

***K*-THEORY OF NON-COMMUTATIVE BERNOULLI SHIFTS**

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ABSTRACT. For a large class of C^* -algebras A , we calculate the K -theory of reduced crossed products $A^{\otimes G} \rtimes_r G$ of Bernoulli shifts by groups satisfying the Baum–Connes conjecture. In particular, we give explicit formulas for finite-dimensional C^* -algebras, UHF-algebras, rotation algebras, and several other examples. As an application, we obtain a formula for the K -theory of reduced C^* -algebras of wreath products $H \wr G$ for large classes of groups H and G . Our methods use a generalization of techniques developed by the second named author together with Joachim Cuntz and Xin Li, and a trivialization theorem for finite group actions on UHF algebras developed in a companion paper by the third and fourth named authors.

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1. INTRODUCTION

Let G be a countable discrete group and let A be a separable unital C^* -algebra. Equip the infinite tensor product $A^{\otimes G}$ with the natural Bernoulli shift action (see Section 2 for the definition). The objective of this paper is

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to compute the K -theory group of the reduced crossed product $A^{\otimes G} \rtimes_r G$ in as many cases as possible.

Building up on the work in [CEL13], Xin Li [Li19] computed this when A is a finite-dimensional C^* -algebra of the form $\mathbb{C} \oplus \bigoplus_{1 \leq k \leq N} M_{p_k}$, assuming the Baum–Connes conjecture with coefficients (referred to as *BCC* below) for G [BCH94]. His motivation was to compute the K -theory of the reduced group C^* -algebra of the wreath product $H \wr G$ for an arbitrary finite group H . For $A = \mathbb{C} \oplus \bigoplus_{1 \leq k \leq N} M_{p_k}$, the K -theory groups $K_*(A^{\otimes G} \rtimes_r G)$ are computed in [Li19] as

$$(1.1) \quad K_*(A^{\otimes G} \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \text{FIN}} \bigoplus_{[S] \in G_F \backslash (\{1, \dots, N\}^F)} K_*(C_r^*(G_S))$$

where FIN is the set of all the finite subsets of G equipped with the left-translation G -action and $G_F := \text{Stab}_G(F)$, resp. $G_S := \text{Stab}_G(S)$, are the stabilizer groups at F , resp. at S , for the action of G on FIN . Note that when G is torsion-free, (1.1) is the direct sum of $K_*(C_r^*(G))$ and infinitely many copies of $K_*(\mathbb{C})$. For wreath products $H \wr G$ with respect to a finite group H , we have $C_r^*(H \wr G) \cong C_r^*(H)^{\otimes G} \rtimes_r G$. In this case, the number N in (1.1) corresponds to the number of non-trivial conjugacy classes of H . Our results were also motivated in part by the paper [Ohh15] by Issei Ohhashi, where he gives K -theory computations for crossed products $A^{\otimes \mathbb{Z}} \rtimes \mathbb{Z}$ for Bernoulli shifts by the integer group \mathbb{Z} .

Our first result computes $K_*(A^{\otimes Z} \rtimes_r G)$ for an arbitrary finite-dimensional C^* -algebra A and for an arbitrary infinite countable G -set Z with the action of G on $A^{\otimes Z}$ induced by the G -action on Z . In what follows we let $\text{FIN}(Z)$ denote the collection of finite subsets of Z and we put $\text{FIN}^\times(Z) := \text{FIN}(Z) \setminus \{\emptyset\}$. In the case $Z = G$ we shall simply write FIN and FIN^\times as above.

Theorem A (Theorem 3.6). *Let G be a discrete group satisfying BCC and let Z be a countably infinite G -set. Let $A = \bigoplus_{0 \leq j \leq N} M_{k_j}$ where k_0, \dots, k_N ($N \geq 1$) are positive integers with $\gcd(k_0, \dots, k_N) = n$. We have an explicit isomorphism*

$$K_*(A^{\otimes Z} \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \text{FIN}(Z)} \bigoplus_{[S] \in G_F \backslash (\{1, \dots, N\}^F)} K_*(C_r^*(G_S))[1/n].$$

For $N = 0$, we have

$$K_*(M_n^{\otimes Z} \rtimes_r G) \cong K_*(C_r^*(G))[1/n].$$

Our strategy of the proof is as follows: First, we consider the case $n = 1$ and construct a unit-preserving KK -equivalence $\mathbb{C}^{N+1} \sim_{KK} A$ using an arithmetic argument about matrices in $\text{SL}(n, \mathbb{Z})$ with non-negative entries. By an argument borrowed from [Izu19, Sza18], this allows us to replace A by \mathbb{C}^{N+1} and apply Xin Li's formula (1.1). Second, we reduce the general

case to the case $n = 1$ with the help of a trivialization theorem for finite group actions on UHF algebras developed in the companion paper [KN22].

One of our main techniques is the following theorem which is inspired by the *regular basis* technique from [CEL13].

Theorem B (Theorem 2.8, Corollary 2.10). *Let G be a discrete group satisfying BCC. Let A be a separable, unital C^* -algebra that satisfies the Universal Coefficient Theorem (UCT) and such that the unital inclusion $\iota: \mathbb{C} \rightarrow A$ induces a split-injection $K_0(\mathbb{C}) \rightarrow K_0(A)$. Then $K_*(A^{\otimes G} \rtimes_r G)$ only depends on G and the cokernel $\tilde{K}_*(A)$ of $\iota_*: K_*(\mathbb{C}) \rightarrow K_*(A)$. For any countable G -set Z we have*

$$K_*(A^{\otimes Z} \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \text{FIN}(Z)} K_*(B^{\otimes F} \rtimes_r G_F),$$

where B is any C^* -algebra satisfying UCT with K -theory isomorphic to $\tilde{K}_*(A)^1$ and where $G_F = \text{Stab}_G(F)$. In particular, if G is torsion-free and the action of G on Z is free, we have

$$K_*(A^{\otimes Z} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times(Z)} K_*(B^{\otimes F}).$$

Note that the second formula can be computed more explicitly using the Künneth theorem. In the important case that $Z = G$ with the left translation action (or, more generally, if G acts properly on Z) the stabilizers G_F for $F \in \text{FIN}^\times(Z)$ are all finite and the formula becomes more explicit once we can compute $K_*(B^{\otimes H} \rtimes_r H)$ for finite groups H and for relevant building blocks for B like \mathbb{C} , $C_0(\mathbb{R})$, the Cuntz-algebras \mathcal{O}_n , and $C_0(\mathbb{R}) \otimes \mathcal{O}_n$. For $H = \mathbb{Z}/2$, these computations have been done by Izumi [Izu19]. For general H , computing these K -groups is a non-trivial, indeed challenging task. For $B = C_0(\mathbb{R})$ however, they are nothing but the equivariant topological K -theory $K_H^*(\mathbb{R}^H)$ for the H -Euclidean space \mathbb{R}^H , i.e., $\mathbb{R}^{|H|}$ with H -action induced by translation of coordinates. The groups $K_H^*(\mathbb{R}^H)$ are quite well-studied ([Kar02] [EP09]). Using these results, we give a more explicit formula for $A = C(S^1)$ (Example 4.2) and for the rotation algebras (or noncommutative tori) $A = A_\theta$ (Example 4.4). More explicitly, we have

Theorem C (Example 4.2). *Let G be a discrete group satisfying BCC. We have*

$$K_*(C(S^1)^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \left(\bigoplus_{\substack{[F] \in G \backslash \text{FIN}^\times, \\ |G_F \backslash F| \text{ even}}} K_*(C_r^*(G_F)) \right) \oplus \left(\bigoplus_{\substack{[F] \in G \backslash \text{FIN}^\times, \\ |G_F \backslash F| \text{ odd}}} K_{G_F}^*(\mathbb{R}^{G_F}) \right).$$

¹Note that this determines B uniquely up to KK -equivalence.

As another application of Theorem B, we obtain a formula for the K -theory of reduced C^* -algebras of many wreath products $H \wr G$:

Theorem D. (Theorem 4.14) *Let G be a discrete group satisfying BCC and let H be a discrete group such that $C_r^*(H)$ satisfies the UCT and such that the inclusion $\mathbb{C} \rightarrow C_r^*(H)$ induces a split injection on K_0 . Then we have*

$$K_*(C_r^*(H \wr G)) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(B^{\otimes F} \rtimes_r G_F),$$

where B is any C^* -algebra satisfying the UCT with K -theory isomorphic to $\hat{K}_*(C_r^*(H))$. In particular, if G is torsion-free, we have

$$K_*(C_r^*(H \wr G)) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(B^{\otimes F}).$$

We note that both the UCT assumption on $C_r^*(H)$ and the split-injectivity of the map $K_0(\mathbb{C}) \rightarrow K_0(C_r^*(H))$ hold for every a- T -menable (in particular every amenable) group H (see Remark 4.15 below).

In Section 4, we also obtain formulas for several C^* -algebras A that are not covered in Theorem B, in particular for Cuntz algebras.

Theorem E (Corollary 4.10, Proposition 4.11). *Let G be a discrete group satisfying BCC and let $n \geq 2$. Then we have*

$$K_* \left(\mathcal{O}_{n+1}^{\otimes G} \rtimes_r G \right) [1/n] = 0,$$

where \mathcal{O}_{n+1} is the Cuntz algebra on $n+1$ generators. If G is finite and $n = p$ prime, then $K_* \left(\mathcal{O}_{p+1}^{\otimes G} \rtimes_r G \right)$ is a finitely generated p -group, that is a group of the form $\bigoplus_{1 \leq j \leq N} \mathbb{Z}/p^{k_j} \mathbb{Z}$.

In Section 5, we obtain a K -theory formula for Bernoulli shifts on unital AF-algebras in terms of colimits over the orbit category $\text{Or}_{\mathcal{FIN}}(G)$. The result also applies to more general examples, in particular to many unital ASH-algebras (see Remark 5.5).

Theorem F (Theorem 5.4). *Let A be a unital AF-algebra, let G be an infinite discrete group satisfying BCC, let Z be a countable proper G -set, and let $S \subseteq \mathbb{Z}$ be the set of all positive integers n such that $[1_A] \in K_0(A)$ is divisible by n . Then, the natural inclusions*

$$\begin{aligned} C_r^*(H) &\rightarrow C_r^*(G), & C_r^*(H) &\rightarrow A^{\otimes Z} \rtimes_r H, \\ C_r^*(G) &\rightarrow A^{\otimes Z} \rtimes_r G, & A^{\otimes Z} \rtimes_r H &\rightarrow A^{\otimes Z} \rtimes_r G, \end{aligned}$$

induce the following pushout diagram

$$\begin{array}{ccc} \operatorname{colim}_{G/H \in \text{Or}_{\mathcal{FLN}}(G)} K_*(C_r^*(H))[S^{-1}] & \longrightarrow & \operatorname{colim}_{G/H \in \text{Or}_{\mathcal{FLN}}(G)} K_*(A^{\otimes Z} \rtimes_r H) \\ \downarrow & & \downarrow \\ K_*(C_r^*(G))[S^{-1}] & \longrightarrow & K_*(A^{\otimes Z} \rtimes_r G). \end{array}$$

In particular, if G is torsion-free, this pushout diagram reads

$$\tilde{K}_*(C_r^*(G))[S^{-1}] \oplus K_*(A^{\otimes Z})_G \cong K_*(A^{\otimes Z} \rtimes_r G),$$

where $\tilde{K}_*(C_r^*(G))$ denotes the cokernel of the map $K_*(\mathbb{C}) \rightarrow K_*(C_r^*(G))$ induced from the unital inclusion $\mathbb{C} \hookrightarrow C_r^*(G)$.

The paper is structured as follows: In Section 2, we develop our main machinery, including Theorem B. We apply this machinery in Section 3 to prove Theorem A. In Section 4, we compute the K -theory of many more examples, including Theorems C, D, and E. Bernoulli shifts on unital AF-algebras are investigated in Section 5 where we prove Theorem F. Based on similar ideas, we obtain in Section 6 some very general K -theory formulas up to inverting an integer k or up to tensoring with the rationals \mathbb{Q} (see Theorem 6.3 and Theorem 6.4). The rational K -theory computations apply to all unital stably finite C^* -algebras satisfying the UCT.

