An identification of the Baum-Connes and Davis-Lück assembly maps

Master thesis supervised by Siegfried Echterhoff

Julian Kranz

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Introduction

The Baum-Connes Conjecture

Problem

Let G be a countable discrete group and A a separable G-C^*-algebra. What is $K_*(A \rtimes_r G)$?

Conjecture (Baum-Connes-Higson)

The assembly map

$$\mu : \textit{left hand side} \to K_*(A \rtimes_r G)$$

is an isomorphism (and the left hand side is easier to compute).

The Baum-Connes conjecture has been proven in many cases and has important applications in many areas of mathematics!

The left hand side

We discuss three equivalent definitions of the assembly map:

$$K_*^G(\mathcal{E}G, A) \to K_*(A \rtimes_r G) \quad \text{(Baum-Connes-Higson)}$$
$$K_*(\tilde{A} \rtimes_r G) \to K_*(A \rtimes_r G) \quad \text{(Meyer-Nest)}$$
$$H_*^G(\mathcal{E}G, \boldsymbol{K}_A^G) \to H_*^G(\mathrm{pt}, \boldsymbol{K}_A^G) \quad \text{(Davis-Lück)}$$

where

$$\mathcal{E}G = \text{ universal proper } G\text{-space}$$
$$K_*^G(-, A) = \text{ equivariant } K\text{-homology with coefficients in}$$
$$\tilde{A} = \text{a "proper" approximation of } A$$
$$H_*^G(-, \boldsymbol{K}_A^G) = \text{ a } G\text{-homology theory with } H_*^G(\mathrm{pt}, \boldsymbol{K}_A^G) = K_*(A \rtimes_r G)$$

The analytic approach of Baum-Connes-Higson

Equivariant KK-theory

Theorem (Thomsen, Meyer)

There is a unique functor

$$KK^G : \text{sep. } G\text{-}C^*\text{-alg.} \to \mathfrak{KK}^G$$

into an additive category $\mathfrak{KK}^G$ satisfying

1. KK^G *is G-homotopy invariant.*
2. KK^G *maps $A \otimes \mathcal{K}(H) \to A \otimes \mathcal{K}(H \oplus H')$ to isomorphisms for all sep. G-Hilbert spaces H, H'.*
3. KK^G *maps split-exact sequences to split-exact sequences.*
4. *For any F : sep. G-C^*-alg. $\to \mathcal{A}$ with the above properties, there is a unique $\tilde{F} : \mathfrak{KK}^G \to \mathcal{A}$ s.t. $F = \tilde{F} \circ KK^G$.*

Some properties of KK^G

1. $KK_n^G(A,B) := KK^G(A, C_0(\mathbb{R}^n)\otimes B)$ is 2-periodic, i.e. $KK_{n+2}^G \cong KK_n^G$.
2. $K_*(A) \cong KK_*(\mathbb{C},A) := KK_*^{\{e\}}(\mathbb{C},A)$
3. $\rtimes_r G$ descends to a functor
$$\rtimes_r G : KK^G(A,B) \to KK(A\rtimes_r G, B\rtimes_r G).$$
Similar: $\mathrm{Ind}_H^G, \mathrm{Res}_G^H, \otimes C$.
4. For any G-semisplit extension $0\to I\to E\to B\to 0$ and any sep. G-C^*-algebra A, there is an exact sequence
$$\begin{array}{ccccc} KK_0^G(A,I) & \longrightarrow & KK_0^G(A,E) & \longrightarrow & KK_0^G(A,B) \\ \uparrow & & & & \downarrow \\ KK_1^G(A,B) & \longleftarrow & KK_1^G(A,E) & \longleftarrow & KK_1^G(A,I) \end{array} .$$

Proper G-spaces

Definition

A proper G-space $\mathcal{E}G$ is *universal* if for any other proper G-space X, there is a G-map $X \to \mathcal{E}G$ which is unique up to G-homotopy.

Lemma

$\mathcal{E}G$ exists and is unique up to unique G-homotopy equivalence.

