THE WEAK CONTAINMENT PROBLEM FOR ÉTALE GROUPOIDS WHICH ARE STRONGLY AMENABLE AT INFINITY

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Abstract. We show that an étale groupoid which is strongly amenable at infinity is amenable whenever its full and reduced C^*-algebras coincide.

1. Introduction

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with a Haar system. We say that $\mathcal{G}$ *has the weak containment property*, if its full and reduced C^*-algebras are isomorphic via the regular representation $\Lambda : C^*\mathcal{G} \to C^*_r\mathcal{G}$. It is a classical result [4, Proposition 6.1.8] that $\mathcal{G}$ has the weak containment property whenever $\mathcal{G}$ is amenable (we will not distinguish between topological and measurewise amenability since they are equivalent for étale groupoids [4, Remark 3.3.9]). The converse is not true as shown by Willett [16]. His counterexample is an étale groupoid which is not inner exact in the sense of [3, Definition 3.7]. However for an exact discrete group G acting on a compact Hausdorff space X, Matsumura [12] showed that amenability of the transformation groupoid $X \rtimes G$ does follow from the weak containment property. This result was recently generalized to actions of locally compact exact groups on locally compact Hausdorff spaces by Buss, Echterhoff and Willett [7] and to partial actions of exact discrete groups on locally compact spaces by Buss, Ferraro and Sehnem [8]. In [3], Anantharaman-Delaroche asked whether under some exactness hypothesis, amenability of a groupoid *does* follow from the weak containment property. In this paper, we give a partial answer to her question. Following [2, Definition 4.1, Proposition 4.8], we call a groupoid $\mathcal{G}$ *strongly amenable at infinity*, if it acts amenably on its fiberwise Stone-Čech compactification $\beta_r\mathcal{G}$. For étale groupoids satisfying some mild assumption, this condition is equivalent to a number of other exactness conditions like exactness of the reduced C^*-algebra [2, Theorem 8.6]. We emphasize that all groupoids considered in this paper are assumed to be Hausdorff. Our main theorem is the following:

Theorem 1.1. *Let $\mathcal{G}$ be an étale groupoid which is strongly amenable at infinity. If $C^*\mathcal{G} = C^*_r\mathcal{G}$ via the regular representation, then $C^*_r\mathcal{G}$ is nuclear.*

In this case, $\mathcal{G}$ is amenable by [4, Corollary 6.2.14, Theorem 3.3.7].

The proof of our main theorem follows the same idea as [12]. The goal is to factor the inclusion of $C^*_r\mathcal{G}$ into its double dual through the nuclear C^*-algebra $C^*_r(\beta_r\mathcal{G} \rtimes \mathcal{G})$. If we want to imitate the construction of [12], we need to extend the

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action of $\mathcal{G}$ on its unit space X to the double dual $C_0(X)^{**}$. This might not be possible since the canonical inclusion $C_0(X) \hookrightarrow C_0(X)^{**}$ is usually degenerate. But since we only consider étale groupoids, we can reformulate the problem in terms of partial actions of inverse semigroups. We show that a partial action of an inverse semigroup on a C^*-algebra naturally extends to a partial action on the double dual. In particular, we get a partial action on $C_0(X)^{**}$ by the inverse semigroup of open bisections of $\mathcal{G}$. We then show that the double dual of a partial action is covariantly represented on its Haagerup standard form [10]. For partial *group* actions, this has already been done in [8]. With the Haagerup standard form at hand, we run essentially the same proof as in [12] to produce a completely positive contractive map

$$C_r^*(\beta_r\mathcal{G} \rtimes \mathcal{G}) \to (C_r^*(\mathcal{G}))^{**}$$

which extends the inclusion on $C_r^*(\mathcal{G})$.

The paper is organized as follows: In Section 2 we fix some notation concerning groupoid actions on C^*-algebras. In Section 3 we translate Section 2 to the context of inverse semigroups. The enveloping von Neumann algebra of a partial action and its Haagerup standard form are introduced in Section 4. In the last section, we prove our main theorem.

Notation. The fiber product of two maps $f : X \to Z, g : Y \to Z$ is denoted by $X \times_{f,Z,g} Y := \{(x, y) \in X \times Y, f(x) = g(y)\}$. If the maps f and g are clear from the context, we omit one or both of them from the notation.

