# Classifiability of crossed products by nonamenable groups

## Noncommutativity in the North

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March 14, 2022

# Classification of $C^*$-algebras

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### Theorem (GGKN)

*For groups with **paradoxical towers**, $\mathcal{Z}$-stability is automatic!*

### Strategy.

paradoxical towers $\overset{GGKN}{\Rightarrow}$ Kerr's dynamical comparison $\overset{Ma}{\Rightarrow}$ purely infinite $\overset{Kirchberg-Phillips}{\Rightarrow}$ $\mathcal{Z}$-stable □

# Paradoxical towers

## Definition (GGKN)

A group $G$ has $n$-**paradoxical towers**, if for every finite $D \Subset G$, there exist $A_1, \dots, A_n \subseteq G$ and $g_1, \dots, g_n \in G$ such that

a) The sets $\{dA_i\}_{d \in D, i=1,\dots,n}$ are pairwise disjoint

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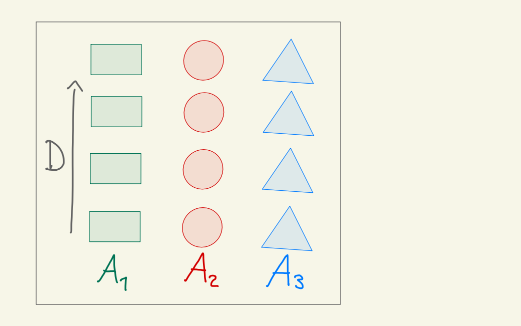

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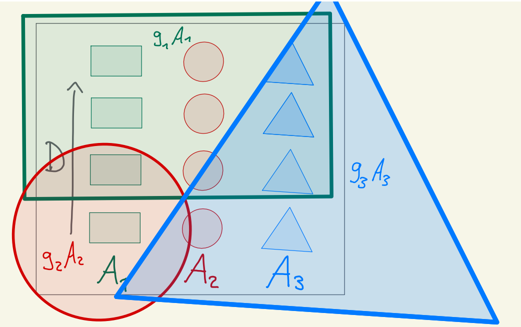

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## Exercise

$F_k \times \mathbb{Z}$ does not have $n$-paradoxical towers for any $n \in \mathbb{N}$.

# A systematic way to construct paradoxical towers

# $n$-filling actions

**Definition (Jolissaint-Robertson)**

An action $G \curvearrowright X$ is called *$n$-filling*, if for every collection of non-empty open subsets $U_1, \ldots, U_n \subseteq X$, there are $g_1, \ldots, g_n \in G$ such that $X = g_1 U_1 \cup \cdots \cup g_n U_n$.

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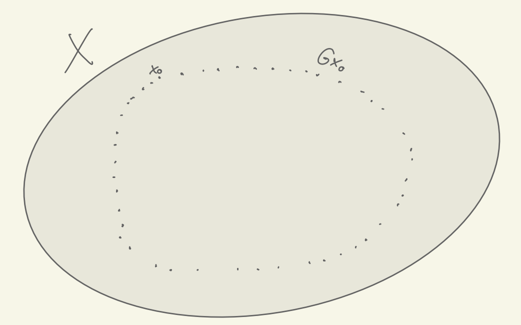

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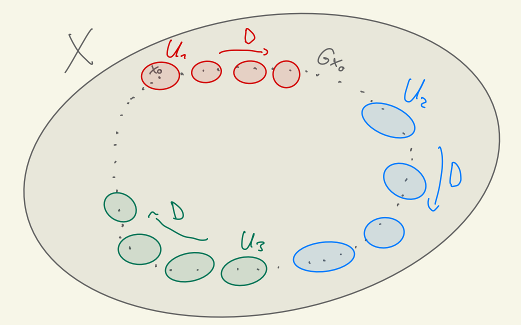

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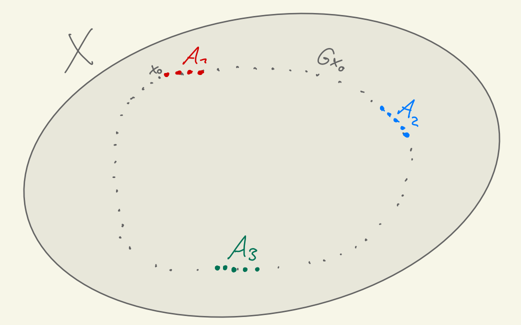

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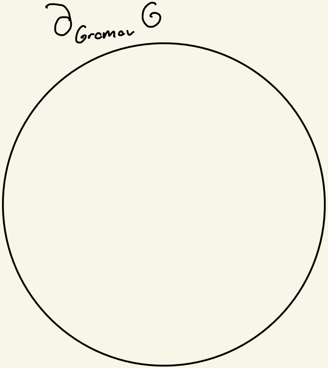

