PARTIAL TENSOR-PRODUCT FUNCTORS AND CROSSED-PRODUCT FUNCTORS

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Abstract. For a given discrete group G, we apply results of Kirchberg on exact and injective tensor products of C^*-algebras to give an explicit description of the minimal exact correspondence crossed-product functor and the maximal injective crossed-product functor for G in the sense of Buss, Echterhoff and Willett. In particular, we show that the former functor dominates the latter.

1. Introduction

A fruitful approach to construct examples of C^*-algebras is to complete $*$-algebras with respect to certain C^*-norms. For instance, if $G \curvearrowright A$ is an action of a discrete group on a C^*-algebra, one can complete the algebraic crossed product $A \rtimes_{\mathrm{alg}} G$ to get the *maximal crossed product* $A \rtimes G$ or the *reduced crossed product* $A \rtimes_r G$.

In the last decade, there has been an increasing interest in exotic completions of $A \rtimes_{\mathrm{alg}} G$, i.e. completions which strictly lie between the maximal and reduced completion. One important motivation comes from the *Baum–Connes conjecture with coefficients* [BCH94] which predicts that the Baum–Connes assembly map

$$\mu\colon K_*^G(\underline{E}G, A) \to K_*(A \rtimes_r G)$$

is an isomorphism. Counterexamples to the conjecture were constructed in [HLS02] by exploiting non-exactness of the functor $- \rtimes_r G$ for certain groups G. Later, in [BGW16] it was suggested to modify the conjecture by replacing the reduced crossed product with the *minimal exact Morita compatible crossed product*. This modification strictly enlarges the class of actions $G \curvearrowright A$ for which the conjecture is known to hold and does not change the statement of the conjecture for exact groups. Other motivations to study *exotic crossed-product functors* come from a-T-menability and property (T) [BG13] or from non-commutative duality [KLQ13, KLQ16, KLQ18, BE14].

General exotic crossed–product functors and their properties were studied systematically by Buss, Echterhoff and Willett [BEW17, BEW18a, BEW18b, BEW20a, BEW20b]. They introduced the *minimal exact crossed-product functor* $- \rtimes_{\mathcal{E}} G$, the *minimal exact correspondence crossed-product functor* $- \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ (which agrees with the minimal exact Morita compatible crossed-product functor of [BGW16] for

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separable G-C^*-algebras [BEW18a, Cor. 8.13]) and the *maximal injective crossed-product functor* $-\rtimes_{\mathrm{inj}} G$. All these functors agree with the reduced crossed product for exact groups, but their interrelations for non-exact groups are still unclear. In particular, it is unclear whether or not $-\rtimes_{\mathcal{E}} G$ and $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ agree. A positive answer to this question would imply that the "new" Baum-Connes conjecture of [BGW16] agrees with the old conjecture of [BCH94] for complex coefficients $A = \mathbb{C}$. The aim of this article is to provide an explicit description of $-\rtimes_{\mathrm{inj}} G$ and $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$. We hope that the interplay of the universal properties and the explicit descriptions of these functors turn out useful in the future. Our main ingredient is the following construction by Kirchberg:

Theorem A ([Kir95])**.** *There is a tensor-product functor $-\otimes_{i,\varepsilon}-$ satisfying the following properties:*

(1) For every C^*-algebra A, $A\otimes_{i,\varepsilon}-$ is the minimal exact partial tensor-product functor for A.
(2) For every C^*-algebra B, $-\otimes_{i,\varepsilon} B$ is the maximal injective partial tensor-product functor for B.

In particular, $-\otimes_{i,\varepsilon}-$ is the unique tensor-product functor which is injective in the first variable and exact in the second variable. Furthermore, $-\otimes_{i,\varepsilon}-$ is functorial for completely positive maps in both variables.

In terms of Kirchberg's tensor product, we can describe $-\rtimes_{\mathrm{inj}} G$ and $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ as follows:

Theorem B (Theorem 4.3)**.** *Let G be a discrete group and let A be a G-C^*-algebra. Then there are injective $*$-homomorphisms*

(1) $A\rtimes_{\mathrm{inj}} G \hookrightarrow (A\rtimes_r G)\otimes_{i,\varepsilon} C^*(G)$
(2) $A\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \hookrightarrow C^*_r(G)\otimes_{i,\varepsilon}(A\rtimes G)$

given by $a\delta_g \mapsto a\delta_g\otimes\delta_g$ and $a\delta_g\mapsto \delta_g\otimes a\delta_g$ respectively.

