K-theory of noncommutative Bernoulli shifts

NYC-NCG Seminar

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joint work in progress with S. Chakraborty, S. Echterhoff, and S. Nishikawa

September 28, 2022

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b) Main results

c) What is KK-theory and the Baum–Connes conjecture?

d) Some techniques appearing in the proofs

Crossed products

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$$\tilde{\alpha}(a)\xi(g) := \alpha_{g^{-1}}(a)\xi(g)$$

$$\lambda_g\xi(h) := \xi(g^{-1}h)$$

for $a \in A, \;\; \xi \in \ell^2(G, A), \;\; g, h \in G.$

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$$\begin{array}{rcl} A \rtimes_r G & \cong & A \otimes C_r^*(G) \\ \lambda_g & \mapsto & u_g \otimes \lambda_g \\ \tilde{\alpha}(a) & \mapsto & a \otimes 1. \end{array}$$

Bernoulli shifts

Definition

Let G be a discrete group and X a compact Hausdorff space. The *Bernoulli shift* of G on X is the shift action

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Example (Wreath Products)

Let G, H be groups, $H \wr G := \left(\bigoplus_{g \in G} H\right) \rtimes G$. Then

$$C_r^*(H \wr G) = C_r^*(H)^{\otimes G} \rtimes_r G.$$

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Theorem (Xin Li)

Let H be a finite group and G a discrete group satisfying the Baum–Connes conjecture with coefficients (e.g. an amenable group). Then

$$K_*(C_r^*(H \wr G)) \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}} \bigoplus_{[S] \in G_F \backslash \{1,\ldots,N\}^F} K_*(C_r^*(G_S)),$$

where FIN denotes the set of finite subsets of G, N is the number of non-trivial conjugacy classes in H, and G_F is the stabilizer of F in G (note that $G_\emptyset = G$).

Roadmap to the proof

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These properties can be abstracted!

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Main results

Theorem (Chakraborty–Echterhoff–K–Nishikawa)

Let $A \cong M_{k_0} \oplus \cdots \oplus M_{k_N}$ be a finite-dimensional C^*-algebra and G a discrete group satisfying the Baum–Connes conjecture with coefficients. Write $n := \gcd(k_0, \ldots, k_N)$. Then we have

$$K_*(A^{\otimes G} \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}} \bigoplus_{[S] \in G_F \backslash \{1,\ldots,N\}^F} K_*(C_r^*(G_S))[1/n].$$

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Theorem (Chakraborty–Echterhoff–K–Nishikawa)

Let H be an amenable group and G as above. Write $B = \ker(1_H \colon C_r^*(H) \to \mathbb{C})$. Then

$$K_*(C_r^*(H \wr G)) \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}} K_*(B^{\otimes F} \rtimes_r G_F),$$

where $B^{\otimes \emptyset} := \mathbb{C}$.

Reminder on Kasparov's KK^G-theory

Let G be a countable discrete group. There is an additive category KK^G with separable G-C^*-algebras as objects, and a functor

$$j\colon \{\text{sep. } G\text{-}C^*\text{-alg.}\} \to KK^G$$

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c) For a subgroup $H \subseteq G$, there are induction and restriction functors

$$\mathrm{Ind}_H^G\colon KK^H \rightleftarrows KK^G\colon \mathrm{Res}_G^H \;.$$

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f) If H is finite, then $KK^H(\mathbb{C}, \mathbb{C})$ is isomorphic to the representation ring $R_{\mathbb{C}}(H)$.

The Baum–Connes conjecture with coefficients

Conjecture (BCC)

Let G be a discrete group, let A, B be G-C^*-algebras and $\varphi \in KK^G(A, B)$ such that the induced map

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is an isomorphism for every finite subgroup $H \subseteq G$.

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BCC holds for all amenable groups (Higson–Kasparov) but not for all groups (Higson–Lafforgue–Skandalis).

An abstract K-theory formula

Theorem (Chakraborty–Echterhoff–K–Nishikawa)

Let G be a discrete group satisfying BCC and let A be a unital C^*-algebra. Denote by $\iota\colon \mathbb{C} \hookrightarrow A$ the unital inclusion. Assume there is a C^*-algebra B and $\phi \in KK(B, A)$ such that $\phi \oplus \iota \in KK(B \oplus \mathbb{C}, A)$ is a KK-equivalence. Then we have

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$$(B \oplus \mathbb{C})^{\otimes G} \cong \bigoplus_{F \in \mathrm{FIN}} B^{\otimes F}.$$

Examples for $B \oplus \mathbb{C} \sim_{KK} A$

The ring $KK(\mathbb{C}^N, \mathbb{C}^N)$ is isomorphic to $M(N \times N, \mathbb{Z})$, the KK-equivalences are given by elements in $GL(N, \mathbb{Z})$.