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2. GENERAL STRATEGY

To avoid technical complications with KK -theory, we assume throughout that all C^* -algebras are separable except for the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H . For a C^* -algebra A and a finite set F , we write $A^{\otimes F}$ to denote the minimal tensor product $\otimes_{x \in F} A$. If A is moreover unital and Z is a (not necessarily finite) countable set, we denote by $A^{\otimes Z}$ the filtered colimit² $\operatorname{colim}_F A^{\otimes F}$ taken over all finite subsets $F \subseteq Z$ with respect to the connecting maps $A^{\otimes F} \ni x \mapsto x \otimes 1 \in A^{\otimes F} \otimes A^{\otimes F' - F} \cong A^{\otimes F'}$, for finite sets F, F' with $F \subseteq F'$. Hence $A^{\otimes Z}$ is the closed linear span of the elementary tensors $\otimes_{z \in Z} a_z$, where $a_z \in A$ for all $z \in Z$, and $a_z = 1$ for all but finitely many z . If G is a discrete group acting on Z , we call the G -action on $A^{\otimes Z}$ given by permutation of the tensor factors the *Bernoulli shift*. More explicitly, $g \left(\otimes_{z \in Z} a_s \right) = \otimes_{z \in Z} a_{g^{-1}z}$ for an elementary tensor $\otimes_{z \in Z} a_z \in A^{\otimes Z}$.

²We use the term ‘filtered colimit’ (which is the standard categorical notion) instead of the terms ‘direct limit’ or ‘inductive limit’ which seem to be more commonly used in C^* -algebra theory. We do this to be consistent with the use of more general colimits of functors in Section 5.

For a countable discrete group G and a G - C^* -algebra A , the *Baum–Connes conjecture with coefficients* (BCC) predicts a formula for the K -theory $K_*(A \rtimes_r G)$ of the reduced crossed product $A \rtimes_r G$, see [BCH94]. The precise formulation of the conjecture is not too important for us. We mostly need one of its consequences recalled in Theorem 2.3 below. Note that the Baum–Connes conjecture with coefficients has been verified for many groups, including all a - T -menable groups [HK01] and all hyperbolic groups [Laf12].

We refer the reader to [Kas88] for the definition and basic properties of the equivariant KK -groups $KK^G(A, B)$. Recall from [MN06] that KK^G can be organized into a triangulated category³ with G - C^* -algebras as objects, the groups $KK^G(A, B)$ as morphism sets, and composition given by the Kasparov product. The construction of KK^G -elements from G -*-homomorphisms may be interpreted as a functor from the category of G - C^* -algebras to the category KK^G . Furthermore, $A \mapsto K_*(A \rtimes_r G)$ factorizes through this functor.

Definition 2.1. A morphism ϕ in $KK^G(A, B)$ is a *weak K -equivalence* in KK^G if its restrictions to $KK^H(A, B)$ induce isomorphisms

$$K_*(A \rtimes_r H) \cong K_*(B \rtimes_r H)$$

for all finite subgroups H of G .

Remark 2.2. Weak K -equivalences are in general weaker than weak equivalences in KK^G in the sense of [MN06]. The latter ones are required to induce a KK^H -equivalence for all finite subgroups H of G .

The following theorem has been shown in [CEOO04] in the setting of locally compact groups (see also [MN06]). A detailed proof in the (easier) discrete case is given in [CELY17, Section 3.5].

Theorem 2.3. *Suppose that the Baum–Connes conjecture holds for G with coefficients in A and B . Then, any weak K -equivalence ϕ in $KK^G(A, B)$ induces an isomorphism*

$$K_*(A \rtimes_r G) \cong K_*(B \rtimes_r G).$$

Let A be a unital C^* -algebra and let G be a countable group satisfying BCC. Our general strategy to compute $K_*(A^{\otimes \mathbb{Z}} \rtimes_r G)$ is to replace $A^{\otimes \mathbb{Z}}$ by a weakly K -equivalent G - C^* -algebra with computable K -theory and then apply Theorem 2.3 above.

The following lemma and corollary are straightforward generalizations of [Izu19, Theorem 2.1] and [Sza18, Corollary 6.9].

Lemma 2.4 (see [Izu19, Theorem 2.1]). *Let A and B be not necessarily unital C^* -algebras, let G be a countable discrete group, and let F be a finite G -set. Then there is a map from $KK(A, B)$ to $KK^G(A^{\otimes F}, B^{\otimes F})$ which*

³Recently, KK^G was even refined to a stable ∞ -category [BEL21].

sends the class of a $*$ -homomorphism $\phi: A \rightarrow B$ to the class of $\phi^{\otimes F}$. Furthermore, this map is compatible with compositions (i.e. Kasparov products) and in particular sends KK -equivalences to a KK^G -equivalences. In particular, the Bernoulli shifts on $A^{\otimes F}$ and $B^{\otimes F}$ are KK^G -equivalent if A and B are KK -equivalent. The analogous statement holds if we replace the minimal tensor product \otimes by the maximal one.

Proof. We recall the description of KK^G in terms of asymptotic morphisms which admit an equivariant c.c.p. lift [Tho99] (see also Appendix in [KS03]). For any G - C^* -algebras A_0 and B_0 , let $[[A_0, B_0]]_{\text{cp}}^G$ be the set of homotopy equivalence classes of completely positive equivariant asymptotic homomorphisms from A_0 to B_0 , see [Tho99, Section 2]. The usual composition law for asymptotic homomorphisms restricts to $[[A_0, B_0]]_{\text{cp}}^G$, [Tho99, Theorem 2]. Let S be the suspension functor $SA_0 = C_0(\mathbb{R}) \otimes A_0$. Denote by $\tilde{\mathcal{K}}_G$ the C^* -algebra of compact operators on $\ell^2(G \times \mathbb{N})$. It is a G - C^* -algebra with respect to the regular representation on $\ell^2(G)$. The assignment $(A_0, B_0) \mapsto \widetilde{KK}^G(A_0, B_0) = [[SA_0 \otimes \tilde{\mathcal{K}}_G, SB_0 \otimes \tilde{\mathcal{K}}_G]]_{\text{cp}}^G$, $(\phi: A_0 \rightarrow B_0) \mapsto S\phi \otimes \text{id}_{\tilde{\mathcal{K}}_G}$ defines a bifunctor from G - C^* -algebras to abelian groups.

Thomsen showed that there is a natural isomorphism $\widetilde{KK}^G(A_0, B_0) \cong KK^G(A_0, B_0)$ of bifunctors that sends the composition product of asymptotic morphisms to the Kasparov product, see [Tho99, Theorem 4.8]. Now, given $\phi \in KK(A, B)$, we represent ϕ as a completely positive asymptotic homomorphism from $SA \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ to $SB \otimes \mathcal{K}(\ell^2(\mathbb{N}))$. Taking the (pointwise) minimal tensor product of ϕ with itself over F , we obtain a completely positive equivariant asymptotic homomorphism $\phi^{\otimes F}$ from $(SA \otimes \mathcal{K}(\ell^2(\mathbb{N})))^{\otimes F}$ to $(SB \otimes \mathcal{K}(\ell^2(\mathbb{N})))^{\otimes F}$. This construction clearly respects homotopy equivalences. Therefore, $[\phi] \mapsto [\phi^{\otimes F}]$ defines a map $KK(A, B) \rightarrow KK^G((SA \otimes \mathcal{K}(\ell^2(\mathbb{N})))^{\otimes F}, (SB \otimes \mathcal{K}(\ell^2(\mathbb{N})))^{\otimes F})$. By construction, this map is compatible with compositions and sends a $*$ -homomorphism $\phi: A \rightarrow B$ to $(S\phi \otimes \text{id}_{\mathcal{K}(\ell^2(\mathbb{N}))})^{\otimes F}$. By the stabilization theorem and by Kasparov's Bott-periodicity theorem⁴ (see [Bla98, Theorem 20.3.2]), the exterior tensor product by the identity on $C_0(\mathbb{R})^{\otimes F} \otimes \mathcal{K}(\ell^2(\mathbb{N}))^{\otimes F}$ induces a natural isomorphism

$$\begin{aligned} & KK^G(A_0, B_0) \\ & \cong KK^G(A_0 \otimes C_0(\mathbb{R})^{\otimes F} \otimes \mathcal{K}(\ell^2(\mathbb{N}))^{\otimes F}, B_0 \otimes C_0(\mathbb{R})^{\otimes F} \otimes \mathcal{K}(\ell^2(\mathbb{N}))^{\otimes F}) \end{aligned}$$

for all G - C^* -algebras A_0 and B_0 . Therefore, the map $[\phi] \mapsto [\phi^{\otimes F}]$ can be naturally regarded as a map from $KK(A, B)$ to $KK^G(A^{\otimes F}, B^{\otimes F})$. By construction, this map is compatible with compositions and sends the class of a $*$ -homomorphism $\phi: A \rightarrow B$ to the class of $\phi^{\otimes F}$. The case of the maximal tensor product is proven in the exact same way. \square

⁴Since the G -action on F factors through the finite group $\text{Sym}(F)$, the theorem is applicable even if G itself is not finite.

We emphasize that the map $\phi \mapsto \phi^{\otimes F}$ is *not* a group homomorphism. For example, it maps n in $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ to the class $[\pi_n]$ in $KK^G(\mathbb{C}, \mathbb{C})$ of the finite dimensional unitary representation π_n of G on $\ell^2(\{1, \dots, n\}^F)$ defined by permutation on the set $\{1, \dots, n\}^F$.

Corollary 2.5 (see [Sza18, Corollary 6.9]). *Let G be a discrete group and let $\phi: A \rightarrow B$ be a unital $*$ -homomorphism which is a KK -equivalence. Then for any countable G -set Z , the map*

$$\phi^{\otimes Z}: A^{\otimes Z} \rightarrow B^{\otimes Z}$$

is a weak K -equivalence in KK^G . In particular, $\phi^{\otimes Z}$ induces an isomorphism

$$K_*(A^{\otimes Z} \rtimes_r G) \rightarrow K_*(B^{\otimes Z} \rtimes_r G)$$

whenever G satisfies BCC.

Proof. For the first statement we may assume that G is finite. Since $K(- \rtimes_r G)$ commutes with filtered colimits, we may as well assume that Z is finite and apply Lemma 2.4. The second statement follows from Theorem 2.3. \square

Definition 2.6. Let G be a discrete group, let Z be a countable G -set and let A_0 and B be C^* -algebras with A_0 unital. We define a G - C^* -algebra $\mathcal{J}_{A_0, B}^Z$ as

$$(2.1) \quad \mathcal{J}_{A_0, B}^Z := \bigoplus_{F \in \text{FIN}(Z)} A_0^{\otimes Z-F} \otimes B^{\otimes F}.$$

The G -action is defined so that a group element $g \in G$ sends $A_0^{\otimes Z-F} \otimes B^{\otimes F}$ to $A_0^{\otimes Z-gF} \otimes B^{\otimes gF}$ by the obvious $*$ -homomorphism.

In the above definition, A_0 should be thought of as either the complex numbers $A_0 = \mathbb{C}$ or a UHF-algebra (see Section 6). For $A_0 = \mathbb{C}$, we just write $\mathcal{J}_B^Z := \mathcal{J}_{\mathbb{C}, B}^Z$. Note that $\mathcal{J}_{A_0, B}^Z$ is a G - $C_0(\text{FIN}(Z))$ -algebra in a natural way. The following lemma is crucial for our computations:

Lemma 2.7. *Let G, Z, A_0 and B be as above. Then $\mathcal{J}_{A_0, B}^Z$ can naturally be identified with the filtered colimit $\text{colim}_S A_0^{\otimes Z-S} \otimes (A_0 \oplus B)^{\otimes S}$ taken over all finite subsets $S \subseteq Z$, with respect to the obvious connecting maps.⁵*

Proof. Just observe that for all finite subsets $S \subseteq Z$ we have canonical isomorphisms

$$A_0^{\otimes Z-S} \otimes (A_0 \oplus B)^{\otimes S} \cong A_0^{\otimes Z-S} \otimes \left(\bigoplus_{F \subseteq S} A_0^{\otimes S-F} \otimes B^{\otimes F} \right) \cong \bigoplus_{F \subseteq S} A_0^{\otimes Z-F} \otimes B^{\otimes F}.$$

\square

⁵If $S \subseteq S'$ the connecting map is the tensor product of the identity on $(A_0 \oplus B)^{\otimes S}$ with the canonical inclusion $A_0^{\otimes Z-S} = A_0^{\otimes Z-S'} \otimes A_0^{\otimes S'-S} \hookrightarrow A_0^{\otimes Z-S'} \otimes (A_0 \oplus B)^{\otimes S'-S}$.