We obtain an even more concrete picture using G-injective G-C^*-algebras (see p.4 for the definition). Note that G-injective G-C^*-algebras are always unital.

Proposition C (Proposition 4.5)**.** *Let G be a discrete group, let A be a G-C^*-algebra and let I be a G-injective G-C^*-algebra (e.g. $I=\ell^\infty(G)$). Then the canonical embedding $A\hookrightarrow A\otimes_{\max} I, a\mapsto a\otimes 1$ induces an injective $*$-homomorphism*

$$A\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G\hookrightarrow (A\otimes_{\max} I)\rtimes G.$$

Note that for $I=\ell^\infty(G)$, this provides a positive solution to a question asked in [BEW18a, Question 9.4] and [BGW16, 8.2]. As an application, we are able to compare $-\rtimes_{\mathrm{inj}} G$ and $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$:

Corollary D (Corollary 4.6)**.** *For any discrete group G, we have $-\rtimes_{\mathrm{inj}} G\le -\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ and $C^*_{\mathrm{inj}}(G) = C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G)$.*

Thus, in order to prove that $-\rtimes_{\mathrm{inj}} G$ and $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ coincide, it would suffice to construct a crossed-product functor which is both exact and injective.

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2. Preliminaries

In this section we fix some terminology regarding crossed-product and tensor-product functors. For definitions and basic properties of crossed products and tensor products we refer to [BO08, Wil07].

Let ${}^*\mathbf{Alg}$ denote the category of $*$-algebras with $*$-homomorphisms as morphisms and let $C^*\mathbf{Alg}$ denote the full subcategory of C^*-algebras. For a discrete group G, we denote by $C^*\mathbf{Alg}_G$ the category of G-C^*-algebras with G-equivariant $*$-homomorphisms.

Let $\mathcal{C}$ be a category. A functor $F^\mu\colon \mathcal{C}\to C^*\mathbf{Alg}$ is a *C^*-completion* of a functor $F\colon \mathcal{C}\to {}^*\mathbf{Alg}$, if for every object X in $\mathcal{C}$, $F^\mu(X)$ is a C^*-completion of $F(X)$ and if for every morphism f in $\mathcal{C}$, $F^\mu(f)$ is an extension of $F(f)$. We define a partial order on the class of C^*-completions of a given functor F by declaring $F^\mu \geq F^\nu$ if for every object X in $\mathcal{C}$, the identity on $F(X)$ extends to a $*$-homomorphism $F^\mu(X)\to F^\nu(X)$.

For two C^*-algebras A and B, we denote by $A\odot B$ the algebraic tensor product, by $A\otimes_{\max} B$ the maximal tensor product and by $A\otimes B$ the minimal tensor product. A *tensor-product functor* $-\otimes_\alpha -$ is a C^*-completion of the functor

$$-\odot -\colon C^*\mathbf{Alg}\times C^*\mathbf{Alg}\to {}^*\mathbf{Alg}.$$

A *partial tensor-product functor* $-\otimes_\alpha B$ *for* B is a C^*-completion of the functor

$$-\odot B\colon C^*\mathbf{Alg}\to {}^*\mathbf{Alg}.$$

A partial tensor-product functor $-\otimes_\alpha B$ is

(1) called *exact* if it maps exact sequences to exact sequences;
(2) called *injective* if it maps injective $*$-homomorphisms to injective $*$-homomorphisms;
(3) said to have the *cp-map property* if for each completely positive map $\varphi\colon A\to C$, the induced map $\varphi\odot \mathrm{id}_B\colon A\odot B\to C\odot B$ extends to a completely positive map $\varphi\otimes_\alpha \mathrm{id}_B\colon A\otimes_\alpha B\to C\otimes_\alpha B$.

Every (partial) tensor-product functor is dominated by the maximal tensor-product functor and dominates the minimal tensor-product functor. For a fixed C^*-algebra B, the functor $-\otimes_{\max} B$ is exact [BO08, Prop. 3.7.1] whereas the functor $-\otimes B$ is injective. Both functors have the cp-map property [BO08, Thm. 3.5.3].