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$$\phi \oplus \iota = \begin{pmatrix} 1 & & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} \in GL(N+1, \mathbb{Z})$$

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Corollary

Assume that G satisfies BCC. Then we have

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Question

Let A be a unital C^*-algebra, denote by $\iota\colon \mathbb{C} \to A$ the unital inclusion. When can you find a C^*-algebra B and an element $\phi \in KK(B, A)$ such that $\phi \oplus \iota \in KK(B \oplus \mathbb{C}, A)$ is a KK-equivalence?

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Answer

Assume the UCT!

The Universal Coefficient Theorem

Definition (Rosenberg–Schochet)

A separable C^*-algebra A satisfies the *Universal Coefficient Theorem* (UCT), if for every separable C^*-algebra C, there is a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(C)) \to KK(A, C) \to \mathrm{Hom}(K_*(A), K_*(C)) \to 0.$$

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$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(C)) \to KK(A, C) \to \mathrm{Hom}(K_*(A), K_*(C)) \to 0.$$

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a) If H is amenable, then $C^*_r(H)$ satisfies the UCT (Tu).

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c) Suppose that A and C satisfy the UCT. Then any isomorphism $\phi \in \mathrm{Hom}(K_*(A), K_*(C))$ is induced by a KK-equivalence $\overline{\phi} \in KK(A, C)$.

Examples for $A \sim_{KK} \mathbb{C} \oplus B$

Let A be a unital C^*-algebra satisfying the UCT such that the unital inclusion $\iota\colon \mathbb{C} \to A$ induces a split injection $K_*(\mathbb{C}) \hookrightarrow K_*(A)$ (for example $A = C_r^*(H)$ with H amenable).

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Corollary

$$K_*(A^{\otimes G} \rtimes_r G) \cong \bigoplus_{[F] \in G \backslash \mathrm{FIN}} K_*(B^{\otimes F} \rtimes_r G_F),$$

where $B^{\otimes \emptyset} := \mathbb{C}$.

More examples for for $A \sim_{KK} \mathbb{C} \oplus B$

There are many unital, separable, C^*-algebras A satisfying the UCT such that the inclusion $\iota\colon \mathbb{C} \to A$ induces a split injection $K_*(\mathbb{C}) \hookrightarrow K_*(A)$:

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Remark

The equivariant topological K-theory groups

$$K_*(C_0(\mathbb{R})^{\otimes F} \rtimes_r G_F) \cong K^*_{G_F}(\mathbb{R}^F)$$

can be computed explicitly (Karoubi, Echterhoff–Pfante).

Finite-dimensional algebras

Theorem (Chakraborty–Echterhoff–K–Nishikawa)

Let $A = M_{k_0} \oplus \cdots \oplus M_{k_N}$ with $\gcd(k_0, \ldots, k_N) = 1$. Then there is a ***unital $*$-homomorphism*** *$\phi\colon \mathbb{C}^{N+1} \to A$ which is a KK-equivalence.*

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Sketch of proof.

It suffices to find a matrix $X \in \mathrm{GL}(N+1, \mathbb{Z})$ such that $X(1, \ldots, 1) = (k_0, \ldots, k_N)$. This exists by the Euclidian algorithm. □

UHF-algebras

Theorem (K-Nishikawa)

Let H be a finite group and Z a countably infinite H-set. Then the canonical inclusions

$$M_n^{\otimes Z} \hookrightarrow M_n^{\otimes Z} \otimes M_n^{\otimes \infty} \hookleftarrow M_n^{\otimes \infty}$$

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Warning: The connecting maps for $\varinjlim_k M_{n^{|H|}}^{\otimes k}$ involve non-trivial KK^H-elements ($\rightsquigarrow$ representation theory). □

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Let G be an infinite discrete group satisfying BCC and let A be a G-C^*-algebra. Then we have

$$K_*((A \otimes M_n^{\otimes G}) \rtimes_r G) \cong K_*((A \rtimes_r G) \otimes M_n^{\otimes \infty}) \cong K_*(A \rtimes_r G)[1/n].$$

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Proof.

Take $A = (M_{k_0/n} \oplus \cdots \oplus M_{k_N/n})^{\otimes G} \Rightarrow A \otimes M_n^{\otimes G} = B^{\otimes G}$. □

More applications

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Let H be a finite group and $\mathcal{D}$ a strongly self-absorbing C^*-algebra satisfying the UCT (e.g. $M_n^{\otimes \infty}$). Then the Bernoulli shift on $\mathcal{D}^{\otimes H} \cong \mathcal{D}$ is KK^H-equivalent to the trivial action on $\mathcal{D}$.

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Corollary

Let $\{e\} \neq H$ be a finite group. Then the Bernoulli shift $H \curvearrowright (M_n^{\otimes \infty})^{\otimes H}$ does not have the Rokhlin property.

Thank you very much!