2. Étale groupoids

A *groupoid* $\mathcal{G}$ is a small category in which every morphism is invertible. We denote the set of all morphisms again by $\mathcal{G}$ and the set of objects by $\mathcal{G}^{(0)}$, considered as a subspace of $\mathcal{G}$ via the identity morphisms. $\mathcal{G}^{(0)}$ is also called the *unit space*. The range and source maps are denoted by $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$. For $x \in \mathcal{G}^{(0)}$, we write $\mathcal{G}^x := r^{-1}(x)$ and $\mathcal{G}_x := s^{-1}(x)$. A *topological groupoid* is a groupoid $\mathcal{G}$ together with a topology on $\mathcal{G}$ such that the range and source maps $\mathcal{G} \to \mathcal{G}^{(0)}$, the inverse map $\mathcal{G} \to \mathcal{G}$ and the composition map $\mathcal{G} \times_{s,\mathcal{G}^{(0)},r} \mathcal{G} \to \mathcal{G}$ are continuous. A topological groupoid $\mathcal{G}$ is called *étale*, if it is locally compact Hausdorff and if the range and source maps are local homeomorphisms. In this case, the unit space is clopen in $\mathcal{G}$ and the fibers $\mathcal{G}^x \subseteq \mathcal{G}, x \in \mathcal{G}^{(0)}$ are discrete. In this article, we only consider étale groupoids. We refer to [15] for an introduction to étale groupoids.

We now introduce actions of étale groupoids on C^*-algebras. A $C_0(X)$-*algebra* is a C^*-algebra A together with a non-degenerate $*$-homomorphism $C_0(X) \to ZM(A)$ into the center of the multiplier algebra of A. Equivalently, A is the section algebra of an upper semicontinuous C^*-bundle over X (see [17, Appendix C] for an account on this perspective). The fiber of this bundle at a point $x \in X$ is given by $A_x := A/(C_0(X \setminus \{x\})A)$. Note that any $C_0(X)$-linear $*$-homomorphism $\phi : A \to B$ of $C_0(X)$-algebras canonically induces $*$-homomorphisms $\pi_x : A_x \to B_x$ on the fibers. We denote by $A \otimes_{C_0(X)} B$ the quotient of the minimal tensor product $A \otimes B$ by the closed two-sided ideal generated by elements of the form $fa \otimes b - a \otimes fb, a \in A, b \in B, f \in C_0(X)$.

Let $\mathcal{G}$ be an étale groupoid with unit space $X = \mathcal{G}^{(0)}$. By pulling back along the maps $r, s : \mathcal{G} \to X$, we equip $C_0(\mathcal{G})$ with the structure of a $C_0(X)$-algebra. The algebras

$$r^*A := C_0(\mathcal{G}) \otimes_{r,C_0(X)} A, \quad s^*A := C_0(\mathcal{G}) \otimes_{s,C_0(X)} A$$

are $C_0(\mathcal{G})$-algebras where the subscript r or s in the tensor product indicates the $C_0(X)$-structure on $C_0(\mathcal{G})$ that we are considering.

Definition 2.1 ([11])**.** A *$\mathcal{G}$-C^*-algebra* (A, α) is a $C_0(X)$-algebra A together with a $C_0(\mathcal{G})$-linear $*$-isomorphism $\alpha : s^*A \to r^*A$ such that for all $(g, h) \in \mathcal{G} \times_{s,X,r} \mathcal{G}$ we have

$$\alpha_{gh} = \alpha_g \circ \alpha_h : A_{s(h)} \to A_{r(g)}.$$

A $C_0(X)$-linear $*$-homomorphism $\phi : A \to B$ between $\mathcal{G}$-C^*-algebras (A, α) and (B, β) is called *equivariant*, if the following diagram commutes:

$$\begin{array}{ccc} s^*A & \xrightarrow{\mathrm{id} \otimes \phi} & s^*B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ r^*A & \xrightarrow{\mathrm{id} \otimes \phi} & r^*B \end{array}$$