For a discrete group G and a G-C^*-algebra A, we denote by $A\rtimes_{\mathrm{alg}} G = A[G]$ the algebraic crossed product, by $A\rtimes G$ the maximal crossed product and by $A\rtimes_r G$ the reduced crossed product. A *crossed-product functor* $-\rtimes_\mu G$ is a C^*-completion of the algebraic crossed-product functor

$$-\rtimes_{\mathrm{alg}} G\colon C^*\mathbf{Alg}_G\to {}^*\mathbf{Alg}$$

which dominates the reduced crossed product. We write $C^*_\mu(G) := \mathbb{C}\rtimes_\mu G$. A crossed-product functor $-\rtimes_\mu G$ is

(1) called *exact* if it maps exact sequences to exact sequences;
(2) called *injective* if it maps injective G-equivariant $*$-homomorphisms to injective $*$-homomorphisms;
(3) said to have the *cp-map property* if for each G-equivariant completely positive map $\varphi\colon A\to B$, the induced map $\varphi\rtimes_{\mathrm{alg}} G\colon A\rtimes_{\mathrm{alg}} G\to B\rtimes_{\mathrm{alg}} G$ extends to a completely positive map $\varphi\rtimes_\mu G\colon A\rtimes_\mu G\to B\rtimes_\mu G$.

Every crossed-product functor is dominated by the maximal crossed product and dominates the reduced crossed product. The maximal crossed product $-\rtimes G$ is exact [Ech17, Prop. 4.8] and the reduced crossed product $-\rtimes_r G$ is injective [EKQR06, Lem. A.16]. Both functors have the cp-map property [BEW20a, Lem. 4.8]. Every injective crossed-product functor has the cp-map property [BEW18a, Thm. 4.9]. Moreover there is a maximal injective crossed-product functor $-\rtimes_{\mathrm{inj}} G$ [BEW20b, Prop. 3.5] and a minimal exact crossed-product functor with the cp-map property $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ [BEW18a, Cor. 8.8].

Remark 2.1. It was shown in [BEW18a, Thm. 4.9] that a crossed-product functor has the cp-map property if and only if it extends to a functor on the G-equivariant correspondence category $\mathfrak{Corr}(G)$ as defined in [BEW18a, Def. 4.4]. Therefore crossed-product functors with the cp-map property are called *correspondence crossed-product functors* in [BEW18a] and $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ is called the minimal exact *correspondence* crossed-product functor. One can prove a similar characterization for partial tensor-product functors.

A G-C^*-algebra I is called *G-injective* if for every injective G-equivariant $*$-homomorphism $\iota\colon A \hookrightarrow B$ and every G-equivariant completely positive contractive map $\varphi\colon A \to I$, there is a G-equivariant completely positive contractive map $\overline{\varphi}\colon B \to I$ such that $\overline{\varphi} \circ \iota = \varphi$. We say that $\overline{\varphi}$ *extends* φ *along* ι. In this case I is unital since there exists a conditional expectation from the unitization $\tilde{I}$ onto I.

3. Exact and injective tensor-product functors

In this section we give a detailed proof of a theorem that was stated in [Kir95] for convenience of the reader. We need a folklore lemma.

Lemma 3.1. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\iota} & A & \xrightarrow{q} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_I} & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \varphi_B} & & \\ 0 & \longrightarrow & I' & \xrightarrow{\iota'} & A' & \xrightarrow{q'} & B' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of C^*-algebras and $*$-homomorphisms. Assume that ι is an ideal inclusion, that the lower row is exact, and that the vertical maps are non-degenerate inclusions. Then we have $\ker q \subseteq \operatorname{im}(\iota)$.

Proof. Let $x \in \ker q$. By exactness, we find $y \in I'$ such that $\iota'(y) = \varphi_A(x)$. Let $(e_\lambda)_\lambda$ be an approximate unit for I. Since φ_I is non-degenerate, $(\varphi_I(e_\lambda))_\lambda$ is an approximate unit for I' and thus $\|\varphi_I(e_\lambda)y - y\| \to 0$. We obtain $\|\varphi_A(\iota(e_\lambda)x - x)\| = \|\iota'(\varphi_I(e_\lambda)y - y)\| \to 0$. This implies $\|\iota(e_\lambda)x - x\| \to 0$ because φ_A is isometric and therefore $x \in \operatorname{im}(\iota)$ since ι is an ideal inclusion. $\square$

