong 0$ is zero.

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