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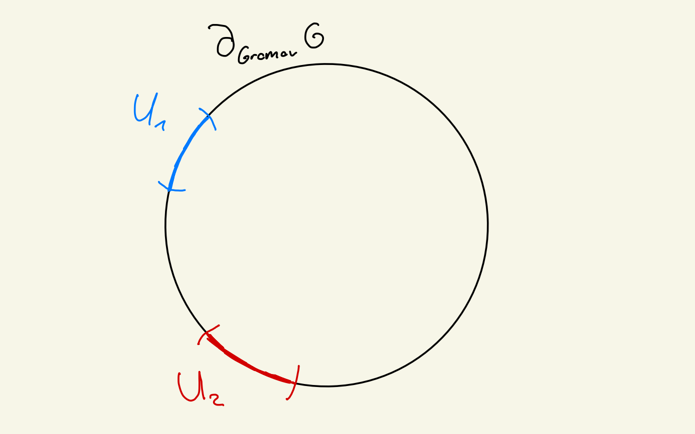


$$\emptyset \neq U_1, U_2 \subseteq X \text{ open}$$

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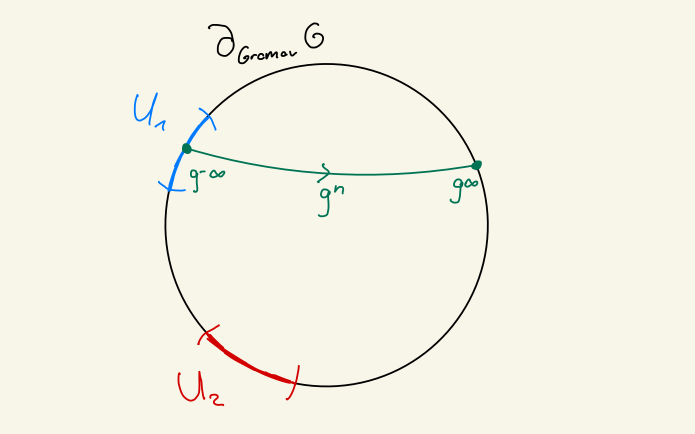


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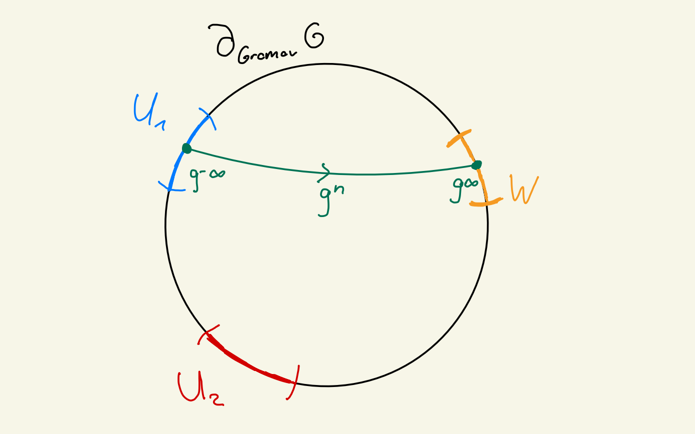


$$g^\infty \in W \text{ open}, \quad g^n(U_1^c) \subseteq W$$

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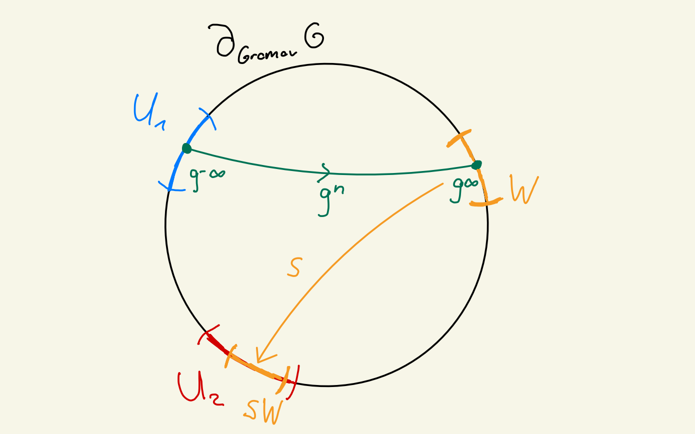


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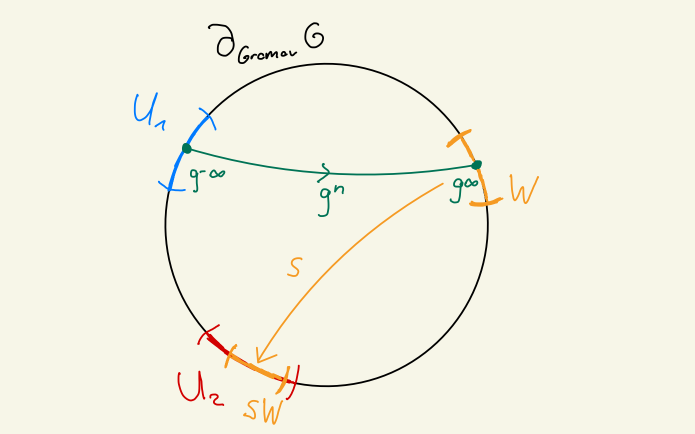


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*Let $N < G$ be a finite normal subgroup. Suppose that $G/N$ has paradoxical towers. Then $G$ has paradoxical towers too.*

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**Corollary**

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Example (modulo caveats)

$F_n, MCG(\Sigma), Out(F_n), SL_n(\mathbb{Z}), BS(n,m)$, RAAGs & RACGs, ...

# Further results

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a) *If $G = H \times K$ where $H$ has paradoxical towers, then $C(X) \rtimes G$ is purely infinite.*

b) *If $G$ contains $F_2$, then $C(X) \rtimes G$ is properly infinite.*

# What's next?

Conjecture

*Let* $G \curvearrowright X$ *be a minimal, amenable, topologically free action of a* ***non-amenable*** *discrete group on a compact space. Then* $C(X) \rtimes G$ *is purely infinite.*

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### Question

Are there groups with paradoxical towers that don't contain $F_2$?

Thank you!