Example 2.2. We can reformulate the definition of a commutative $\mathcal{G}$-C^*-algebra $A = C_0(Y)$ in terms of Gelfand duals as follows: The non-degenerate $*$-homomorphism $C_0(X) \to ZM(C_0(Y)) = C_b(Y)$ corresponds to a continuous map $p : Y \to X$ which is also called the *anchor map*. The $*$-isomorphism $\alpha : s^*C_0(Y) \to r^*C_0(Y)$ corresponds to a continuous map $\alpha' : \mathcal{G} \times_{s,X,p} Y \to \mathcal{G} \times_{r,X,p} Y$ which commutes with the projection onto $\mathcal{G}$ and which satisfies

$$\alpha'_{gh} = \alpha'_g \circ \alpha'_h : p^{-1}(s(h)) \to p^{-1}(r(g))$$

for all $(g, h) \in \mathcal{G} \times_{s,X,r} \mathcal{G}$. Here α'_g denotes the restriction of α' to the preimage of g under the projection of $\mathcal{G} \times_{s,X,p} Y$ resp. $\mathcal{G} \times_{r,X,p} Y$ onto $\mathcal{G}$. We also say that (Y, p, α') is a *$\mathcal{G}$-space*.

Example 2.3. The unit space X of $\mathcal{G}$ itself is a $\mathcal{G}$-space where the action $\mathcal{G} \times_{s,X} X \to \mathcal{G} \times_{r,X} X$ is given by $(g, s(g)) \mapsto (g, r(g))$.

Definition 2.4 ([1, 2])**.** The *fiberwise Stone-Čech compactification of* $\mathcal{G}$ is the Gelfand dual $\beta_r\mathcal{G}$ of the commutative C^*-algebra of all continuous bounded functions $f : \mathcal{G} \to \mathbb{C}$ such that for every $\varepsilon > 0$ there exists a compact subset $C \subseteq X$ satisfying $|f(g)| < \varepsilon$ for all $g \notin r^{-1}(C)$. We define a $\mathcal{G}$-action on $C_0(\beta_r\mathcal{G})$ by taking the canonical inclusion

$$\iota : C_0(X) \hookrightarrow C_0(\beta_r\mathcal{G}), \quad \iota(f)(g) := f(r(g)), \quad f \in C_0(X), g \in \mathcal{G}$$

for the $C_0(X)$-structure and by defining the action

$$\alpha : C_0(\mathcal{G}) \otimes_{s,C_0(X),r} C_0(\beta_r\mathcal{G}) \to C_0(\mathcal{G}) \otimes_{r,C_0(X),r} C_0(\beta_r\mathcal{G})$$

via the formular $\alpha(f \otimes f')(g, h) := f(g)f'(g^{-1}h)$. Here we identify the codomain of α with certain functions on $\mathcal{G} \times_{r,X,r} \mathcal{G}$.

Note that the inclusion $\iota : C_0(X) \hookrightarrow C_0(\beta_r\mathcal{G})$ is $\mathcal{G}$-equivariant. Despite of its name, the fiberwise Stone-Čech compactification $\beta_r\mathcal{G}$ is in general *not* compact and its fibers might not agree with the Stone-Čech compactifications of the range fibers of $\mathcal{G}$. However in the case that $\mathcal{G}$ is a group (or more generally if X is compact), $\beta_r\mathcal{G}$ agrees with the usual Stone-Čech compactification of $\mathcal{G}$.

Definition 2.5. Let (A, α) be a $\mathcal{G}$-C^*-algebra. Denote by $C_c(\mathcal{G}, A) := C_c(\mathcal{G})r^*A$ the space of compactly supported continuous sections of the upper semicontinuous C^*-bundle $\mathcal{G} \times_{r,X} A$. We define a multiplication and involution on $C_c(\mathcal{G}, A)$ by

$$f * f'(g) := \sum_{h \in \mathcal{G}^{r(g)}} f(h)\alpha_h(f'(h^{-1}g)), \quad f^*(g) := \alpha_g(f(g^{-1})^*)$$

for $f, f' \in C_c(\mathcal{G}, A)$ and $g \in \mathcal{G}$.

The *full crossed product* $A \rtimes \mathcal{G}$ is by definition the enveloping C^*-algebra of $C_c(\mathcal{G}, A)$ (c.f. [13, Proposition 3.2]). To define the reduced crossed product, fix $x \in X$ and consider the Hilbert-A_x-module $\ell^2(\mathcal{G}_x, A_x)$. We define a representation

$$\Lambda_x : C_c(\mathcal{G}, A) \to \mathcal{L}(\ell^2(\mathcal{G}_x, A_x)), \quad \Lambda_x(f)\xi(h) := \sum_{g \in \mathcal{G}^{r(h)}} \alpha_{h^{-1}}(f(g))\xi(g^{-1}h).$$

The *reduced crossed product* $A \rtimes_r \mathcal{G}$ is defined as the completion of $C_c(\mathcal{G}, A)$ by the norm

$$\|f\|_r := \sup_{x \in X} \|\Lambda_x(f)\|, \quad f \in C_c(\mathcal{G}, A).$$

We get a canonical quotient map $\Lambda : A \rtimes \mathcal{G} \to A \rtimes_r \mathcal{G}$. In the case $A = C_0(X)$, we simply write $C^*\mathcal{G} := C_0(X) \rtimes \mathcal{G}$ and $C^*_r\mathcal{G} := C_0(X) \rtimes_r \mathcal{G}$.