Theorem 2.8 (Theorem B). *Let A , A_0 and B be C^* -algebras with A and A_0 unital and let $\iota: A_0 \rightarrow A$ be a unital $*$ -homomorphism. Let $\phi \in KK(B, A)$ be an element such that $\iota \oplus \phi \in KK(A_0 \oplus B, A)$ is a KK -equivalence. Then for each countable G -set Z , there is a weak K -equivalence in $KK^G(\mathcal{J}_{A_0, B}^Z, A^{\otimes Z})$. If G moreover satisfies BCC , there is an isomorphism*

$$\begin{aligned} K_* (A^{\otimes Z} \rtimes_r G) &\cong K_* (\mathcal{J}_{A_0, B}^Z \rtimes_r G) \\ &\cong \bigoplus_{[F] \in G \backslash \text{FIN}(Z)} K_* \left((A_0^{\otimes Z-F} \otimes B^{\otimes F}) \rtimes_r G_F \right), \end{aligned}$$

where $\text{FIN}(Z)$ denotes the set of finite subsets of Z and where G_F denotes the stabilizer of F for the action of G on $\text{FIN}(Z)$.

For the proof, we need the following well-known lemma:

Lemma 2.9. *Let Z be a countable G -set and let A_Z be a G - $C_0(Z)$ -algebra. For any G - C^* -algebra D and any choice of representatives z for $[z] \in G \backslash Z$, there is a natural isomorphism*

$$\Psi = (\Psi_z)_{[z]}: KK^G(A_Z, D) \xrightarrow{\cong} \prod_{[z] \in G \backslash Z} KK^{G_z}(A_z, D)$$

where G_z is the stabilizer of z and A_z is the fiber of A_Z at $z \in Z$. The map Ψ is independent of the choice of representatives in the sense that for any $z \in Z$ and $g \in G$, the diagram

$$\begin{array}{ccc} KK^G(A_Z, D) & \xrightarrow{\Psi_z} & KK^{G_z}(A_z, D) \\ & \searrow \Psi_{gz} & \cong \downarrow g \\ & & KK^{G_{gz}}(A_{gz}, D) \end{array}$$

commutes where $g: KK^{G_z}(A_z, D) \rightarrow KK^{G_{gz}}(A_{gz}, D)$ is given by conjugation with g .

Proof. Recall that for a subgroup $H \subseteq G$, and an H - C^* -algebra B , the induced G - C^* -algebra $\text{Ind}_H^G B$ is defined as

$$\text{Ind}_H^G B := \left\{ f \in C_b(G, B) \mid \begin{array}{l} h(f(s)) = f(sh^{-1}), s \in G, h \in H \\ \text{and } sH \mapsto \|f(s)\| \in C_0(G/H) \end{array} \right\}$$

equipped with the left-translation G -action (see [CE01, Section 2] for example). Since Z is discrete, we have a natural isomorphism

$$(2.2) \quad A_Z = \bigoplus_{z \in Z} A_z \cong \bigoplus_{[z] \in G \backslash Z} \text{Ind}_{G_z}^G A_z,$$

where the last isomorphism follows from [CELY17, Theorem 3.4.13]. Now let

$$\Psi = (\Psi_z)_{[z]}: KK^G(A_Z, D) \rightarrow \prod_{[z] \in G \backslash Z} KK^{G_z}(A_z, D)$$

be the map with components

$$\Psi_z: KK^G(A_Z, D) \xrightarrow{\text{Res}_G^{Gz}} KK^{Gz}(A_Z, D) \xrightarrow{\iota_z^*} KK^{Gz}(A_z, D),$$

where $\iota_z: A_z \hookrightarrow A_Z$ is the inclusion. The second part of the lemma follows from the construction of Ψ and the fact that conjugation by any $g \in G$ acts trivially on $KK^G(A_Z, D)$. We show that Ψ is an isomorphism. By (2.2) and [Kas88, Theorem 2.9], we can identify Ψ with the product, taken over all $[z] \in G \backslash Z$, of the compression maps

$$\text{comp}_{G_z}^G: KK^G(\text{Ind}_{G_z}^G A_z, D) \rightarrow KK^{Gz}(A_z, D).$$

These are isomorphisms by [CE01, Proposition 5.14] (see also [MN06, (20)]). \square

Proof of Theorem 2.8. For each orbit $[F]$ in $G \backslash \text{FIN}(Z)$ with stabilizer G_F , we define an element

$$\Phi_F \in KK^{G_F}(A_0^{\otimes Z-F} \otimes B^{\otimes F}, A^{\otimes Z})$$

as the composition

$$A_0^{\otimes Z-F} \otimes B^{\otimes F} \rightarrow A_0^{\otimes Z-F} \otimes (A_0 \oplus B)^{\otimes F} \rightarrow A_0^{\otimes Z-F} \otimes A^{\otimes F} \rightarrow A^{\otimes Z}.$$

Here the first and the third map are the obvious maps and the second map is the tensor product of the identity on $A_0^{\otimes Z-F}$ and the KK^{G_F} -equivalence $(\iota \oplus \phi)^{\otimes F}$ obtained in Lemma 2.4. When F is the empty set, we define Φ_F as the unital map $\iota^{\otimes Z}: A_0^{\otimes Z} \rightarrow A^{\otimes Z}$. By Lemma 2.9 the family $\{\Phi_F : [F] \in G \backslash \text{FIN}(Z)\}$ defines an element $\Phi \in KK^G(J_{A_0, B}^Z, A^{\otimes Z})$ that does not depend on the choice of representatives F for each $[F]$ (since we have $\Phi_{gF} = g(\Phi_F)$ for every $g \in G$).

We show that Φ is a weak K -equivalence. By construction, for any finite subgroup H of G and for any finite H -subset S of Z , the element Φ may be restricted to

$$\Phi_S: \bigoplus_{F \in \text{FIN}(Z), F \subset S} A_0^{\otimes Z-F} \otimes B^{\otimes F} \rightarrow A_0^{\otimes Z-S} \otimes A^{\otimes S}$$

in KK^H . Using Lemma 2.7, Φ_S can be identified with the tensor product of the identity on $A_0^{\otimes Z-S}$ and the KK^H -equivalence $(\iota \oplus \phi)^{\otimes S} \in KK^H((A_0 \oplus B)^{\otimes S}, A^{\otimes S})$ via the isomorphism

$$(A_0 \oplus B)^{\otimes S} \cong \bigoplus_{F \in \text{FIN}(Z), F \subset S} A_0^{\otimes S-F} \otimes B^{\otimes F}.$$

Thus each Φ_S induces an isomorphism

$$K_* \left((A_0^{\otimes Z-S} \otimes (A_0 \oplus B)^{\otimes S}) \rtimes H \right) \cong K_* \left((A_0^{\otimes Z-S} \otimes A^{\otimes S}) \rtimes H \right).$$

By taking the filtered colimit over finite H -subsets $S \subseteq Z$, we see that Φ is a weak K -equivalence.

The K -theory computation follows from the fact that $\mathcal{J}_{A_0, B}^Z$ is a G - $C_0(\text{FIN}(Z))$ -algebra for the discrete G -set $\text{FIN}(Z)$ together with Green's imprimitivity Theorem (e.g., see [CEL13, Remark 3.13] for details). \square

The following special case of Theorem 2.8 will be used to compute examples in Section 4. Its main advantage is that we do not have to construct the element ϕ in order to apply it.

Corollary 2.10. *Let G be a discrete group satisfying BCC. Let Z be a countable G -set and let A be a unital C^* -algebra satisfying the UCT such that the unital inclusion $\mathbb{C} \rightarrow A$ induces a split injection $K_*(\mathbb{C}) \rightarrow K_*(A)$. Denote by $\tilde{K}_*(A)$ its cokernel and let B be any C^* -algebra satisfying the UCT with $K_*(B) \cong \tilde{K}_*(A)$. Then we have*

$$K_*(A^{\otimes Z} \rtimes_r G) \cong K_*(\mathcal{J}_B^Z \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \text{FIN}(Z)} K_*(B^{\otimes F} \rtimes_r G_F).$$

In particular, if $Z = G$ equipped with the left translation, then $K_*(A^{\otimes G} \rtimes_r G)$ only depends on G and $\tilde{K}_*(A)$.

Note that B exists and is uniquely determined up to KK -equivalence (see [Bla98, Corollary 23.10.2]).

Proof. By the UCT, the inclusion map $K_*(B) \cong \tilde{K}_*(A) \rightarrow K_*(A)$ is induced by an element $\phi \in KK(B, A)$. By construction, $\iota \oplus \phi \in KK(\mathbb{C} \oplus B, A)$ is a KK -equivalence, so that we can apply Theorem 2.8. \square

As another special case of Theorem 2.8, we recover

Corollary 2.11 ([CEL13, Example 3.17], [Li19, Proposition 2.4]). *Let G be a discrete group satisfying BCC and let Z be a countable G -set. Then for $n \geq 1$, we have*

$$K_*(C(\{0, 1, \dots, n\}^Z) \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \text{FIN}(Z)} \bigoplus_{[S] \in G_F \backslash (\{1, \dots, n\}^F)} K_*(C_r^*(G_S)).$$

Moreover, if $Z = G$ with the left translation action, we get

$$\begin{aligned} & K_*(C(\{0, 1, \dots, n\}^G) \rtimes_r G) \\ & \cong K_*(C_r^*(G)) \oplus \bigoplus_{[C] \in \mathcal{C}} \bigoplus_{[X] \in N_C \backslash F(C)} \bigoplus_{[S] \in C \backslash \{1, \dots, n\}^{C \cdot X}} K_*(C_r^*(C_S)). \end{aligned}$$

Here \mathcal{C} denotes the set of all conjugacy classes of finite subgroups of G , $F(C)$ the nonempty finite subsets of $C \backslash G$, $N_C = \{g \in G : gCg^{-1} = C\}$ the normalizer of C in G , and $C_S = G_S \cap C$ the stabilizer of S in C .

Proof. Let $A = C(\{0, \dots, n\})$ and $B = C(\{1, \dots, n\})$ and let $\phi: B \rightarrow A$ be the canonical inclusion. The first isomorphism follows from Theorem 2.8. The second isomorphism is obtained by analyzing the orbit structure of $\bigsqcup_{[F] \in \text{FIN}(G)} \{1, \dots, n\}^F$ (see [Li19, Proposition 2.4] for details). \square

3. FINITE-DIMENSIONAL ALGEBRAS

In this section we compute the K -theory of crossed products of the form $A^{\otimes Z} \rtimes_r G$ where $A = \bigoplus_{0 \leq j \leq N} M_{k_j}$ is a finite-dimensional C^* -algebra, Z is a countable G -set, and where G is a group satisfying BCC. This generalizes the case $k_0 = 1$ from [Li19]. We denote by $\gcd(k_0, \dots, k_N)$ the greatest common divisor of k_0, \dots, k_N . We believe that the following theorem is known to experts. In lack of a reference, we give a detailed proof here.

Theorem 3.1. *Let k_0, \dots, k_N be positive integers with $\gcd(k_0, \dots, k_N) = 1$. Then there is a unital $*$ -homomorphism*

$$\phi: \mathbb{C}^{N+1} \rightarrow \bigoplus_{0 \leq j \leq N} M_{k_j}$$

that induces a KK -equivalence. Moreover when $N = 1$, there are exactly two such $*$ -homomorphisms up to unitary equivalence.