Example (Baum-Connes-Higson)

$$\mathcal{E}G := \left\{ f : G \to [0,1] : \quad |\operatorname{supp}(f)| < \infty, \quad \sum_{g \in G} f(g) = 1 \right\} \subseteq \ell^1(G)$$

The Baum-Connes assembly map

Definition (Baum-Connes-Higson)

The *equivariant K-homology of $\mathcal{E}G$ with coefficients in A* is

$$K_*^G(\mathcal{E}G, A) := \operatorname*{colim}_{\substack{Y \subseteq \mathcal{E}G \\ G-cpt.}} KK_*^G(C_0(Y), A).$$

The *assembly map*

$$\mu : K_*^G(\mathcal{E}G, A) \to K_*(A \rtimes_r G)$$

is the colimit of the maps

$$KK_*^G(C_0(Y), A) \xrightarrow{\rtimes_r G} KK_*(C_0(Y) \rtimes_r G, A \rtimes_r G) \xrightarrow{p_Y^*} KK_*(\mathbb{C}, A \rtimes_r G),$$

where $p_Y \in C_0(Y) \rtimes_r G$ is a certain projection.

The localization approach of Meyer-Nest

Localizing subcategories

Definition

A full subcategory $\mathcal{C} \subseteq \mathfrak{KK}^G$ is *localizing*, if it is closed under

1. KK^G-equivalence
2. Suspension
3. G-semisplit extensions
4. Countable direct sums

Remark

For any $\mathcal{C} \subseteq \mathfrak{KK}^G$ there is a smallest localizing subcategory $\langle \mathcal{C} \rangle \subseteq \mathfrak{KK}^G$ such that $\mathcal{C} \subseteq \langle \mathcal{C} \rangle$.

The Dirac morphism

Definition

Let $\mathcal{CI} \subseteq \mathfrak{KK}^G$ be the full subcategory of all G-C^*-algebras of the form

$$\operatorname{Ind}_H^G B := \left\{ f \in C_0(G,B) : \quad f(gh) = h^{-1}(f(g)) \quad \forall g \in G, h \in H \right\},$$

where $H < G$ is a **finite** subgroup and B a separable H-C^*-algebra.

Theorem (Meyer-Nest)

For any sep. G-C^*-algebra A, there is a sep. G-C^*-algebra $\tilde{A} \in \langle \mathcal{CI} \rangle$ and a morphism $D \in KK^G(\tilde{A}, A)$, such that $\operatorname{Res}_G^H(D)$ is a KK^H-equivalence for every finite subgroup $H < G$. The pair $(\tilde{A}, D)$ is unique up to KK^G-equivalence.

D is called the *Dirac morphism*.

The Meyer-Nest assembly map

Definition (Meyer-Nest)

The *assembly map* is the the map

$$D_* : K_*(\tilde{A} \rtimes_r G) \to K_*(A \rtimes_r G)$$

induced by the Dirac morphism $D \in KK^G(\tilde{A}, A)$.

Theorem (Meyer-Nest)

The indicated maps in the following diagram are isomorphisms:

$$\begin{array}{ccc} K_*^G(\mathcal{E}G, \tilde{A}) & \xrightarrow[\cong]{\mu} & K_*(\tilde{A} \rtimes_r G) \\ \cong \downarrow D_* & & \downarrow D_* \\ K_*^G(\mathcal{E}G, A) & \xrightarrow{\mu} & K_*(A \rtimes_r G) \end{array}$$

The topological approach of Davis-Lück

The Davis-Lück assembly map

Davis-Lück construct a G-homology theory $H_*^G(-, \boldsymbol{K}_A^G)$, whose values on **discrete** G-spaces X are given by

$$H_*^G(X, \boldsymbol{K}_A^G) \cong K_*(C_0(X, A) \rtimes_r G).$$

For $X = G/H$, this means

$$H_*^G(G/H, \boldsymbol{K}_A^G) \cong K_*(C_0(G/H, A) \rtimes_r G) \overset{\text{Green}}{\cong} K_*(A \rtimes_r H).$$

Definition (Davis-Lück)

The *assembly map* is the map

$$\mathrm{pr}_* : H_*^G(\mathcal{E}G, \boldsymbol{K}_A^G) \to H_*^G(\mathrm{pt}, \boldsymbol{K}_A^G).$$

induced by the projection $\mathcal{E}G \to \mathrm{pt}$.