3. Inverse semigroups

Definition 3.1. An *inverse semigroup* is a semigroup S such that for every $s \in S$, there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^* = s^*ss^*$.

A unit in S is an element $1 \in S$ such that $s = 1s = s1$ for all $s \in S$. In this article, all inverse semigroups are assumed to have a unit. Of course groups are examples of inverse semigroups. Our main example is the following.

Example 3.2. Let $\mathcal{G}$ be a topological groupoid. A subset $U \subseteq \mathcal{G}$ is called a *bisection*, if the restrictions of the range and source maps to U are homeomorphisms onto their images. If U and V are bisections, their product

$$UV := \{gh : g \in U, h \in V, r(h) = s(g)\}$$

is again a bisection. Also the inverse $U^* := U^{-1}$ of a bisection is again a bisection. With these operations, the collection of all open bisections of $\mathcal{G}$ becomes an inverse semigroup. It has a unit given by the whole unit space X.

Note that a locally compact Hausdorff groupoid is étale if and only if its topology has a basis consisting of open bisections.

Definition 3.3. Let S be an inverse semigroup with unit $1 \in S$ and let X be a set. A partial action $\theta = ((X_s)_{s \in S}, (\theta_s)_{s \in S})$ of S on X consists of a collection $(X_s)_{s \in S}$ of subsets of X together with bijections

$$\theta_s : X_{s^*} \to X_s, \quad s \in S$$

satisfying

(1) $X_1 = X$ and $\theta_1 = \mathrm{id}$.
(2) For every $s, t \in S$, we have $\theta_{s^*}(X_s \cap X_{t^*}) \subseteq X_{(ts)^*}$.
(3) θ_{ts} extends $\theta_t\theta_s$ on $\theta_{s^*}(X_s \cap X_{t^*})$.

Definition 3.4. Let S be an inverse semigroup with partial actions θ and ω on sets X and Y. A map $f : X \to Y$ is called *equivariant* if we have $f(X_s) \subseteq Y_s$ and $\omega_s \circ f = f \circ \theta_s$ on X_{s^*} for every $s \in S$.

Depending on the additional structure that X carries, we require some extra conditions on θ: If X is a C^*-algebra, all the X_s are required to be closed two-sided ideals and the θ_s are required to be $*$-isomorphisms. If X is a von Neumann algebra, the X_s are required to be ultraweakly closed two-sided ideals and the θ_s are required to be $*$-isomorphisms. If X is a topological space, the X_s are required to be open subsets and the θ_s are required to be homeomorphisms. If X is a Hilbert space, the X_s are required to be closed linear subspaces and the θ_s are required to be isometries.

Note that partial actions on commutative C^*-algebras and locally compact spaces can be identified with one another via Gelfand duality. We will also call a partial action of S on a Hilbert space H a *partial representation* and identify it with a map $S \to \mathcal{B}(H)$ acting by partial isometries.

Example 3.5. Let $\mathcal{G}$ be an étale groupoid with unit space X and let (A, α) be a $\mathcal{G}$-C^*-algebra. Let S be the inverse semigroup of open bisections of $\mathcal{G}$. There is a partial action $\theta = ((A_U)_{U\in S}, (\theta_U)_{U\in S})$ of S defined as follows: For an open bisection $U \subseteq \mathcal{G}$, define A_U to be the ideal $C_0(r(U))A \subseteq A$. Note that we can identify A_U with $C_0(U)r^*A$ and A_{U^*} with $C_0(U)s^*A$. Under this identification, we define θ_U to be the restriction

$$\theta_U := \alpha|_U : C_0(U)s^*A \xrightarrow{\simeq} C_0(U)r^*A.$$

of α to U. In particular, there is a canonical partial action of S on $A = C_0(X)$. In this case, we have $A_U = C_0(r(U))$ and $A_{U^*} = C_0(s(U))$ and θ_U is induced by the canonical homeomorphism $s(U) \xrightarrow{s|_U^{-1}} U \xrightarrow{r} r(U)$.