The main ingredient for the proof is the following arithmetic fact:

Proposition 3.2. *For any pair of positive integers $k_1, k_2 \in \mathbb{N}$, there is a unique matrix X in $\mathrm{SL}(2, \mathbb{Z})$ with non-negative entries such that*

$$X \begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

where $n = \gcd(k_1, k_2)$. If we allow X to be in $\mathrm{GL}(2, \mathbb{Z})$, then there are exactly two such X : the one X_0 in $\mathrm{SL}(2, \mathbb{Z})$ and $X_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof. Existence: Let $f: \mathbb{N}_+^2 \rightarrow \mathbb{N}_+^2$ be given by

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} k_1 - k_2 \\ k_2 \end{bmatrix}, & \text{if } k_1 > k_2 \\ \begin{bmatrix} k_1 \\ k_2 - k_1 \end{bmatrix}, & \text{if } k_1 < k_2 \\ \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, & \text{if } k_1 = k_2. \end{cases}$$

The Euclidian algorithm precisely says that there exists a $k \in \mathbb{N}$ such that $f^k \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix}$. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then the map f defined

above is given by

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \mapsto \begin{cases} A^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, & \text{if } k_1 > k_2 \\ B^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, & \text{if } k_1 < k_2 \\ \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, & \text{if } k_1 = k_2. \end{cases}$$

Now the above formulation of the Euclidean algorithm gives us integers $a_1, a_2, \dots, a_l, b_1, b_2, \dots, b_l \geq 1$ such that

$$A^{a_1} B^{b_1} A^{a_2} B^{b_2} \dots A^{a_l} B^{b_l} \begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Then $X := A^{a_1} B^{b_1} A^{a_2} B^{b_2} \dots A^{a_l} B^{b_l} \in \mathrm{SL}(2, \mathbb{Z})$ is the required matrix.

Uniqueness: Suppose there are two such matrices X_1 and X_2 . Then, $Y = X_2^{-1} X_1 \in \mathrm{SL}(2, \mathbb{Z})$ satisfies

$$Y \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

From this, we have

$$Y = \begin{bmatrix} a & 1-a \\ a-1 & 2-a \end{bmatrix}$$

for $a \in \mathbb{Z}$. Now, we get

$$X_1 = X_2 \begin{bmatrix} a & 1-a \\ a-1 & 2-a \end{bmatrix},$$

but since both X_1 and X_2 have non-negative entries and since they are non-singular, it is not hard to see that a must be 1. Thus, $X_1 = X_2$. The last assertion is immediate. \square

Remark 3.3. It follows from the proof that the subsemigroup in $\mathrm{SL}(2, \mathbb{Z})$ consisting of matrices with non-negative entries is the free monoid of two generators A and B .

Corollary 3.4. *For any positive integers k_0, \dots, k_N with $\mathrm{gcd}(k_0, \dots, k_N) = n$, there is a matrix X in $\mathrm{SL}(N+1, \mathbb{Z})$ with non-negative entries such that*

$$X \begin{bmatrix} n \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} k_0 \\ \vdots \\ k_N \end{bmatrix}.$$

Proof. We give the proof by induction on N . The case $N = 0$ is clear. Let $N \geq 1$ and let $k_0, \dots, k_N \geq 1$ be positive integers with $\mathrm{gcd}(k_0, \dots, k_N) = n$. Set $l := \mathrm{gcd}(k_0, \dots, k_{N-1})$. By induction, we may assume that there exists a

matrix $\tilde{X} \in \mathrm{SL}(N, \mathbb{Z})$ with non-negative entries such that $\tilde{X} \begin{bmatrix} l \\ \vdots \\ l \end{bmatrix} = \begin{bmatrix} k_0 \\ \vdots \\ k_{N-1} \end{bmatrix}$.

Since $\gcd(l, k_N) = \gcd(l, n) = n$, it follows from Proposition 3.2 that there are matrices $X_0, X_N \in \mathrm{SL}(2, \mathbb{Z})$ with non-negative entries such that $X_0 \begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} l \\ n \end{bmatrix}$ and $X_N \begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} l \\ k_N \end{bmatrix}$. Now for $i \in \{1, \dots, N-1\}$, let Y_i denote the matrix with X_0 as the $(i, i+1)$ -th diagonal block, with ones in all other diagonal entries and with zeros elsewhere, and let $Y_N = \begin{bmatrix} I_{N-1} & 0 \\ 0 & X_N \end{bmatrix}$. Write $Y := \begin{bmatrix} \tilde{X} & 0 \\ 0 & 1 \end{bmatrix}$. Then $X := Y \cdot Y_N \cdots Y_1 \in \mathrm{SL}(N+1, \mathbb{Z})$ has non-negative entries and

$$X \begin{bmatrix} n \\ \vdots \\ n \end{bmatrix} = Y \left(Y_N \cdots Y_1 \begin{bmatrix} n \\ \vdots \\ n \end{bmatrix} \right) = Y \begin{bmatrix} l \\ \vdots \\ l \\ k_N \end{bmatrix} = \begin{bmatrix} k_0 \\ \vdots \\ k_N \end{bmatrix}$$

as desired. \square

Proof of Theorem 3.1. Unital $*$ -homomorphisms from \mathbb{C}^{N+1} to $\bigoplus_{0 \leq j \leq N} M_{k_j}$ are classified up to unitary equivalence by their induced maps on ordered K_0 -groups with units. If we identify the K_0 -groups of \mathbb{C}^{N+1} and $\bigoplus_{0 \leq j \leq N} M_{k_j}$ with \mathbb{Z}^{N+1} , we may represent the induced maps on K_0 -groups by $(N+1)$ -

square matrices with non-negative integer entries that send $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ to $\begin{bmatrix} k_0 \\ \vdots \\ k_N \end{bmatrix}$.

Such a matrix is an isomorphism on K -theory if and only if it is in $\mathrm{GL}(N+1, \mathbb{Z})$. All the assertions now follow from Proposition 3.2 and Corollary 3.4. \square

We are now ready to prove Theorem A. For the proof, we need a result from [KN22]. We formulate it in full generality here since this will be needed in later sections of the paper. Recall that a *supernatural number* is a formal product $\mathfrak{n} = \prod_p p^{n_p}$ of prime powers with $n_p \in \{0, 1, \dots, \infty\}$. We denote by $M_{\mathfrak{n}} := \otimes_p M_p^{\otimes n_p}$ the associated *UHF-algebra*. $M_{\mathfrak{n}}$ and \mathfrak{n} are called *of infinite type* if $n_p \in \{0, \infty\}$ for all p and $n_p \neq 0$ for at least one p . For an abelian group L , we denote by

$$L[1/\mathfrak{n}] := \mathrm{colim}(L \xrightarrow{p_1} L \xrightarrow{p_2} \dots)$$

the *localization at \mathfrak{n}* where (p_1, p_2, \dots) is a sequence of primes containing every p with $n_p \geq 1$ infinitely many times. In other words, $L[1/\mathfrak{n}]$ is the localization at the set of all primes dividing \mathfrak{n} . If \mathfrak{n} is finite, this definition recovers the usual localization at a natural number.

Theorem 3.5 ([KN22, Corollary 2.11]). *Let G be a discrete group satisfying BCC, let Z a countable G -set, let A a G - C^* -algebra and let $M_{\mathfrak{n}}$ a UHF-algebra. Assume that Z is infinite or that \mathfrak{n} is of infinite type. Then the inclusion $A \rightarrow A \otimes M_{\mathfrak{n}}^{\otimes Z}$ induces an isomorphism*

$$K_*(A \rtimes_r G)[1/\mathfrak{n}] \cong K_*((A \otimes M_{\mathfrak{n}}^{\otimes Z}) \rtimes_r G).$$

In particular, the right-hand side is a $\mathbb{Z}[1/\mathfrak{n}]$ -module.

Theorem 3.6 (Theorem A). *Let G be a discrete group satisfying BCC. Let Z be a countably infinite G -set and let $A = \bigoplus_{0 \leq j \leq N} M_{k_j}$ where k_0, \dots, k_N ($N \geq 1$) are positive integers with $\gcd(k_0, \dots, k_N) = n$. Then*

$$\begin{aligned} K_*(A^{\otimes Z} \rtimes_r G) &\cong K_*(C(\{0, \dots, N\}^Z) \rtimes_r G)[1/n] \\ &\cong \bigoplus_{[F] \in G \setminus \text{FIN}(Z)} \bigoplus_{[S] \in G_F \setminus (\{1, \dots, N\}^F)} K_*(C_r^*(G_S))[1/n]. \end{aligned}$$

Proof. Write $B = \bigoplus_{0 \leq j \leq N} M_{k_j/n}$ so that B satisfies the assumptions of Theorem 3.1 and so that $A \cong B \otimes M_n$. By Theorem 3.5, the inclusion $B^{\otimes Z} \hookrightarrow A^{\otimes Z}$ induces an isomorphism

$$K_*(B^{\otimes Z} \rtimes_r G)[1/n] \cong K_*(A^{\otimes Z} \rtimes_r G).$$

By Theorem 3.1 and Corollary 2.5, we furthermore have

$$K_*(B^{\otimes Z} \rtimes_r G) \cong K_*(C(\{0, \dots, N\}^Z) \rtimes_r G).$$

This proves the isomorphism in the first line of the theorem. The isomorphism in the second line follows from Corollary 2.11. \square

4. MORE EXAMPLES

In this section, we compute the K -theory of Bernoulli shifts in more examples. We mostly restrict ourselves to the case $Z = G$ with the left translation action, but some of the results have straightforward generalizations to arbitrary countable G -sets. Recall that for $Z = G$ we write $\text{FIN} = \text{FIN}(G)$ for the collection of finite subsets of G and we put $\mathcal{J}_B := \mathcal{J}_{\mathbb{C}, B}^G$ as in Definition 2.6 for any C^* -algebra B . We start with some easy applications of Corollary 2.5.

Example 4.1. Let \mathcal{A} be the Fibonacci algebra [Dav96, Example III 2.6], which is the filtered colimit of $(M_{m_k} \oplus M_{n_k})_{k \in \mathbb{N}}$ where $m_1 = n_1 = 1$ and where the connecting maps are given by repeated use of the partial embedding matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Since this matrix belongs to $\text{GL}(2, \mathbb{Z})$, the unital embedding $\mathbb{C} \oplus \mathbb{C} \hookrightarrow \mathcal{A}$ is a KK -equivalence.

Let \mathcal{K}^+ be the unitization of the algebra of compact operators on $\ell^2(\mathbb{N})$ and let $p \in \mathcal{K}$ be a rank-1 projection. Then the unital embedding $\mathbb{C} \oplus \mathbb{C} \hookrightarrow \mathcal{K}^+$ given by $(\lambda, \mu) \mapsto \lambda(1 - p) + \mu p$ is a KK -equivalence.