Theorem 3.2 ([Kir95])**.** *There is a tensor-product functor $-\otimes_{i,\varepsilon}-$ satisfying the following properties:*

1. *For every C^*-algebra A, $A\otimes_{i,\varepsilon}-$ is the minimal exact partial tensor-product functor for A.*
2. *For every C^*-algebra B, $-\otimes_{i,\varepsilon} B$ is the maximal injective partial tensor-product functor for B.*

In particular, $-\otimes_{i,\varepsilon}-$ is the unique tensor-product functor which is injective in the first variable and exact in the second variable. Furthermore, $-\otimes_{i,\varepsilon}-$ has the cp-property in both variables.

Proof. Let A and B be C^*-algebras and let $\iota\colon A\hookrightarrow\mathcal{B}(H)$ be an embedding into the bounded operators on a Hilbert space. We define

$$A\otimes_{i,\varepsilon}B:=\iota\otimes\mathrm{id}_B(A\otimes_{\max}B)\subseteq\mathcal{B}(H)\otimes_{\max}B. \tag{1}$$

To show that $-\otimes_{i,\varepsilon}-$ has the desired properties, we verify the following claims:

Claim 1. *Up to canonical isomorphism, the definition of $A\otimes_{i,\varepsilon}B$ is independent of ι.*

Let $\iota'\colon A\hookrightarrow\mathcal{B}(H')$ be another embedding. Then by Arveson's extension theorem there exist completely positive contractive maps $\Psi\colon\mathcal{B}(H)\to\mathcal{B}(H')$ extending ι' along ι and $\Phi\colon\mathcal{B}(H')\to\mathcal{B}(H)$ extending ι along ι'. Then $\Psi\otimes_{\max}\mathrm{id}_B$ and $\Phi\otimes_{\max}\mathrm{id}_B$ restrict to mutually inverse $*$-isomorphisms

$$\iota\otimes\mathrm{id}_B(A\otimes_{\max}B)\cong\iota'\otimes\mathrm{id}_B(A\otimes_{\max}B).$$

Claim 2. *$A\otimes_{i,\varepsilon}B$ is functorial for completely positive maps in both variables.*

Functoriality for completely positive maps in B follows immediately from the definition. To see functoriality in A, let $\varphi\colon A_1\to A_2$ be a completely positive map and let $\iota_j\colon A_j\hookrightarrow\mathcal{B}(H_j), j=1,2$ be embeddings. Let $\Psi\colon\mathcal{B}(H_1)\to\mathcal{B}(H_2)$ be a completely positive map extending $\iota_2\circ\varphi$ along ι_1. Then $\Psi\otimes_{\max}\mathrm{id}_B$ restricts to a completely positive map $A_1\otimes_{i,\varepsilon}B\to A_2\otimes_{i,\varepsilon}B$ extending the canonical map $\varphi\odot\mathrm{id}_B\colon A_1\odot B\to A_2\odot B$.

Claim 3. *The functor $-\otimes_{i,\varepsilon}B$ is the maximal injective partial tensor-product functor for B.*

Let $\varphi\colon A_1\hookrightarrow A_2$ be an injective $*$-homomorphism and let $\iota\colon A_2\hookrightarrow\mathcal{B}(H)$ be an embedding. Then $\varphi\circ\iota\colon A_1\hookrightarrow\mathcal{B}(H)$ is an embedding too. Inserting this embedding into (1) shows that $\varphi\otimes\mathrm{id}_B\colon A_1\otimes_{i,\varepsilon}B\to A_2\otimes_{i,\varepsilon}B$ is isometric and therefore injective. Now let $-\otimes_\alpha B$ be another injective partial tensor-product functor for B and let $A\hookrightarrow\mathcal{B}(H)$ be an embedding. Then the canonical quotient map $\mathcal{B}(H)\otimes_{\max}B\to\mathcal{B}(H)\otimes_\alpha B$ restricts to a quotient map $A\otimes_{i,\varepsilon}B\to A\otimes_\alpha B$. Thus, $-\otimes_{i,\varepsilon}B$ is maximal.