The next two definitions are due to [14].

Definition 3.6. Let θ be a partial action of an inverse semigroup S on a C^*-algebra A. A *covariant representation* of (A, S, θ) on a Hilbert space H is given by a pair (π, v) where $\pi : A \to \mathcal{B}(H)$ is a $*$-homomorphism and $v : S \to \mathcal{B}(H)$ is a partial representation such that

(1) $\pi(A_s)H = v_{ss^*}H$ for every $s \in S$.
(2) For every $s \in S$ and $a \in A_{s^*}$, we have
$$\pi(\theta_s(a)) = v_s\pi(a)v_{s^*}.$$

Definition 3.7. Let θ be a partial action of an inverse semigroup S on a C^*-algebra A. Let $C_c(S, A)$ be the set of all finite formal linear combinations $\sum_{s\in S} a_s u_s$ where $a_s \in A_s$. We define a product and involution on $C_c(S, A)$ by linear extension of the formulas

$$(au_s)(bu_t) := \theta_s(\theta_{s^*}(a)b)u_{st}, \quad (au_s)^* := \theta_{s^*}(a^*)u_{s^*}, \quad s, t \in S, a \in A_s, b \in A_t.$$

Let (π, v) be a covariant representation of (A, S, θ) on H. The *integrated form* of (π, v) is the representation

$$\pi \rtimes v : C_c(S, A) \to \mathcal{B}(H), \quad \pi \rtimes v(au_s) := \pi(a)v_s.$$

Denote by $A \rtimes S$ the Hausdorff completion of $C_c(S, A)$ by the seminorm

$$\|f\| := \sup \|\pi \rtimes v(f)\|, \quad f \in C_c(S, A)$$

where the supremum runs over all covariant representations (π, v) of (A, S, θ).

Remark 3.8. Since we assumed S to have a unit $1 \in S$, we can consider A as a subalgebra of $A \rtimes S$ via the inclusion $a \mapsto au_1$. For $a \in A_s$, we can even identify au_1 with au_{ss^*} since the integrated forms of all covariant representations agree on these two elements.

As in the group case, we have

Proposition 3.9 ([14, Proposition 4.8]). *Every non-degenerate representation of $A \rtimes S$ is the integrated form of a covariant representation.*

The following theorem allows us to translate Theorem 1.1 to the inverse semigroup setting.

Theorem 3.10 ([13, Theorem 7.2]). *Let $\mathcal{G}$ be an étale groupoid and A a $\mathcal{G}$-C^*-algebra. Equip A with the canonical partial action of the inverse semigroup S of open bisections of $\mathcal{G}$ as in Example 3.5. Then there is a canonical isomorphism*

$$A \rtimes \mathcal{G} \cong A \rtimes S.$$

4. The Haagerup standard form

As in [7, 8, 12], a key ingredient for Theorem 1.1 is Haagerup's standard form of von Neumann algebras. Recall that a cone $P \subseteq H$ in a Hilbert space H is called *self-dual*, if it coincides with its dual $P^\circ := \{\xi \in H : \langle \xi, \eta \rangle \geq 0 \quad \forall \eta \in P\}$.

Theorem 4.1 ([10, Theorem 1.6]). *Let M be a von Neumann algebra. Then there is a Hilbert space H, an embedding $M \subseteq \mathcal{B}(H)$, a conjugate linear isometric involution $J : H \to H$ and a self-dual cone $P \subseteq H$ such that the following properties are satisfied:*

1. $JMJ = M'$.
2. $JcJ = c^*$ *for all* $c \in Z(M)$.
3. $J\xi = \xi$ *for all* $\xi \in P$.
4. $aJaJ(P) \subseteq P$ *for all* $a \in M$.

The quadruple (M, H, J, P) is called a *standard form*. It is unique in the following sense:

Theorem 4.2 ([10, Theorem 2.3]). *Let (M_i, H_i, J_i, P_i) be standard forms for $i = 1, 2$ and let $\phi : M_1 \to M_2$ be a $*$-isomorphism. Then there is a unique unitary $U : H_1 \to H_2$ such that*

1. $\phi(x) = UxU^*$ *for all* $x \in M_1$.
2. $J_2 U = UJ_1$.
3. $P_2 = U(P_1)$.

The following Lemma is the inverse semigroup analogue of [8, Proposition 3.4].