Now let A be either \mathcal{A} or \mathcal{K}^+ and let G be a discrete group satisfying BCC. Then by Corollaries 2.5 and 2.11, we have

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(C_r^*(G_F)).$$

In particular, if G is torsion free, we have

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(\mathbb{C}).$$

Example 4.2 (Theorem C). Consider $A = C(S^1)$. Note that the canonical inclusion $\phi: C_0(\mathbb{R}) \rightarrow C(S^1)$ together with the unital inclusion $\iota: \mathbb{C} \rightarrow C(S^1)$ induces a KK -equivalence $\iota \oplus \phi \in KK(\mathbb{C} \oplus C_0(\mathbb{R}), A)$. By Theorem 2.8, we obtain a weak K -equivalence $\Phi: \mathcal{J}_{C_0(\mathbb{R})} \rightarrow A^{\otimes G}$. We can compute the K -theory of $\mathcal{J}_{C_0(\mathbb{R})} \rtimes_r G$ as

$$K_*(\mathcal{J}_{C_0(\mathbb{R})} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(C_0(\mathbb{R})^{\otimes F} \rtimes_r G_F).$$

In this expression, each nonempty finite subset $F \subseteq G$ can be written as $F = G_F \cdot L_F$ where L_F is a complete set of representatives for $G_F \backslash F$. If the cardinality of L_F is even, the G_F -action on $C_0(\mathbb{R})^{\otimes F} = C_0(\mathbb{R}^F)$ is KK^{G_F} -equivalent to the trivial action on \mathbb{C} by Kasparov's Bott-periodicity theorem (see [Bla98, Theorem 20.3.2]). When the cardinality of L_F is odd, then

$$C_0(\mathbb{R})^{\otimes F} \cong (C_0(\mathbb{R})^{\otimes_{G_F}})^{\otimes |L_F| - 1} \otimes C_0(\mathbb{R})^{\otimes_{G_F}}$$

is KK^{G_F} -equivalent to $C_0(\mathbb{R})^{\otimes_{G_F}} = C_0(\mathbb{R}^{G_F})$. Therefore, we have

$$K_*(C_0(\mathbb{R})^{\otimes F} \rtimes_r G_F) \cong \begin{cases} K_*(C_r^*(G_F)), & \text{if } |G_F \backslash F| \text{ is even} \\ K_*(C_0(\mathbb{R}^{G_F}) \rtimes_r G_F), & \text{if } |G_F \backslash F| \text{ is odd} \end{cases}.$$

In general, for any finite group H and for any orthogonal representation $H \rightarrow O(V)$ on a finite-dimensional Euclidean space V , the K -theory of $C_0(V) \rtimes_r H$ is the well-studied equivariant topological K -theory $K_H^*(V)$ of V (see [Kar02], [EP09]). It is known to be a finitely generated free abelian group with $\text{rank}_{\mathbb{Z}} K_H^*(V)$ equal to the number of conjugacy classes $\langle g \rangle$ of H which are oriented and even/odd respectively (see [Kar02, Theorem 1.8]). Here, a conjugacy class $\langle g \rangle$ of H is oriented if the centralizer C_g of g acts on the g -fixed points V^g of V by oriented automorphisms. The class $\langle g \rangle$ is even/odd if the dimension of V^g is even/odd respectively. For example for any cyclic group $\mathbb{Z}/m\mathbb{Z}$, using [EP09, Example 4.2], we have

$$K_{\mathbb{Z}/m\mathbb{Z}}^*(\mathbb{R}^m) = \begin{cases} \mathbb{Z}^m, & * = 1 \\ 0, & * = 0 \end{cases},$$

for odd $m \geq 1$, and

$$K_{\mathbb{Z}/m\mathbb{Z}}^*(\mathbb{R}^m) = \begin{cases} \mathbb{Z}^{m/2}, & * = 1 \\ 0, & * = 0 \end{cases}$$

for even $m \geq 2$. We summarize our discussion as follows:

Theorem 4.3. *Let G be a discrete group satisfying BCC. We have*

$$\begin{aligned} & K_* (C(S^1)^{\otimes G} \rtimes_r G) \\ & \cong K_* (C_r^*(G)) \oplus \left(\bigoplus_{\substack{[F] \in G \backslash \text{FIN}^\times, \\ |G_F \backslash F| \text{ even}}} K_* (C_r^*(G_F)) \right) \oplus \left(\bigoplus_{\substack{[F] \in G \backslash \text{FIN}^\times, \\ |G_F \backslash F| \text{ odd}}} K_{G_F}^* (\mathbb{R}^{G_F}) \right). \end{aligned}$$

Example 4.4. Let A_θ be the rotation algebra for $\theta \in \mathbb{R}$, the universal C^* -algebra generated by two unitaries u and v satisfying $uv = vue^{2\pi i\theta}$. It is well-known that $K_0(A_\theta) \cong \mathbb{Z}^2$, that $K_1(A_\theta) \cong \mathbb{Z}^2$, and that the unital inclusion $\iota: \mathbb{C} \rightarrow A_\theta$ induces a split injection on K_0 . Since A_θ satisfies the UCT, we can apply Corollary 2.10 to $B = \mathbb{C} \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$ and obtain

$$\begin{aligned} & K_* (A_\theta^{\otimes G} \rtimes_r G) \\ & \cong K_* (C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_* ((\mathbb{C} \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R}))^{\otimes F} \rtimes_r G_F). \end{aligned}$$

Using Theorem 2.8 for $B = C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$, each summand for $F \in \text{FIN}^\times$ may be computed as

$$\begin{aligned} & K_* ((\mathbb{C} \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R}))^{\otimes F} \rtimes_r G_F) \\ & \cong \bigoplus_{[X_1, X_2]} K_* ((C_0(\mathbb{R}^{X_1}) \otimes C_0(\mathbb{R}^{X_2})) \rtimes_r G_{X_{1,2}}), \end{aligned}$$

where the sum is taken over all ordered equivalence classes $[X_1, X_2]$ of pairs of ordered disjoint subsets X_1, X_2 of F modulo the action of the stabilizer G_F of F , and where we set $G_{X_{1,2}} := G_F \cap G_{X_1} \cap G_{X_2}$. As in Example 4.2, the computation of $K_* ((C_0(\mathbb{R}^{X_1}) \otimes C_0(\mathbb{R}^{X_2})) \rtimes_r G_{X_{1,2}})$ either reduces to $K_* (C_r^*(G_{X_{1,2}}))$ or to that of $K_* (C_0(\mathbb{R}^{G_{X_{1,2}}}) \rtimes_r G_{X_{1,2}}) = K_{G_{X_{1,2}}}^* (\mathbb{R}^{G_{X_{1,2}}})$ in general. We summarize the discussion as follows.

Theorem 4.5. *Let G be a discrete group satisfying BCC and let $\theta \in \mathbb{R}$. We have*

$$K_* (A_\theta^{\otimes G} \rtimes_r G) \cong K_* (C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_F$$

where for all $F \in \text{FIN}^\times$ we set

$$K_F := \left(\bigoplus_{\substack{[X_1, X_2], X_1 \sqcup X_2 \subset F, \\ |G_{X_{1,2}} \backslash (X_1 \sqcup X_2)| \text{ even}}} K_* (C_r^*(G_{X_{1,2}})) \right) \oplus \left(\bigoplus_{\substack{[X_1, X_2], X_1 \sqcup X_2 \subset F, \\ |G_{X_{1,2}} \backslash (X_1 \sqcup X_2)| \text{ odd}}} K_{G_{X_{1,2}}}^* (\mathbb{R}^{G_{X_{1,2}}}) \right).$$

Cuntz algebras. For $n \in \{2, 3, \dots, \infty\}$ we denote by \mathcal{O}_n the Cuntz algebra on n generators.

Example 4.6. [Sza18, Corollary 6.9] Let G be a discrete group satisfying BCC. Then the unital inclusion $\mathbb{C} \rightarrow \mathcal{O}_\infty$ induces an isomorphism

$$K_*(C_r^*(G)) \cong K_*(\mathcal{O}_\infty^{\otimes G} \rtimes_r G).$$

Proof. Combine Corollary 2.5 and the fact that the unital inclusion $\mathbb{C} \rightarrow \mathcal{O}_\infty$ is a KK -equivalence. \square

Example 4.7. Let G be a discrete group satisfying BCC. Then we have

$$K_*(\mathcal{O}_2^{\otimes G} \rtimes_r G) = 0.$$

Proof. Combine Lemma 2.4, Theorem 2.3 and the fact that \mathcal{O}_2 is KK -equivalent to 0. \square

Example 4.8. Let G be a discrete group satisfying BCC. Let $A = \mathbb{C} \oplus \mathcal{O}_n$ for $n \geq 3$. By Theorem 2.8, we have

$$K_*(A^{\otimes G} \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \setminus \text{FIN}^\times} K_*(\mathcal{O}_n^{\otimes F} \rtimes_r G_F).$$

Each summand for $F = G_F \cdot L_F$ becomes $K_*\left((\mathcal{O}_n^{\otimes L_F})^{\otimes G_F} \rtimes_r G_F\right)$. It follows from the UCT and an inductive application of the Künneth theorem that $\mathcal{O}_n^{\otimes L_F}$ is KK -equivalent to $\mathcal{O}_n \otimes (C_0(\mathbb{R}) \oplus \mathbb{C})^{|L_F|-1}$. Thus, by Lemma 2.4, we may express $K_*(\mathcal{O}_n^{\otimes F} \rtimes_r G_F)$ explicitly in terms of

$$K_*(\mathcal{O}_n^{\otimes H} \rtimes_r H) \text{ and } K_*((C_0(\mathbb{R}) \otimes \mathcal{O}_n)^{\otimes H} \rtimes_r H)$$

for finite subgroups H of G_F . These groups for $H = \mathbb{Z}/2$ are nicely computed in [Izu19]. According to [Izu19], we have

$$K_*\left(\mathcal{O}_{n+1}^{\otimes \mathbb{Z}/2} \rtimes_r \mathbb{Z}/2\right) = \begin{cases} \mathbb{Z}/n \oplus \mathbb{Z}/n, & * = 0 \\ 0, & * = 1 \end{cases}$$

for odd n ,

$$K_*\left(\mathcal{O}_{n+1}^{\otimes \mathbb{Z}/2} \rtimes_r \mathbb{Z}/2\right) = \begin{cases} \mathbb{Z}/\frac{n}{2} \oplus \mathbb{Z}/2n, & * = 0, \\ 0 & * = 1 \end{cases}$$

for even n , and

$$K_*\left((C_0(\mathbb{R}) \otimes \mathcal{O}_{n+1})^{\otimes \mathbb{Z}/2} \rtimes_r \mathbb{Z}/2\right) = \begin{cases} 0, & * = 0 \\ \mathbb{Z}/n \oplus \mathbb{Z}/n, & * = 1 \end{cases}$$

for odd n ,

$$K_*\left((C_0(\mathbb{R}) \otimes \mathcal{O}_{n+1})^{\otimes \mathbb{Z}/2} \rtimes_r \mathbb{Z}/2\right) = \begin{cases} 0, & * = 0 \\ \mathbb{Z}/\frac{n}{2} \oplus \mathbb{Z}/2n, & * = 1 \end{cases}$$

for even n . Using similar methods as in [Izu19], we can compute the case $H = \mathbb{Z}/3$. We omit the very technical computations. For general H , the computations become increasingly complicated as the order of H increases.

Question 4.9. Is $K_*(\mathcal{O}_n^{\otimes H} \rtimes_r H)$ computable for all finite groups H or at least for all cyclic groups?

Although we do not know how to compute $K_*(\mathcal{O}_n^{\otimes H} \rtimes_r H)$ in general, we can say something about its structure. The following Corollary is a combination of Theorem 3.5 and Corollary 2.5.

Corollary 4.10. *Let G be a discrete group satisfying BCC, Z a countable G -set, and A a C^* -algebra. Let M_n be a UHF-algebra of infinite type such that $A \otimes M_n$ is KK-equivalent to zero (for instance $A = \mathcal{O}_{n+1}$ and $M_n = M_{n^\infty}$ for some $n \geq 2$). Then*

$$K_*(A^{\otimes Z} \rtimes_r G)[1/n] \cong 0.$$

For $A = \mathcal{O}_{n+1}$ and a finite group H , we can say a bit more:

Proposition 4.11. *Let H be a finite group, let Z be a finite H -set and let $n \geq 2$. Then $K_*(\mathcal{O}_{n+1}^{\otimes Z} \rtimes_r H)$ is a finitely generated abelian group L such that $L[1/n] = 0$. That is, any element in L is annihilated by n^k for some k . In particular, if $n = p$ is prime, then L is isomorphic to the direct sum of finitely many p -groups $\mathbb{Z}/p^k\mathbb{Z}$.*