$H_*^G(X, \boldsymbol{K}_A^G)$ for discrete X

Definition (naive)

$H_*^G(X, \boldsymbol{K}_A^G) := K_*(C_0(X, A) \rtimes_r G)$

Problem: $C_0(X, A) \rtimes_r G$ is not functorial in X.

Theorem (Joachim, Mitchener)

There is a functor

$$G\text{-sets} \to C^*\text{-alg.}, \quad X \mapsto \mathfrak{C}_f^*(A \rtimes_r \overline{X})$$

such that $\mathfrak{C}_f^*(A \rtimes_r \overline{X}) \sim_{KK} C_0(X, A) \rtimes_r G$.

Definition

$H_*^G(X, \boldsymbol{K}_A^G) := K_*(\mathfrak{C}_f^*(A \rtimes_r \overline{X}))$

$H_*^G(X, \boldsymbol{K}_A^G)$ for non-discrete X

To pass from discrete X to arbitrary X, we need to take a detour through spectra (a.k.a infinite loop spaces).

Definition

A *spectrum* is a sequence of pointed spaces $E_n, n \geq 0$ together with weak homotopy equivalences

$$E_n \simeq \Omega E_{n+1} := \mathrm{Map}_*(S^1, E_{n+1}).$$

For $n \in \mathbb{Z}$, its n-th homotopy group is

$$\pi_n(E) := \pi_{n+k}(E_k), \quad n + k \geq 0.$$

$H_*^G(X, \boldsymbol{K}_A^G)$ for non-discrete X

Theorem (Davis-Lück)

Let $\mathrm{Or}(G)$ *be the category of homogeneous* G*-sets* G/H. *Every functor*

$$\boldsymbol{E} : \mathrm{Or}(G) \to \mathit{Spectra}$$

has a unique (up to weak homotopy equivalence) extension

$$\boldsymbol{E}_{\%} : G\text{-}\mathit{CW}\text{-}\mathit{Complexes} \to \mathit{Spectra}$$

such that

$$H_*^G(-, \boldsymbol{E}) := \pi_*(\boldsymbol{E}_{\%}(-))$$

is a G*-homology theory.*

$H_*^G(X, \boldsymbol{K}_A^G)$ for non-discrete X

Theorem (Joachim)

There is a functor $\boldsymbol{K}$: sep. C^*-alg. $\to$ Spectra, such that

$$\pi_*(\boldsymbol{K}(A)) \cong K_*(A).$$

Definition

$$\boldsymbol{K}_A^G : \mathrm{Or}(G) \to \mathrm{Spectra}, \quad \boldsymbol{K}_A^G(G/H) := \boldsymbol{K}(\mathfrak{C}_f^*(A \rtimes_r \overline{G/H}))$$

$$\Rightarrow H_*^G(-, \boldsymbol{K}_A^G) = \pi_*((\boldsymbol{K}_A^G)_\%(-))$$

The identification

Remark

We can identify the Meyer-Nest assembly map

$$D_* : K_*(\tilde{A} \rtimes_r G) \to K_*(A \rtimes_r G)$$

with the map

$$D_* : H_*^G(\mathrm{pt}, \boldsymbol{K}_{\tilde{A}}^G) \to H_*^G(\mathrm{pt}, \boldsymbol{K}_A^G).$$

Theorem (K.)

The indicated maps in the following diagram are isomorphisms.

$$\begin{array}{ccc} H_*^G(\mathcal{E}G, \boldsymbol{K}_{\tilde{A}}^G) & \xrightarrow[\cong]{\mathrm{pr}_*} & H_*^G(\mathrm{pt}, \boldsymbol{K}_{\tilde{A}}^G) \\ \cong \big\downarrow D_* & & \big\downarrow D_* \\ H_*^G(\mathcal{E}G, \boldsymbol{K}_A^G) & \xrightarrow{\mathrm{pr}_*} & H_*^G(\mathrm{pt}, \boldsymbol{K}_A^G) \end{array}$$

The left hand isomorphism

Lemma

The map $H_*^G(\mathcal{E}G, \boldsymbol{K}_{\tilde{A}}^G) \to H_*^G(\mathcal{E}G, \boldsymbol{K}_A^G)$ *is an isomorphism.*

Proof (Sketch).