Lemma 4.3. *Let θ be a partial action of an inverse semigroup S on a von Neumann algebra M. Let (M, H, J, P) be its standard form and denote by $\iota : M \hookrightarrow \mathcal{B}(H)$ the inclusion. Then there is a canonical partial representation $v : S \to \mathcal{B}(H)$ such that (ι, v) is a covariant representation.*

Proof. For $s \in S$ denote by $p_s \in M$ the central projection such that $M_s = p_s M$. It follows from Lemma 2.6 of [10] that $(p_s M, p_s H, p_s J p_s, p_s(P))$ again is a standard form. Theorem 4.2 applied to $\theta_s : p_{s^*} M \to p_s M$ provides us with a unique isometry $v_s : p_{s^*} H \to p_s H$ satisfying

(1) $\theta_s(a) = v_s a v_{s^*}$
(2) $p_s J v_s = v_s J p_{s^*}$
(3) $p_s(P) = v_s(p_{s^*}(P))$

for all $s \in S$ and $a \in M_{s^*}$. Another application of Theorem 4.2 shows that the map $s \mapsto v_s$ defines a partial representation of S on H. It immediately follows from the above properties that (ι, v) is covariant. □

We will later apply the above lemma to the following construction:

Definition 4.4. Let θ be a partial action of an inverse semigroup S on a C^*-algebra A. We extend θ to a partial action θ^{**} on the enveloping von Neumann algebra $M := A^{**}$ as follows. For $s \in S$, let $M_s := (A_s)^{**}$ be the enveloping von Neumann algebra of A_s, considered as an ultraweakly closed ideal in M. Define $\theta_s^{**} : A_{s^*}^{**} \to A_s^{**}$ as the normal extension of θ_s.

By uniqueness of the various normal extensions, θ^{**} is indeed a partial action.

5. Proof of the main theorem

The following application of Stinespring's theorem can be proved exactly as [6, Lemma 4.8].

Lemma 5.1. *Let θ be a partial action of an inverse semigroup S on a C^*-algebra A. Let (ϕ, v) be a completely positive covariant representation of A on a Hilbert space H (i.e. the same conditions as in Definition 3.6 hold with "$*$-homomorphism" replaced by "completely positive map"). Then the map*

$$\tilde{\phi} : C_c(S, A) \to \mathcal{B}(H), \quad au_s \mapsto \phi(a)v_s$$

extends to a completely positive map $\tilde{\phi} : A \rtimes S \to \mathcal{B}(H)$. If ϕ is contractive, then so is $\tilde{\phi}$.

Corollary 5.2. *Let A and B be C^*-algebras equipped with partial actions of an inverse semigroup S. Let $\phi : A \to B$ be an equivariant completely positive map. Then the map*

$$\tilde{\phi} : C_c(S, A) \to C_c(S, B), \quad au_s \mapsto \phi(b)u_s$$

extends to a completely positive map $A \rtimes S \to B \rtimes S$. If ϕ is contractive, then so is $\tilde{\phi}$.

Proof. Take a non-degenerate and faithful representation $B \rtimes S \subseteq \mathcal{B}(H)$. By Proposition 3.9, any such representation is the integrated form of a covariant representation (π, v). Now apply the above lemma to the pair $(\pi \circ \phi, v)$. □

Lemma 5.3 ([5, Proposition 1.5.7])**.** *Let $\phi : A \to B$ be a completely positive contractive map. Then there is a largest subalgebra $A_\phi \subseteq A$ such that $\phi|_{A_\phi}$ is a $*$-homomorphism. Furthermore, we have*

$$\phi(ab) = \phi(a)\phi(b), \quad \phi(ba) = \phi(b)\phi(a), \quad \forall a \in A, b \in A_\phi.$$

The subalgebra $A_\phi \subseteq A$ is called the *multiplicative domain* of A_ϕ.

Proposition 5.4. *Let $\mathcal{G}$ be an étale groupoid with unit space X. Denote by S its inverse semigroup of open bisections. Suppose that $C^*\mathcal{G} = C_r^*\mathcal{G}$. Then there is an S-equivariant completely positive contractive map*

$$C_0(\beta_r\mathcal{G}) \to C_0(X)^{**}$$

which extends the inclusion on $C_0(X)$.