Proof. The method used in [Izu19], at an abstract level, tells us that $K_*(\mathcal{O}_{n+1}^{\otimes Z} \rtimes_r H)$ is finitely generated. To see this, let \mathcal{T}_{n+1} be the universal C^* -algebra generated by isometries $s_1 \cdots, s_{n+1}$ with mutually orthogonal range projections $q_i = s_i s_i^*$ and we let $p = 1 - \sum_{1 \leq j \leq n+1} q_j$. The ideal generated by p is isomorphic to \mathcal{K} and we have the following exact sequences

$$(4.1) \quad 0 \longrightarrow I_1 \rtimes H \longrightarrow \mathcal{T}_{n+1}^{\otimes Z} \rtimes H \longrightarrow \mathcal{O}_{n+1}^{\otimes Z} \rtimes H \longrightarrow 0,$$

$$(4.2) \quad 0 \longrightarrow I_{m+1} \rtimes H \longrightarrow I_m \rtimes H \longrightarrow (I_m/I_{m+1}) \rtimes H \longrightarrow 0,$$

where for $1 \leq m \leq |Z|$, an H -ideal I_m of $\mathcal{T}_{n+1}^{\otimes Z}$ is defined as the ideal generated by $\mathcal{K}^{\otimes F} \otimes \mathcal{T}_{n+1}^{\otimes Z-F}$ for subsets F of H with $|F| = m$. In particular $I_{|Z|} = \mathcal{K}^{\otimes Z}$. By Lemma 2.4, or by the stabilization theorem, $K_*(\mathcal{K}^{\otimes Z} \rtimes H) \cong K_*(C^*(H))$. Moreover, for each $1 \leq m < |Z|$, the quotient I_m/I_{m+1} is the direct sum of $\mathcal{K}^{\otimes F} \otimes \mathcal{O}_{n+1}^{\otimes Z-F}$ over the subsets F of Z with $|F| = m$. Thus, $I_m/I_{m+1} \rtimes H$ is Morita-equivalent to the direct sum of $\mathcal{K}^{\otimes F} \otimes \mathcal{O}_{n+1}^{\otimes Z-F} \rtimes H_F$ over $[F] \in H \setminus \{F \subset Z \mid |F| = m\}$ where H_F is the stabilizer of F in H . By induction on the size $|Z|$ of Z (for all finite groups H at the same time), $K_*(\mathcal{K}^{\otimes F} \otimes \mathcal{O}_{n+1}^{\otimes Z-F} \rtimes H_F) \cong K_*(\mathcal{O}_{n+1}^{\otimes Z-F} \rtimes H_F)$ is finitely generated. Using the six-term exact sequences on K -theory associated to (4.2), we see that $K_*(I_m \rtimes H)$ are all finitely-generated. By Lemma 2.4,

$K_*\left(\mathcal{T}_{n+1}^{\otimes Z} \rtimes H\right) \cong K_*(C^*(H))$ since \mathcal{T}_{n+1} is KK -equivalent to \mathbb{C} by [Pim97]. Using the six-term exact sequence on K -theory associated to (4.1), we now see that $K_*\left(\mathcal{O}_{n+1}^{\otimes Z} \rtimes H\right)$ is finitely-generated.

On the other hand, we have

$$K_*\left(\left(\mathcal{O}_{n+1}^{\otimes Z} \otimes M_{n^\infty}^{\otimes Z}\right) \rtimes_r H\right) \cong K_*\left(\mathcal{O}_{n+1}^{\otimes Z} \rtimes_r H\right) [1/n]$$

by Theorem 3.5. The assertion follows from Lemma 2.4 since $\mathcal{O}_{n+1} \otimes M_{n^\infty}$ is KK -equivalent to 0 by the UCT. \square

For an infinite G -set Z , Corollary 4.10 for $A = \mathcal{O}_{n+1}$ may also be deduced from the combination of Theorem 3.5 and the following result (for $A = \mathcal{O}_{n+1} \otimes M_n$).

Theorem 4.12. *Let G be a discrete group satisfying BCC and let Z be a countably infinite G -set. Let A be any unital C^* -algebra such that $[1_{A^{\otimes r}}] = 0 \in K_0(A^{\otimes r})$ for some $r \geq 1$. Then we have*

$$K_*(A^{\otimes Z} \rtimes_r G) \cong 0.$$

Proof. By Theorem 2.3 it is enough to show $K_*(A^{\otimes Z} \rtimes_r H) \cong 0$ for all finite subgroups H of G . Since Z is infinite, it contains infinitely many orbits of type H/H_0 for some fixed subgroup $H_0 \subset H$. Denote Z_0 the union of orbits of type H/H_0 . Let L be a (necessarily infinite) complete set of representatives for $H \setminus Z_0$. We have

$$A^{\otimes Z_0} \cong (A^{\otimes L})^{\otimes H/H_0}.$$

By assumption, the unital inclusion $\mathbb{C} \rightarrow A^{\otimes r}$ induces the zero element in $K_0(A^{\otimes r}) = KK(\mathbb{C}, A^{\otimes r})$. In particular, the maps

$$A^{\otimes N} \rightarrow A^{\otimes N+r},$$

induce the zero map in KK -theory since, on the level of KK -theory, they are given by the exterior Kasparov product with $[0] = [1] \in KK(\mathbb{C}, A^{\otimes r})$. It follows from Lemma 2.4, that the unital inclusions

$$(A^{\otimes N})^{\otimes H/H_0} \rightarrow (A^{\otimes N+r})^{\otimes H/H_0}$$

are zero in KK^H . Writing $A^{\otimes Z} = A^{\otimes Z_0} \otimes A^{\otimes Z-Z_0}$, we have

$$K_*\left(A^{\otimes Z} \rtimes_r H\right) \cong \varinjlim_N K_*\left(\left((A^{\otimes N})^{\otimes H/H_0} \otimes A^{\otimes Z-Z_0}\right) \rtimes_r H\right).$$

The right-hand side is zero by the preceding argument. \square

Remark 4.13. Theorem 4.12 can be applied whenever A is Morita equivalent to \mathcal{O}_{n^r+1} such that $[1_A] = n \in K_0(A) \cong \mathbb{Z}/n^r$, or whenever $K_0(A) \cong \mathbb{Q}/\mathbb{Z}$. In the second case, this is due to the fact that $[1]^{\otimes 2} \in K_0(A^{\otimes 2})$ is in the image of $K_0(A) \otimes_{\mathbb{Z}} K_0(A) \cong \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

Wreath products. For discrete groups G and H , the wreath product $H \wr G$ is defined as the semi-direct product $(\bigoplus_{g \in G} H) \rtimes G$, where G acts by left translation. We have a canonical isomorphism

$$C_r^*(H \wr G) \cong C_r^*(H)^{\otimes G} \rtimes_r G.$$

An application of Corollary 2.10 gives the following result which generalizes [Li19].

Theorem 4.14 (Theorem D). *Let G be a group satisfying BCC and let H be a group for which $C_r^*(H)$ satisfies the UCT and for which the unital inclusion $\mathbb{C} \rightarrow C_r^*(H)$ induces a split injection $K_*(\mathbb{C}) \rightarrow K_*(C_r^*(H))$. Denote its cokernel by $\tilde{K}_*(C_r^*(H))$. Then we have*

$$K_*(C_r^*(H \wr G)) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(B^{\otimes F} \rtimes_r G_F)$$

where B is any C^* -algebra satisfying UCT with $K_*(B) \cong \tilde{K}_*(C_r^*(H))$. In particular, if G is torsion-free, we have

$$K_*(C_r^*(H \wr G)) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(B^{\otimes F}).$$

Remark 4.15. The assumptions on H in the above theorem are not very restrictive: If H is a discrete group for which the Baum–Connes assembly map is split-injective (for instance if H satisfies BCC, or if H is exact by [Hig00, Theorem 1.1]), then the unital inclusion $\mathbb{C} \rightarrow C_r^*(H)$ induces a split injection $K_*(\mathbb{C}) \rightarrow K_*(C_r^*(H))$ since the corresponding map $K_*^{\text{top}}(\{e\}) \rightarrow K_*^{\text{top}}(H)$ always splits.

On the other hand, it follows from Tu’s [Tu99, Proposition 10.7] that $C_r^*(H)$ satisfies the UCT for every a- T -menable (in particular every amenable) group H , or, more generally, if H satisfies the strong Baum–Connes conjecture in the sense of [MN06].

Example 4.16. Let G be a group which satisfies BCC and consider the wreath product $\mathbb{F}_n \wr G$ with \mathbb{F}_n the free group in n generators. Since \mathbb{F}_n is known to be a- T -menable it follows that Theorem 4.14 applies. Since $K_0(C_r^*(\mathbb{F}_n)) = \mathbb{Z}$ and $K_1(C_r^*(\mathbb{F}_n)) = \mathbb{Z}^n$ we may choose $B = \bigoplus_{i=1}^n C_0(\mathbb{R})$ so that Theorem 4.14 implies

$$K_*(C_r^*(\mathbb{F}_n \wr G)) \cong K_*(C_r^*(G)) \oplus \bigoplus_{F \in G \backslash \text{FIN}^\times} K_*(B^{\otimes F} \rtimes_r G_F),$$

where each summand $K_*(B^{\otimes F} \rtimes_r G_F)$ decomposes into a direct sum of equivariant K -theory groups of the form $K_H^*(V)$ for certain subgroups H of G_F and certain euclidean H -spaces V . A more precise analysis can be done, at least for $n = 2$, along the lines of Example 4.4.

In particular, if G is torsion free, we get

$$K_*(C_r^*(\mathbb{F}_n \wr G)) \cong K_*(C_r^*(G)) \oplus \bigoplus_{[F] \in G \backslash \text{FIN}^\times} K_*(B^{\otimes F})$$

with $B^{\otimes F} \cong \bigoplus_{i=1}^{n^{|F|}} C_0(\mathbb{R}^{|F|})$. Therefore each G -orbit of a nonempty finite set $F \subseteq G$ provides $n^{|F|}$ copies of \mathbb{Z} as direct summands of K_0 if $|F|$ is even and of K_1 if $|F|$ is odd.

We close this section with

Corollary 4.17. *Suppose that G and H are as in Theorem 4.14, such that the K -theory of both $C_r^*(G)$ and $C_r^*(H)$ is free abelian. Then the K -theory of $C_r^*(H \wr G)$ is free abelian as well.*

Proof. Note that $\tilde{K}_*(C_r^*(H))$ is free abelian as it is the direct summand of the free abelian group $K_*(C_r^*(H))$. Therefore, in Theorem 4.14, B can be taken as the direct sum of, possibly infinitely many, \mathbb{C} and $C_0(\mathbb{R})$. The assertion follows from the fact that equivariant K -theory $K_{G_0}^*(\mathbb{R}^{G_0})$ (more generally $K_{G_0}^*(V)$ for any G_0 -Euclidean space V) is a (finitely-generated) free abelian group for any finite group G_0 by [Kar02] (or [EP09]). \square

5. AF-ALGEBRAS

Let $A = \varinjlim_n A_n$ be a unital AF-algebra (with unital connecting maps) and let G be a discrete group satisfying BCC. If Z is a countable G -set, then in principle, we can try to compute

$$K_*(A^{\otimes Z} \rtimes_r G) \cong \varinjlim_n K_*(A_n^{\otimes Z} \rtimes_r G)$$

using the decomposition from Theorem 3.6. In general, such a computation can be challenging, even in relatively simple cases like $A = M_2 \oplus M_{3\infty}$. Instead of trying to compute the connecting maps, we now provide an abstract approach to calculating $K_*(A^{\otimes Z} \rtimes_r G)$ for an arbitrary unital AF-algebra A , a discrete group G satisfying BCC and a countable proper G -set Z (for example $Z = G$ with the left translation action).

We first need some preparation. Let \mathcal{F} be a family of subgroups, i.e. a non-empty set of subgroups of G closed under taking conjugates and subgroups. The orbit category $\text{Or}_{\mathcal{F}}(G)$ has as objects homogeneous G -spaces G/H for each $H \in \mathcal{F}$ and as morphisms G -maps (see [DL98, Definition 1.1]). We will mainly use the family \mathcal{FIN} of finite subgroups. For any G - C^* -algebra A , we have a functor from $\text{Or}_{\mathcal{F}}(G)$ to the category of graded abelian groups that sends G/H to $K_*(A \rtimes_r H)$ and a morphism $G/H_0 \rightarrow G/H_1$ given by $H_0 \mapsto gH_1$ (so that $H_0 \subseteq gH_1g^{-1}$) to the map $K_*(A \rtimes_r H_0) \rightarrow K_*(A \rtimes_r H_1)$ induced by the composition

$$A \rtimes_r H_0 \hookrightarrow A \rtimes_r gH_1g^{-1} \cong A \rtimes_r H_1,$$

where the second map is given by conjugation with g^{-1} inside $A \rtimes_r G$.⁶ We denote by

$$\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H)$$

the colimit of the functor $G/H \mapsto K_*(A \rtimes_r H)$ from $\operatorname{Or}_{\mathcal{F}}(G)$ to the category of graded abelian groups. The inclusions $A \rtimes_r H \rightarrow A \rtimes_r G$ induce a natural homomorphism

$$(5.1) \quad \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H) \rightarrow K_*(A \rtimes_r G).$$

Remark 5.1. The map (5.1) should not be confused with the assembly map

$$\operatorname{asmb}_{\mathcal{F}}: H_*^G(E_{\mathcal{F}}G, \mathbb{K}_A^{\operatorname{top}}) \rightarrow K_*(A \rtimes_r G),$$

which corresponds to taking the homotopy colimit at the level of K -theory spectra instead of taking the ordinary colimit at the level of K -theory groups, see [DL98, Section 5.1]. By the involved universal properties, there is a natural commuting diagram

$$\begin{array}{ccc} \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H) & \longrightarrow & K_*(A \rtimes_r G) , \\ \downarrow & \nearrow & \\ H_*^G(E_{\mathcal{F}}G, \mathbb{K}_A^{\operatorname{top}}) & & \end{array}$$

but the vertical map is neither injective nor surjective in general. One can think of the map (5.1) as the best approximation of $K_*(A \rtimes_r G)$ using 0-dimensional G - \mathcal{F} -CW-complexes. Likewise, the best approximation by r -dimensional G - \mathcal{F} -CW-complexes can be defined by taking the colimit of $H_*^G(X, \mathbb{K}_A^{\operatorname{top}})$ over the category of r -dimensional G - \mathcal{F} -CW-complexes.