Excision reduces the statement to $\mathcal{E}G = G/H$ for $H < G$ finite. Since D is a KK^H-equivalence, the map

$$H_*^G(G/H, \boldsymbol{K}_{\tilde{A}}^G) \cong K_*(\tilde{A} \rtimes_r H) \xrightarrow{D_*} K_*(A \rtimes_r H) \cong H_*^G(G/H, \boldsymbol{K}_A^G)$$

is an isomorphism. □

Reducing to $\tilde{A} = \operatorname{Ind}_H^G B$

Lemma (K.)

The class of G-C^*-algebras $\tilde{A}$ for which the assembly map

$$H_*^G(\mathcal{E}G, \boldsymbol{K}_{\tilde{A}}^G) \to H_*^G(\mathrm{pt}, \boldsymbol{K}_{\tilde{A}}^G)$$

is an isomorphism is localizing. In particular, we may assume $\tilde{A} = \operatorname{Ind}_H^G B \in \mathcal{CI}$.

Idea of Proof.

KK^G	$\rightsquigarrow$	$H_*^G(-, \boldsymbol{K}_{\tilde{A}}^G)$
KK^G-equivalence	$\rightsquigarrow$	natural isomorphism
suspension	$\rightsquigarrow$	shift
G-semisplit extensions	$\rightsquigarrow$	long exact sequences
countable direct sums	$\rightsquigarrow$	countable direct sums

□

The induction isomorphism

Proposition (K.)

Let X be a G-CW-complex, $H < G$ a subgroup and B a separable H-C^*-algebra. There is a natural induction isomorphism

$$H_*^H(X|_H, \boldsymbol{K}_B^H) \xrightarrow{\cong} H_*^G(X, \boldsymbol{K}^G_{\mathrm{Ind}_H^G B}).$$

Proof (for discrete X).

By Green's imprimitivity theorem, there is a full corner embedding

$$C_0(X, B) \rtimes_r H \hookrightarrow C_0(X, \mathrm{Ind}_H^G B) \rtimes_r G.$$

On K-theory, this gives an isomorphism

$$\begin{aligned} H_*^H(X|_H, \boldsymbol{K}_B^H) &\cong K_*(C_0(X, B) \rtimes_r H) \\ &\cong K_*(C_0(X, \mathrm{Ind}_H^G B) \rtimes_r G) \cong H_*^G(X, \boldsymbol{K}^G_{\mathrm{Ind}_H^G B}). \end{aligned}$$

Corollary

The map

$$H_*^G(\mathcal{E}G, \boldsymbol{K}_{\tilde{A}}^G) \to H_*^G(\mathrm{pt}, \boldsymbol{K}_{\tilde{A}}^G)$$

is an isomorphism.

Proof.

We may assume $\tilde{A} = \mathrm{Ind}_H^G B$ for $H < G$ finite. Consider the diagram

$$\begin{array}{ccc} H_*^G(\mathcal{E}G, \boldsymbol{K}^G_{\mathrm{Ind}_H^G B}) & \longrightarrow & H_*^G(\mathrm{pt}, \boldsymbol{K}^G_{\mathrm{Ind}_H^G B}) \\ \cong\uparrow & & \cong\uparrow \\ H_*^H(\mathcal{E}G|_H, \boldsymbol{K}_B^H) & \longrightarrow & H_*^H(\mathrm{pt}, \boldsymbol{K}_B^H) \end{array}$$

Since H is finite, we have $\mathcal{E}G|_H \simeq \mathcal{E}H \simeq \mathrm{pt}$ and all maps are isomorphisms. □

Thank you for your attention!

References

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