Claim 4. *The functor $A\otimes_{i,\varepsilon}-$ is exact.*

Let $0\to I\xrightarrow{\iota}B\xrightarrow{\pi}Q\to 0$ be an exact sequence of C^*-algebras. Assume first that A is unital and choose a unital embedding $A\hookrightarrow\mathcal{B}(H)$. Then the upper row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & A\otimes_{i,\varepsilon}I \arrow[r, "\mathrm{id}_A\otimes\iota"] \arrow[d, hook] & A\otimes_{i,\varepsilon}B \arrow[r, "\mathrm{id}_A\otimes\pi"] \arrow[d, hook] & A\otimes_{i,\varepsilon}Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{B}(H)\otimes_{\max}I \arrow[r, "\mathrm{id}_{\mathcal{B}(H)}\otimes\iota"] & \mathcal{B}(H)\otimes_{\max}B \arrow[r, "\mathrm{id}_{\mathcal{B}(H)}\otimes\pi"] & \mathcal{B}(H)\otimes_{\max}Q \arrow[r] & 0
\end{tikzcd}$$

is exact by Lemma 3.1. Now assume that A is not unital and denote by $\tilde{A}$ its unitization. By the above, the middle and lower row of the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \otimes_{i,\varepsilon} I \arrow[r] \arrow[d] & A \otimes_{i,\varepsilon} B \arrow[r] \arrow[d] & A \otimes_{i,\varepsilon} Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{A} \otimes_{i,\varepsilon} I \arrow[r] \arrow[d] & \tilde{A} \otimes_{i,\varepsilon} B \arrow[r] \arrow[d] & \tilde{A} \otimes_{i,\varepsilon} Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \otimes_{i,\varepsilon} I \arrow[r] \arrow[d] & \mathbb{C} \otimes_{i,\varepsilon} B \arrow[r] \arrow[d] & \mathbb{C} \otimes_{i,\varepsilon} Q \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2}$$

are exact. Since the extension $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$ splits, the columns of (2) are exact as well. Now exactness of the upper row of (2) follows from the 3×3-Lemma.

Claim 5. *The functor $A \otimes_{i,\varepsilon} -$ is the minimal exact partial tensor-product functor.*

Let $A \otimes_\alpha -$ be another exact partial tensor-product functor and fix a C^*-algebra B. Assume first that B is unital and pick a surjective $*$-homomorphism $C^*(F_X) \to B$ where F_X denotes the free group on a set X of unitaries generating B. Denote by I the kernel of $C^*(F_X) \to B$ and choose an embedding $\iota\colon A \hookrightarrow \mathcal{B}(H)$. By [Kir94, Cor. 1.2] (see also [Pis96]), there is a unique C^*-norm on $\mathcal{B}(H) \odot C^*(F_X)$. In particular, we have a canonical $*$-homomorphism

$$A \otimes_\alpha C^*(F_X) \to A \otimes C^*(F_X) \xrightarrow{\iota \otimes \mathrm{id}} \mathcal{B}(H) \otimes C^*(F_X) = \mathcal{B}(H) \otimes_{\max} C^*(F_X)$$

mapping $A \otimes_\alpha I$ to $\mathcal{B}(H) \otimes_{\max} I$. By exactness of both $A \otimes_\alpha -$ and $\mathcal{B}(H) \otimes_{\max} -$, we can fill the following diagram with the dashed $*$-homomorphism ψ:

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_\alpha I \arrow[r] \arrow[d] & A \otimes_\alpha C^*(F_X) \arrow[r] \arrow[d] & A \otimes_\alpha B \arrow[r] \arrow[d, dashed, "\psi"] & 0 \\
0 \arrow[r] & \mathcal{B}(H) \otimes_{\max} I \arrow[r] & \mathcal{B}(H) \otimes_{\max} C^*(F_X) \arrow[r] & \mathcal{B}(H) \otimes_{\max} B \arrow[r] & 0
\end{tikzcd}$$

By definition, we have $\psi(A \otimes_\alpha B) = A \otimes_{i,\varepsilon} B$. If B is a non-unital C^*-algebra, we can apply the same argument to its unitization and use exactness to produce a canonical quotient map $A \otimes_\alpha B \to A \otimes_{i,\varepsilon} B$. This proves maximality. □