Let us give two examples of how to compute $\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)} K_*(A \rtimes_r H)$ when the structure or $\operatorname{Or}_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)$ is understood. Recall that the *coinvariants* of a G -module L are given by

$$L_G := \operatorname{colim}_G L \cong L / \langle x - gx \mid g \in G, x \in L \rangle.$$

Here the group G is considered as a category with one object with morphisms given by the elements of G , and L is considered as a functor from G to the category of (graded) abelian groups.

Example 5.2. If G is torsion-free, then $\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)} K_*(A \rtimes_r H)$ can be identified with the coinvariants $K_*(A)_G$.

⁶This is a well-defined functor since different choices for g give the same map on K -theory. On the other hand, $G/H \mapsto A \rtimes_r H$ does not define a functor. This is why it requires more care to upgrade this to a functor taking values in the category of spectra, see [DL98, Section 2].

Example 5.3. If G has only one non-trivial conjugacy class $[H]$ of finite subgroups and if $N_G(H)$ denotes the normalizer of H in G , then the colimit $\operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{FLN}}(G)} K_*(A \rtimes_r H)$ is given by the pushout

$$\begin{array}{ccc} K_*(A) & \longrightarrow & K_*(A \rtimes_r H)_{N_G(H)} \\ \downarrow & & \downarrow \\ K_*(A)_G & \longrightarrow & \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{FLN}}(G)} K_*(A \rtimes_r H). \end{array}$$

For a G -module M and a set S of positive integers, we denote by

$$M[S^{-1}] := \operatorname{colim}_{\rightarrow} (M \xrightarrow{\cdot s_1} M \xrightarrow{\cdot s_2} \dots) \cong M \otimes_{\mathbb{Z}} \mathbb{Z}[S^{-1}]$$

its localization at S , where (s_1, s_2, \dots) is a sequence containing every element of S infinitely many times. Note that localization at S commutes with taking coinvariants since both constructions are colimits.

Theorem 5.4 (Theorem F). *Let A be a unital C^* -algebra of the form $A = \operatorname{colim}_{\rightarrow n} M_{k_n}(A_n)$ (with unital connecting maps) where $(k_n)_{n \in \mathbb{N}}$ is a sequence of positive integers and where each A_n is a unital C^* -algebra satisfying the assumptions of Theorem 2.8 for the unital inclusion $\mathbb{C} \hookrightarrow A_n$. Let G be an infinite discrete group satisfying BCC, let Z be a countable proper G -set, and let $S \subseteq \mathbb{Z}$ be the set of all positive integers n such that $[1_A] \in K_0(A)$ is divisible by n . Then, the natural inclusions*

$$(5.2) \quad \begin{array}{l} C_r^*(H) \rightarrow C_r^*(G), \quad C_r^*(H) \rightarrow A^{\otimes Z} \rtimes_r H, \\ C_r^*(G) \rightarrow A^{\otimes Z} \rtimes_r G, \quad A^{\otimes Z} \rtimes_r H \rightarrow A^{\otimes Z} \rtimes_r G, \end{array}$$

induce the following pushout diagram

$$\begin{array}{ccc} \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{FLN}}(G)} K_*(C_r^*(H))[S^{-1}] & \longrightarrow & \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{FLN}}(G)} K_*(A^{\otimes Z} \rtimes_r H) \\ \downarrow & & \downarrow \\ K_*(C_r^*(G))[S^{-1}] & \longrightarrow & K_*(A^{\otimes Z} \rtimes_r G). \end{array}$$

In particular, if G is torsion-free, this pushout diagram reads

$$\tilde{K}_*(C_r^*(G))[S^{-1}] \oplus K_*(A^{\otimes Z})_G \cong K_*(A^{\otimes Z} \rtimes_r G),$$

where $\tilde{K}_*(C_r^*(G))$ denotes the cokernel of $K_*(\mathbb{C}) \rightarrow K_*(C_r^*(G))$ induced from the unital inclusion $\mathbb{C} \hookrightarrow C_r^*(G)$.

Remark 5.5. By Theorem 3.1, Theorem 5.4 applies to all unital AF-algebras. More generally, Theorem 5.4 applies whenever A is of the form $\operatorname{colim}_{\rightarrow n} M_{k_n}(A_n)$ (with unital connecting maps) where each A_n is one of the following examples:

- (1) The unitization B^+ of a C^* -algebra B , e.g. $\mathbb{C} \oplus B$ for unital B ;

- (2) A C^* -algebra of the form $\bigoplus_{1 \leq j \leq N} C(X_j) \otimes M_{k_j}$ for nonempty compact metric spaces X_j and $\gcd(k_1, \dots, k_N) = 1$ (use Theorem 3.1);
- (3) A reduced group C^* -algebra $C_r^*(\Gamma)$ of a countable group Γ that satisfies the UCT and such that the map $K_*(\mathbb{C}) \rightarrow K_*(C_r^*(\Gamma))$ is a split-injection (see Remark 4.15 and Theorem 4.14).

It would be interesting to know if Theorem 5.4 also holds for unital ASH-algebras, i.e. when A is a filtered colimit of algebras of the form $p(C(X) \otimes M_n)p$ for a projection $p \in C(X) \otimes M_n$.

For the proof of Theorem 5.4, we need the following Lemma.

Lemma 5.6. *Let G be any discrete group, let \mathcal{F} be a family of subgroups of G , let $H_i \in \mathcal{F}$, and let A_i be H_i - C^* -algebras for $i \in I$. Denote by $A = \bigoplus_{i \in I} \text{Ind}_{H_i}^G(A_i)$ the direct sum of the induced G - C^* -algebras. Then, the natural map*

$$\text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H) \rightarrow K_*(A \rtimes_r G)$$

is an isomorphism.

Proof. By additivity we may assume $A = \text{Ind}_{H_0}^G A_0$ for $H_0 \in \mathcal{F}$ and an H_0 - C^* -algebra A_0 . Write

$$A = \bigoplus_{[z] \in G/H_0} A_z$$

where A_z is the fiber of A at $[z] \in G/H_0$. Note that A_z is a zH_0z^{-1} - C^* -algebra, in particular, A_e is the H_0 - C^* -algebra A_0 . By [CELY17, Proposition 2.6.8], the inclusion $A_e \rightarrow A$ induces an isomorphism

$$K_*(A_e \rtimes_r H_0) \cong K_*(A \rtimes_r G).$$

This isomorphism factors through $\text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H)$ as

$$K_*(A_e \rtimes_r H_0) \rightarrow K_*(A \rtimes_r H_0) \rightarrow \text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H) \rightarrow K_*(A \rtimes_r G).$$

Our claim follows once we show that the map

$$(5.3) \quad K_*(A_e \rtimes_r H_0) \rightarrow \text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H)$$

is surjective. Fix a subgroup $H \in \mathcal{F}$. Decomposing G/H_0 into H -orbits and using the decomposition of (2.2), we obtain a decomposition

$$(5.4) \quad K_*(A \rtimes_r H) \cong \bigoplus_{[z] \in H \backslash G/H_0} K_*(A_z \rtimes_r H_z),$$

where $H_z := H \cap zH_0z^{-1} \in \mathcal{F}$. In (5.4), the inclusions $K_*(A_z \rtimes_r H_z) \subseteq K_*(A \rtimes_r H)$ are induced by the natural inclusions $A_z \rtimes_r H_z \subseteq A \rtimes_r H$. In the colimit $\text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H)$, the summand

$$K_*(A_z \rtimes_r H_z) \subseteq K_*(A \rtimes_r H)$$

gets via conjugation with z^{-1} identified with the summand

$$K_*(A_e \rtimes_r H'_z) \subset K_*(A \rtimes_r H'_z)$$

corresponding to $[e] \in G/H_0$ and $H'_z = z^{-1}Hz \cap H_0$. But the elements in $\text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H)$ coming from $K_*(A_e \rtimes_r H'_z)$ are certainly in the image of the map in (5.3) since

$$K_*(A_e \rtimes_r H'_z) \rightarrow \text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_*(A \rtimes_r H)$$

factors through the K -theory map of the inclusion $A_e \rtimes_r H'_z \subseteq A_e \rtimes_r H_0$. \square

As a direct consequence of Lemma 5.6, we get

Corollary 5.7. *Let G be any discrete group, let Z be a countable G -set, and let A_0 and B be C^* -algebras with A_0 unital. Let $\dot{\mathcal{J}}_{A_0, B}^Z$ be the G - C^* -algebra*

$$\dot{\mathcal{J}}_{A_0, B}^Z = \bigoplus_{F \in \text{FIN}^\times(Z)} A_0^{\otimes Z-F} \otimes B^{\otimes F},$$

so that $\mathcal{J}_{A_0, B}^Z = A_0^{\otimes Z} \oplus \dot{\mathcal{J}}_{A_0, B}^Z$ as in Definition 2.6. Then the natural map

$$\text{colim}_{G/H \in \text{Or}_{\mathcal{F}}(G)} K_* \left(\dot{\mathcal{J}}_{A_0, B}^Z \rtimes_r H \right) \rightarrow K_* \left(\dot{\mathcal{J}}_{A_0, B}^Z \rtimes_r G \right)$$

is an isomorphism for any family \mathcal{F} of subgroups of G containing all the stabilizers of the G -action on $\text{FIN}^\times(Z)$. This applies in particular when Z is a proper G -set and $\mathcal{F} = \mathcal{FLN}$ is the family of finite subgroups. \square

Proof of Theorem 5.4. We prove Theorem 5.4 by considering three successively more general cases:

Case 1. A satisfies the assumptions of Theorem 2.8 for the unital inclusion $\iota: \mathbb{C} \rightarrow A$.

By assumption, there is a C^* -algebra B and an element $\phi \in KK(B, A)$ such that $\iota \oplus \phi \in KK(\mathbb{C} \oplus B, A)$ is a KK -equivalence. Using the weak K -equivalence of \mathcal{J}_B^Z and $A^{\otimes Z}$ constructed in Theorem 2.8, we may identify the maps in (5.2) with the natural maps

$$(5.5) \quad C_r^*(G) \rightarrow \mathcal{J}_B^Z \rtimes_r G, \quad \mathcal{J}_B^Z \rtimes_r H \rightarrow \mathcal{J}_B^Z \rtimes_r G,$$

where the first map is induced from the (non-unital) inclusion $\mathbb{C} \hookrightarrow \mathcal{J}_B^Z$ corresponding to $F = \emptyset$. We may therefore replace $A^{\otimes Z}$ by \mathcal{J}_B^Z throughout the proof. The corresponding statement for \mathcal{J}_B^Z then follows from Corollary 5.7 since by the decomposition $\mathcal{J}_B^Z = \mathbb{C} \oplus \dot{\mathcal{J}}_B^Z$, we have

$$K_*(\mathcal{J}_B^Z \rtimes_r G) \cong K_*(C_r^*(G)) \oplus \text{colim}_{G/H \in \text{Or}_{\mathcal{FLN}}(G)} K_*(\dot{\mathcal{J}}_B^Z \rtimes_r H)$$

and consequently a pushout diagram

$$\begin{array}{ccc} \text{colim}_{G/H \in \text{Or}_{\mathcal{FLN}}(G)} K_*(C_r^*(H)) & \longrightarrow & \text{colim}_{G/H \in \text{Or}_{\mathcal{FLN}}(G)} K_*(\mathcal{J}_B^Z \rtimes_r H) \\ \downarrow & & \downarrow \\ K_*(C_r^*(G)) & \longrightarrow & K_*(\mathcal{J}_B^Z \rtimes_r G). \end{array}$$

In the torsion free case this becomes

$$K_*(\mathcal{J}_B^Z \rtimes_r G) \cong K_*(C_r^*(G)) \oplus_{K_*(\mathbb{C})} K_*(\mathcal{J}_B^Z)_G \cong \tilde{K}_*(C_r^*(G)) \oplus K_*(\mathcal{J}_B^Z)_G.$$

Case 2. $A = M_n \otimes D$ where D is as in Case 1.