Proof. Let $\tilde{\pi} : C_0(X)^{**} \hookrightarrow \mathcal{B}(H)$ be the standard form of $C_0(X)^{**}$ and $(\tilde{\pi}, v)$ the associated covariant representation as in Lemma 4.3. Denote by π the restriction of $\tilde{\pi}$ to $C_0(X)$. Then (π, v) integrates to a representation

$$\pi \rtimes v : C_r^*\mathcal{G} = C^*\mathcal{G} = C_0(X) \rtimes S \to \mathcal{B}(H).$$

Recall that reduced crossed products preserve inclusions. The proof of this fact in [9, Lemma A.16] for groups can easily be adapted to groupoids. Thus, the inclusion $\iota : C_0(X) \to C_0(\beta_r\mathcal{G})$ induces an inclusion

$$\iota \rtimes_r \mathcal{G} : C_r^*\mathcal{G} \hookrightarrow C_0(\beta_r\mathcal{G}) \rtimes_r \mathcal{G}.$$

By Arveson's extension theorem, there is a completely positive contractive map $\tilde{\phi}$ such that the following diagram commutes.

$$\begin{array}{ccc} C_0(\beta_r\mathcal{G}) \rtimes_r \mathcal{G} & & \\ {\scriptstyle \iota \rtimes_r \mathcal{G}} \big\uparrow & \overset{\tilde{\phi}}{\searrow} & \\ C_r^*\mathcal{G} & \xrightarrow{\pi \rtimes v} & \mathcal{B}(H) \end{array}$$

Since $C_0(\beta_r\mathcal{G}) \subseteq C_0(\beta_r\mathcal{G}) \rtimes_r \mathcal{G}$ is commutative, it follows from Lemma 5.3 that $\tilde{\phi}(C_0(\beta_r\mathcal{G}))$ is contained in the commutant $\pi(C_0(X))' \cong C_0(X)^{**}$. We claim that the restriction ϕ of $\tilde{\phi}$ to $C_0(\beta_r\mathcal{G})$ is S-equivariant. Again by Lemma 5.3, ϕ is $C_0(X)$-linear and thus preserves the domains of the partial actions by S.

Now denote the canonical partial action of S on $C_0(\beta_r\mathcal{G})$ as well as its restriction to $C_0(X)$ by θ. To see that ϕ is equivariant, fix elements $s \in S$ and $a \in C_0(\beta_r\mathcal{G})_{s^*}$. We have to show that

$$\phi(\theta_s(a)) = \theta_s^{**}(\phi(a)) \tag{1}$$

holds. We identify a with its image au_{s^*s} in the crossed product $C_0(\beta_r\mathcal{G}) \rtimes S$ as in Remark 3.8. Similarly, we identify $\theta_s(a)$ with $\theta_s(a)u_{ss^*}$. Fix an approximate identity $(x_\lambda)_\lambda$ for $C_0(X)_s$. Since the inclusion $C_0(X)_s \to C_0(\beta_r\mathcal{G})_s$ is non-degenerate, the image of $(x_\lambda)_\lambda$ in $C_0(\beta_r\mathcal{G})_s$ is again an approximate identity. We denote it by $(x_\lambda)_\lambda$ as well. Similarly, $(\theta_{s^*}(x_\lambda))_\lambda$ is an approximate identity for $C_0(\beta_r\mathcal{G})_{s^*}$. A calculation in the crossed product $C_0(\beta_r\mathcal{G}) \rtimes S$ shows that we have

$$x_\lambda u_s a \theta_{s^*}(x_\lambda) u_{s^*} = x_\lambda \theta_s(a) x_\lambda u_{ss^*}. \tag{2}$$

Now reinterpret $\tilde{\phi}$ as a map on $C_0(\beta_r\mathcal{G}) \rtimes S$ by precomposing it with the quotient map

$$C_0(\beta_r\mathcal{G}) \rtimes S \cong C_0(\beta_r\mathcal{G}) \rtimes \mathcal{G} \to C_0(\beta_r\mathcal{G}) \rtimes_r \mathcal{G}.$$

Since the elements $x_\lambda u_s$ and $\theta_{s^*}(x_\lambda)u_{s^*}$ belong to the multiplicative domain of $\tilde{\phi}$ and since $\tilde{\phi}$ extends $\pi \rtimes v$, we obtain

$$\begin{aligned} \phi(\theta_s(a)) &= \tilde{\phi}(\theta_s(a)u_{ss^*}) = \lim \tilde{\phi}(x_\lambda \theta_s(a) x_\lambda u_{ss^*}) \\ &\overset{(2)}{=} \lim \tilde{\phi}(x_\lambda u_s a \theta_{s^*}(x_\lambda) u_{s^*}) = \lim \pi(x_\lambda) v_s \phi(a) \pi(\theta_{s^*}(x_\lambda)) v_{s^*} = v_s \phi(a) v_{s^*} \\ &= \theta_s^{**}(\phi(a)). \end{aligned}$$