Remark 3.3. Let F be a non-amenable free group and H an infinite-dimensional Hilbert space. Then the flip isomorphism $\mathcal{B}(H) \odot C_r^*(F) \cong C_r^*(F) \odot \mathcal{B}(H)$ does *not* extend to an isomorphism $\mathcal{B}(H) \otimes_{i,\varepsilon} C_r^*(F) \cong C_r^*(F) \otimes_{i,\varepsilon} \mathcal{B}(H)$. Therefore, Kirchberg's tensor-product functor $- \otimes_{i,\varepsilon} -$ is not symmetric. Indeed, we have $C_r^*(F) \otimes_{i,\varepsilon} \mathcal{B}(H) = C_r^*(F) \otimes \mathcal{B}(H)$ since $C_r^*(F)$ is exact and $\mathcal{B}(H) \otimes_{i,\varepsilon} C_r^*(F) = \mathcal{B}(H) \otimes_{\max} C_r^*(F)$ by construction. But the identity map on $\mathcal{B}(H) \odot C_r^*(F)$ does not extend to an isomorphism $\mathcal{B}(H) \otimes_{\max} C_r^*(F) \cong \mathcal{B}(H) \otimes C_r^*(F)$ since $C_r^*(F)$ does not have the local lifting property [BO08, Cor. 3.7.12, Thm. 13.1.6, Cor. 13.2.5].

4. Application to crossed products

Throughout this section, let G be a *discrete* group. We recall a version of Fell's absorption principle from [ABES21].

Proposition 4.1 ([ABES21, Prop. 2.8])**.** *Let $-\rtimes_\mu G$ be a crossed-product functor with the cp-map property and let A be a C^*-algebra equipped with the trivial G-action. Then the canonical map $A \odot C^*_\mu(G) \to A \rtimes_\mu G$ is injective. In particular, $A \mapsto A \otimes_\mu C^*_\mu(G) := A \rtimes_\mu G$ is a partial tensor-product functor for $C^*_\mu(G)$.*

Although only stated for $\rho = \max$ in [ABES21], the proof of the following lemma works verbatim for every crossed-product functor $-\rtimes_\rho G$:

Lemma 4.2 ([ABES21, Lem 2.10])**.** *Let $-\rtimes_\mu G$ be a crossed-product functor with the cp-map property and let $-\rtimes_\rho G$ be any crossed-product functor. Then for every G-C^*-algebra A, there is an injective $*$-homomorphism*

$$A \rtimes_\mu G \hookrightarrow (A \rtimes_\rho G) \otimes_\mu C^*_\mu(G)$$

given by $a\delta_g \mapsto a\delta_g \otimes \delta_g$ for $a \in A, g \in G$.

Theorem 4.3. *For every G-C^*-algebra A, there are injective $*$-homomorphisms*

1. *$A \rtimes_{\mathrm{inj}} G \hookrightarrow (A \rtimes_r G) \otimes_{i,\varepsilon} C^*(G), \quad a\delta_g \mapsto a\delta_g \otimes \delta_g$.*
2. *$A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \hookrightarrow C^*_r(G) \otimes_{i,\varepsilon} (A \rtimes G), \quad a\delta_g \mapsto \delta_g \otimes a\delta_g$.*

Proof. We first prove the statement for $-\rtimes_{\mathrm{inj}} G$. Denote by $A \rtimes_\alpha G$ the image of $A \rtimes G$ in $(A \rtimes_r G) \otimes_{i,\varepsilon} C^*(G)$ under the map $a\delta_g \mapsto a\delta_g \otimes \delta_g$. Then $-\rtimes_\alpha G$ is an injective crossed-product functor and therefore $-\rtimes_\alpha G \leq -\rtimes_{\mathrm{inj}} G$. On the other hand, Lemma 4.2 gives us an embedding

$$A \rtimes_{\mathrm{inj}} G \hookrightarrow (A \rtimes_r G) \otimes_{\mathrm{inj}} C^*_{\mathrm{inj}}(G), \quad a\delta_g \mapsto a\delta_g \otimes \delta_g.$$

Since $-\otimes_{i,\varepsilon} C^*_{\mathrm{inj}}(G)$ is the maximal injective partial tensor-product functor for $C^*_{\mathrm{inj}}(G)$, we have

$$-\otimes_{i,\varepsilon} C^*(G) \geq -\otimes_{i,\varepsilon} C^*_{\mathrm{inj}}(G) \geq -\otimes_{\mathrm{inj}} C^*_{\mathrm{inj}}(G)$$

and therefore $-\rtimes_\alpha G \geq -\rtimes_{\mathrm{inj}} G$.