By applying Case 1 and localizing at n , we see that the maps in (5.2) (for D instead of A) induce a pushout diagram

$$\begin{array}{ccc} \operatorname{colim}_{G/H \in \operatorname{Or}_{FLN}(G)} K_*(C_r^*(H))[1/n] & \longrightarrow & \operatorname{colim}_{G/H \in \operatorname{Or}_{FLN}(G)} K_*(D^{\otimes Z} \rtimes_r H)[1/n] \\ \downarrow & & \downarrow \\ K_*(C_r^*(G))[1/n] & \longrightarrow & K_*(D^{\otimes Z} \rtimes_r G)[1/n], \end{array}$$

and in the torsion-free case an isomorphism

$$\tilde{K}_*(C_r^*(G))[1/n] \oplus K_*(D^{\otimes Z})[1/n]_G \cong K_*(D^{\otimes Z} \rtimes_r G)[1/n].$$

Here we have used that localization at n commutes with taking colimits. By Theorem 3.5, the unital inclusion $D \rightarrow M_n \otimes D = A$ induces an isomorphism $K_*(D^{\otimes Z} \rtimes_r H)[1/n] \cong K_*(A^{\otimes Z} \rtimes_r H)$ for every subgroup H of G . This finishes the proof of Case 2.

Case 3. $A = \operatorname{colim}_{\rightarrow k} A_k$ where each A_k is as in Case 2.

For each k , denote by $S_k \subseteq S$ the set of positive integers n such that the unit $[1] \in K_0(A_k)$ is divisible by n . Note that we have $S = \bigcup_k S_k$. By Case 2, the conclusion of the theorem holds if we replace A by A_k and S by S_k . Now the general case follows by taking the filtered colimit along k . \square

6. RATIONAL AND k -ADIC COMPUTATIONS

In this section we give systematic tools to compute the K -theory of non-commutative Bernoulli shifts up to localizing at a supernatural number \mathfrak{n} (i.e. at the set of prime factors of \mathfrak{n}). The results apply to unital C^* -algebras A for which the inclusion $\iota: \mathbb{C} \rightarrow A$ does not induce a split injection $K_0(\mathbb{C}) \rightarrow K_0(A)$ integrally, but a split injection

$$(6.1) \quad K_0(\iota)[1/\mathfrak{n}]: K_0(\mathbb{C})[1/\mathfrak{n}] \hookrightarrow K_0(A)[1/\mathfrak{n}]$$

after localizing at a supernatural number \mathfrak{n} . Important special cases are $\mathfrak{n} = k^\infty$ for $k \in \mathbb{N}$ (Example 6.1), or when $\mathfrak{n} = \prod_p p^\infty$ (Example 6.2). The latter case amounts to rational K -theory computations since in this case we have $L[1/\mathfrak{n}] \cong L \otimes_{\mathbb{Z}} \mathbb{Q}$ for any abelian group L . We give two examples of when one of these situations naturally occurs:

Example 6.1 ($M_{\mathfrak{n}} = M_{k^\infty}$). Let A be a unital C^* -algebra that admits a finite-dimensional representation $A \rightarrow M_k(\mathbb{C})$ for some k . Then the unital inclusion $\iota: \mathbb{C} \rightarrow A$ induces a split-injection $K_*(\mathbb{C})[1/k] \rightarrow K_*(A)[1/k]$. Concrete examples are unital continuous trace C^* -algebras or subhomogeneous C^* -algebras.

Example 6.2 ($M_{\mathfrak{n}} = \mathcal{Q}$). Let A be a unital C^* -algebra for which the unit $[1] \in K_0(A)$ is not torsion, for instance let A be unital and stably finite. Then the inclusion $\iota: \mathbb{C} \rightarrow A$ induces a split injection $K_*(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 6.3 (c.f. Theorem 2.8). *Let G be a discrete group satisfying BCC, let Z be a countable G -set and let \mathfrak{n} be a supernatural number. Let A be a unital C^* -algebra satisfying the UCT such that the unital inclusion $\mathbb{C} \rightarrow A$ induces a split injection $K_*(\mathbb{C})[1/\mathfrak{n}] \rightarrow K_*(A)[1/\mathfrak{n}]$. Denote by $\tilde{K}_*(A)$ the cokernel of the injection $K_*(\mathbb{C}) \rightarrow K_*(A)$ and let B be any C^* -algebra satisfying the UCT with $K_*(B) \cong \tilde{K}_*(A)$. Then we have*

$$\begin{aligned} K_*(A^{\otimes Z} \rtimes_r G)[1/\mathfrak{n}] \\ \cong K_*(\mathcal{J}_B^Z \rtimes_r G)[1/\mathfrak{n}] \cong \bigoplus_{[F] \in G \setminus \text{FIN}(Z)} K_*(B^{\otimes F} \rtimes_r G_F)[1/\mathfrak{n}]. \end{aligned}$$

Proof. Replacing \mathfrak{n} by \mathfrak{n}^∞ , we may assume that \mathfrak{n} is of infinite type. By the UCT, there is a UCT C^* -algebra B whose K -theory is isomorphic to the cokernel $\tilde{K}_*(A)$ of $K_*(\iota)$, and an element $\phi \in KK(B \otimes M_{\mathfrak{n}}, A \otimes M_{\mathfrak{n}})$ which together with the inclusion $\iota_{\mathfrak{n}}: M_{\mathfrak{n}} \rightarrow A \otimes M_{\mathfrak{n}}$ induces a KK -equivalence

$$\iota_{\mathfrak{n}} \oplus \phi \in KK((\mathbb{C} \oplus B) \otimes M_{\mathfrak{n}}, A \otimes M_{\mathfrak{n}}).$$

We can thus apply Theorem 2.8 for $A \otimes M_{\mathfrak{n}}$ in place of A , for $A_0 = M_{\mathfrak{n}}$ and for $B \otimes M_{\mathfrak{n}}$ in place of B . We get

$$\begin{aligned} K_*(A^{\otimes Z} \rtimes_r G)[1/\mathfrak{n}] &\cong K_*\left((A \otimes M_{\mathfrak{n}})^{\otimes Z} \rtimes_r G\right) \\ &\cong K_*\left(\mathcal{J}_{M_{\mathfrak{n}}, B \otimes M_{\mathfrak{n}}}^Z \rtimes_r G\right) \\ &\cong \bigoplus_{[F] \in G \setminus \text{FIN}(Z)} K_*\left((M_{\mathfrak{n}}^{\otimes Z-F} \otimes (B \otimes M_{\mathfrak{n}})^{\otimes F}) \rtimes_r G_F\right) \\ &\cong \bigoplus_{[F] \in G \setminus \text{FIN}(Z)} K_*(B^{\otimes F} \rtimes_r G_F)[1/\mathfrak{n}] \end{aligned}$$

where the first and last isomorphisms are obtained from Theorem 3.5. \square

Theorem 5.4 has a counter-part as well:

Theorem 6.4. *Let G be a discrete group satisfying BCC, let Z be a countable, proper G -set, and let \mathfrak{n} a supernatural number. Let A be a unital C^* -algebra which is a unital filtered colimit of C^* -algebras A_k satisfying the assumptions of Theorem 6.3. Then, the natural inclusions*

$$\begin{aligned} C_r^*(H) \rightarrow C_r^*(G), \quad C_r^*(H) \rightarrow A^{\otimes Z} \rtimes_r H, \\ C_r^*(G) \rightarrow A^{\otimes Z} \rtimes_r G, \quad A^{\otimes Z} \rtimes_r H \rightarrow A^{\otimes Z} \rtimes_r G, \end{aligned}$$

induce a pushout diagram

$$\begin{array}{ccc} \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{FIN}}(G)} K_*(C_r^*(H))[1/\mathfrak{n}] & \longrightarrow & \operatorname{colim}_{G/H \in \operatorname{Or}_{\mathcal{FIN}}(G)} K_*(A^{\otimes Z} \rtimes_r H)[1/\mathfrak{n}] \\ \downarrow & & \downarrow \\ K_*(C_r^*(G))[1/\mathfrak{n}] & \longrightarrow & K_*(A^{\otimes Z} \rtimes_r G)[1/\mathfrak{n}]. \end{array}$$

In particular, if G is torsion-free, this pushout diagram reads

$$\tilde{K}_*(C_r^*(G))[1/\mathfrak{n}] \oplus K_*(A^{\otimes Z})_G[1/\mathfrak{n}] \cong K_*(A^{\otimes Z} \rtimes_r G)[1/\mathfrak{n}].$$

Proof. Without loss of generality we may assume that \mathfrak{n} is of infinite type and A satisfies the assumption of Theorem 6.3. Let B, ϕ and $\iota_{\mathfrak{n}}$ be as in the proof of Theorem 6.3. In particular, Theorem 2.8 applies so that $(A \otimes M_{\mathfrak{n}})^{\otimes Z}$ is weakly K -equivalent to $\mathcal{J}_{M_{\mathfrak{n}}, B \otimes M_{\mathfrak{n}}}^Z$. As in the proof of Theorem 5.4, the Theorem now follows from the combination of Corollary 5.7 and Theorem 3.5. \square

Example 6.5. Let $p \in C(X) \otimes M_n$ be a rank- k projection over $C(X)$ corresponding to some vector bundle on X . Let $A = p(C(X) \otimes M_n)p$ be the associated homogeneous C^* -algebra. Of course, A is Morita-equivalent to $C(X)$ but the unit-inclusion $\iota: \mathbb{C} \rightarrow A$ is not split-injective in general; it corresponds to the inclusion $\mathbb{Z}[p] \rightarrow K^0(X)$. On the other hand, A has a k -dimensional irreducible representation and therefore satisfies the assumptions of Theorem 6.3. Therefore, we get

$$K_*(A^{\otimes G} \rtimes_r G)[1/k] \cong \bigoplus_{[F] \in G \setminus \mathcal{FIN}} K_*(B^{\otimes F} \rtimes_r G_F)[1/k]$$

where B is any UCT C^* -algebra whose K -theory is isomorphic to the cokernel of $K_*(\iota)$.

As an application to Theorem 6.3, we obtain a proof of the fact that Bernoulli shifts by finite groups rarely have the Rokhlin property (see [Izu04, Definition 3.1]). This result is possibly known to experts and we would like to thank N. C. Phillips for mentioning it to us.

Corollary 6.6. *Let A be a unital C^* -algebra satisfying the UCT such that $[1] \in K_0(A)$ is not torsion (for instance, suppose that A is stably finite). Let $G \neq \{e\}$ be a finite group and let Z a G -set. Then the Bernoulli shift of G on $A^{\otimes Z}$ does not have the Rokhlin property.*

Proof. It follows from Theorem 6.3 that the inclusion $C_r^*(G) \hookrightarrow A^{\otimes Z} \rtimes_r G$ induces a split injection

$$(6.2) \quad K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow K_*(A^{\otimes Z} \rtimes_r G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Now assume that the action of G on $A^{\otimes Z}$ has the Rokhlin property. Using [Phi11, Theorem 2.6], we can find a unital equivariant $*$ -homomorphism $C(G) \rightarrow A^{\otimes Z}$. But then we can factor the map in (6.2) as the composition

$$K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C(G) \rtimes_r G) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(A^{\otimes Z} \rtimes_r G) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which is never injective unless $G = \{e\}$. □

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