This proves (1). □

Lemma 5.5. *Let θ be a partial action of an inverse semigroup S on a C^*-algebra A. Then there is a $*$-homomorphism*

$$A^{**} \rtimes S \to (A \rtimes S)^{**}$$

such that the composition

$$A \rtimes S \to A^{**} \rtimes S \to (A \rtimes S)^{**}$$

is the canonical inclusion.

Proof. Represent $(A \rtimes S)^{**} \subseteq \mathcal{B}(H)$ faithfully, normally and non-degenerately on a Hilbert space H. The restriction of this inclusion to $A \rtimes S$ is an integrated form of a covariant representation (π, v) by Proposition 3.9. Denote by π^{**} the unique normal extension of π to A^{**}. We claim that (π^{**}, v) is again a covariant representation whose integrated form $\pi^{**} \rtimes v$ maps into $(A \rtimes S)^{**}$. Indeed, using normality of π^{**} we get

$$\pi^{**}(A_s^{**})H = \pi(A_s)''H = \overline{\pi(A_s)H} = H_s$$

for all $s \in S$. For an element $a \in A_s^{**}$ which is the ultraweak limit of a net $a_\lambda \in A_s$, we have

$$\pi^{**}(\theta_s^{**}(a)) = \lim \pi(\theta_s(a_\lambda)) = \lim v_s \pi(a_\lambda) v_s^* = v_s \pi^{**}(a) v_s^*,$$

again using normality of π^{**} and θ_s^{**}. Observe that we also have

$$\pi^{**} \rtimes v(au_s) = \lim \pi(a_\lambda) v_s \in (A \rtimes S)^{**}.$$

Thus the map $\pi^{**} \rtimes v$ has the desired properties. □

We can now prove our main theorem:

Theorem 5.6. *Let $\mathcal{G}$ be an étale groupoid which is strongly amenable at infinity. If $C_r^*\mathcal{G} = C^*\mathcal{G}$, then $C_r^*\mathcal{G}$ is nuclear. In particular $\mathcal{G}$ is amenable.*

Proof. Denote by S the inverse semigroup of open bisections of $\mathcal{G}$. By Proposition 5.4, there is an S-equivariant completely positive contractive map

$$\phi : C_0(\beta_r \mathcal{G}) \to C_0(X)^{**}$$

extending the inclusion of $C_0(X)$. By Corollary 5.2, ϕ extends to a completely positive contractive map

$$\tilde{\phi} : C_0(\beta_r \mathcal{G}) \rtimes S \to C_0(X)^{**} \rtimes S.$$

By Lemma 5.5, there is a $*$-homomorphism

$$C_0(X)^{**} \rtimes S \to (C_0(X) \rtimes S)^{**}$$

extending the inclusion on $C_0(X) \rtimes S$. Putting things together, we can express the inclusion $C_0(X) \rtimes S \hookrightarrow (C_0(X) \rtimes S)^{**}$ as the following composition of completely positive contractive maps:

$$C_0(X) \rtimes S \to C_0(\beta_r \mathcal{G}) \rtimes S \to C_0(X)^{**} \rtimes S \to (C_0(X) \rtimes S)^{**}. \tag{3}$$

Since $\mathcal{G}$ was assumed to be strongly amenable at infinity, the C^*-algebra $C_0(\beta_r \mathcal{G}) \rtimes S \cong C_0(\beta_r \mathcal{G}) \rtimes \mathcal{G} \cong C_0(\beta_r \mathcal{G}) \rtimes_r \mathcal{G}$ is nuclear [2, Proposition 7.2]. Thus the map (3) is nuclear. Recall that by [5, Proposition 2.3.8], a C^*-algebra is nuclear if and only if its inclusion into its double dual is nuclear. Therefore $C_0(X) \rtimes S \cong C^*\mathcal{G} \cong C_r^*\mathcal{G}$ is nuclear. Now amenability of $\mathcal{G}$ follows from [4, Corollary 6.2.14, Theorem 3.3.7]. □

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