We now prove the statement for $-\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$. Denote by $A \rtimes_\beta G$ the image of $A \rtimes G$ in $C^*_r(G) \otimes_{i,\varepsilon} (A \rtimes G)$ under the map $a\delta_g \mapsto \delta_g \otimes a\delta_g$. Then $-\rtimes_\beta G$ is an exact crossed-product functor with the cp-map property by Lemma 3.1 and therefore $-\rtimes_\beta G \geq -\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$. On the other hand, Lemma 4.2 gives us an embedding

$$A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \hookrightarrow C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G) \otimes_{\mathcal{E}_{\mathfrak{Corr}}} (A \rtimes G).$$

Since $C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G) \otimes_{i,\varepsilon} -$ is the minimal exact partial tensor-product functor for $C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G)$, we get

$$C^*_r(G) \otimes_{i,\varepsilon} - \leq C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G) \otimes_{i,\varepsilon} - \leq C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G) \otimes_{\mathcal{E}_{\mathfrak{Corr}}} -$$

and therefore $-\rtimes_\beta G \leq -\rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$. □

Remark 4.4. The statement of the above theorem remains true if we replace $A \rtimes G$ by $A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$, $C^*(G)$ by $C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G)$, $A \rtimes_r G$ by $A \rtimes_{\mathrm{inj}} G$, or $C^*_r(G)$ by $C^*_{\mathrm{inj}}(G)$. Indeed, the only properties of the maximal (resp. reduced) crossed product that we used in the proof were exactness (resp. injectivity) and the cp-map property.

Proposition 4.5. *Let I be a G-injective G-C^*-algebra and let A be any G-C^*-algebra. Then the canonical embedding $A \hookrightarrow A \otimes_{\max} I, a \mapsto a \otimes 1$ induces an embedding*

$$A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \hookrightarrow (A \otimes_{\max} I) \rtimes G.$$

Proof. Denote by $A \rtimes_\alpha G$ the image of $A \rtimes G$ in $(A \otimes_{\max} I) \rtimes G$ under the map $a\delta_g \mapsto (a \otimes 1)\delta_g$. Then $- \rtimes_\alpha G$ is an exact crossed-product functor with the cp-map property by Lemma 3.1 and therefore $- \rtimes_\alpha G \geq - \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$. It remains to prove the converse inequality. Consider $\mathcal{B}(\ell^2(G))$ as a G-C^*-algebra equipped with the conjugation action of the left regular representation $\lambda\colon G \to \mathcal{U}(\ell^2(G))$. By G-injectivity, there is a G-equivariant unital completely positive map $\varphi\colon \mathcal{B}(\ell^2(G)) \to I$. Consider the "untwisting isomorphism"

$$\Psi\colon \mathcal{B}(\ell^2(G)) \otimes_{\max} (A \rtimes G) \xrightarrow{\cong} (\mathcal{B}(\ell^2(G)) \otimes_{\max} A) \rtimes G, \quad T \otimes a\delta_g \mapsto T\lambda_{g^{-1}} \otimes a\delta_g$$

and denote by κ the following composition of contractive maps.

$$\begin{tikzcd} A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \arrow[r, hook, "\text{Thm.4.3}"] & C^*_r(G) \otimes_{i,\varepsilon} (A \rtimes G) \arrow[r, hook, "\lambda\otimes\mathrm{id}"] & \mathcal{B}(\ell^2(G)) \otimes_{\max} (A \rtimes G) \arrow[d, "\Psi"] \\ & (I \otimes_{\max} A) \rtimes G & (\mathcal{B}(\ell^2(G)) \otimes_{\max} A) \rtimes G \arrow[l, "(\varphi\otimes\mathrm{id})\rtimes G"'] \end{tikzcd}$$

A straightforward computation shows that $\kappa(a\delta_g) = (a \otimes 1)\delta_g$ for $a \in A$ and $g \in G$. Thus, we have $\kappa(A \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G) = A \rtimes_\alpha G$ and therefore $- \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G \geq - \rtimes_\alpha G$. □

Corollary 4.6. *For any discrete group G, we have $- \rtimes_{\mathrm{inj}} G \leq - \rtimes_{\mathcal{E}_{\mathfrak{Corr}}} G$ and $C^*_{\mathrm{inj}}(G) = C^*_{\mathcal{E}_{\mathfrak{Corr}}}(G)$.*

Proof. Let A, I be G-C^*-algebras where I is G-injective. The embedding $A \hookrightarrow A \otimes_{\max} I$ induces an embedding $A \rtimes_{\mathrm{inj}} G \hookrightarrow (A \otimes_{\max} I) \rtimes_{\mathrm{inj}} G$. The first statement now follows from Proposition 4.5. The second statement follows from the same argument and the fact that $I \rtimes G = I \rtimes_{\mathrm{inj}} G$ [BEW20b, Cor. 3.3]. □

References

[ABES21] Paolo Antonini, Alcides Buss, Alexander Engel, and Timo Siebenand. Strong Novikov conjecture for low degree cohomology and exotic group C^*-algebras. *Trans. Amer. Math. Soc.*, 374(7):5071–5093, 2021.

[BCH94] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.

[BE14] Alcides Buss and Siegfried Echterhoff. Universal and exotic generalized fixed-point algebras for weakly proper actions and duality. *Indiana Univ. Math. J.*, 63(6):1659–1701, 2014.

[BEW17] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. Exotic crossed products. In *Operator algebras and applications—the Abel Symposium 2015*, volume 12 of *Abel Symp.*, pages 67–114. Springer, [Cham], 2017.

[BEW18a] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. Exotic crossed products and the Baum-Connes conjecture. *J. Reine Angew. Math.*, 740:111–159, 2018.

[BEW18b] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. The minimal exact crossed product. *Doc. Math.*, 23:2043–2077, 2018.

[BEW20a] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. Injectivity, crossed products, and amenable group actions. In *K-theory in algebra, analysis and topology*, volume 749 of *Contemp. Math.*, pages 105–137. Amer. Math. Soc., [Providence], RI, 2020.

[BEW20b] Alcides Buss, Siegfried Echterhoff, and Rufus Willett. The maximal injective crossed product. *Ergodic Theory Dynam. Systems*, 40(11):2995–3014, 2020.

[BG13] Nathanial P. Brown and Erik P. Guentner. New C*-completions of discrete groups and related spaces. *Bull. Lond. Math. Soc.*, 45(6):1181–1193, 2013.

[BGW16] Paul Baum, Erik Guentner, and Rufus Willett. Expanders, exact crossed products, and the Baum-Connes conjecture. *Ann. K-Theory*, 1(2):155–208, 2016.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Ech17] Siegfried Echterhoff. Crossed products and the Mackey–Rieffel–Green machine. In *K-Theory for Group C*-Algebras and Semigroup C*-Algebras*, pages 5–79. Springer, 2017.

[EKQR06] Siegfried Echterhoff, S. Kaliszewski, John Quigg, and Iain Raeburn. A categorical approach to imprimitivity theorems for C^*-dynamical systems. *Mem. Amer. Math. Soc.*, 180(850):viii+169, 2006.

[HLS02] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.

[Kir94] Eberhard Kirchberg. Commutants of unitaries in UHF algebras and functorial properties of exactness. *J. Reine Angew. Math.*, 452:39–77, 1994.

[Kir95] Eberhard Kirchberg. Exact C^*-algebras, tensor products, and the classification of purely infinite algebras. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 943–954. Birkhäuser, Basel, 1995.

[KLQ13] S. Kaliszewski, Magnus B. Landstad, and John Quigg. Exotic group C^*-algebras in noncommutative duality. *New York J. Math.*, 19:689–711, 2013.

[KLQ16] S. Kaliszewski, Magnus B. Landstad, and John Quigg. Coaction functors. *Pacific J. Math.*, 284(1):147–190, 2016.

[KLQ18] S. Kaliszewski, Magnus B. Landstad, and John Quigg. Coaction functors, II. *Pacific J. Math.*, 293(2):301–339, 2018.

[Pis96] Gilles Pisier. A simple proof of a theorem of Kirchberg and related results on C^*-norms. *J. Operator Theory*, 35(2):317–335, 1996.

[Wil07] Dana P. Williams. *Crossed products of C^*-algebras*, volume 134 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.