Cantor minimal systems seminar: talks & abstracts
University of Münster,
Summer Semester 2022

Below is the list of abstracts for the seminar *Cantor minimal systems* running in the summer semester of 2022. The plan is to follow Putnam's book with the same name [3]. The talks roughly correspond to chapters in the book, with the main points highlighted in the abstracts. We choose not to give specific references in this list (except for the last talk), since all relevant papers are cited in our main source, [3].

The organizers,
Rafaela Gesing
Grigoris Kopsacheilis
Julian Kranz
Petr Naryshkin

Talk 1: Topological preliminaries, Bratteli diagrams and path spaces

Date: 07/04/2022
Speaker: Petr Naryshkin

In the first half of this talk, we build a topological toolkit (chapter 2), exploring the elementary properties of the Cantor space, and review basic definitions of topological dynamics, including the orbit relation and the orbit equivalence of two dynamical systems (def 2.19, 2.20). In the second half, we introduce Bratteli diagrams (def. 3.1), show how to construct the associated path space for each Bratteli diagram (def. 3.3, prop. 3.5), and define the AF-equivalence relation on the path space (def 3.8, prop. 3.9).

Talk 2: The Bratteli-Vershik model

Date: 21/04/2022
Speaker: Rafaela Gesing

We present examples of Bratteli diagrams and their associated Cantor spaces and introduce the method of telescoping (chapter 3). We introduce properly ordered Bratteli diagrams (def. 4.1, 4.6) and define the Vershik map (def. 4.4) on the path space. We characterize the minimality of the $\mathbb{Z}$-action implemented by the Vershik homeomorphism in terms of the Bratteli diagram and the AF-equivalence relation on the path space (prop. 4.13) and discuss how the Bratteli-Vershik model generalizes odometer actions.

Talk 3: Any minimal Cantor system is isomorphic to a Bratteli-Vershik model

Date: 28/04/2022
Speaker: Julian Kranz

In this talk we prove in detail that any minimal homeomorphism of the Cantor space is topologically conjugate to the Vershik homeomorphism on the path space of some simple, proper Bratteli diagram (prop. 5.1). This result reduces the classification of minimal $\mathbb{Z}$-actions on the Cantor space to the classification of Bratteli-Vershik models.

Talk 4: Local actions and étale equivalence relations

Date: 05/05/2022
Speaker: Petr Naryshkin

We introduce some new notations, local actions (def. 6.3) and étale equivalence relations (def. 6.6, def. 6.10). We see basic properties of étale equivalence relations and how they generalize both AF-equivalence relations on the path space of a Bratteli diagram (prop. 6.14) and orbit relations of free Cantor dynamics (prop. 6.19).

Talk 5: The D-invariant (part I)

Date: 12/05/2022
Speaker: Grigoris Kopsacheilis

We define the category of ordered abelian groups and see examples (prop 7.7, 7.8). We then introduce the D-invariant of an étale equivalence relation on a compact totally disconnected space (def. 7.10, prop. 7.12) and the dimension group of a Bratteli diagram (def. 7.23, prop. 7.29, 7.30, 7.31).

Talk 6: The D-invariant (part II)

Date: 19/05/2022
Speaker: Alessandro Codenotti

We show that the D-invariant of the AF-equivalence relation on a Bratteli diagram is canonically isomorphic to the dimension group (prop. 7.32) and that the D-invariant of the orbit relation of the Vershik homeomorphism on the path space of a properly ordered Bratteli diagram is also canonically isomorphic to the dimension group (prop. 7.35). The rest of the talk is devoted to explicit examples and computations.

Talk 7: Intertwining our way to the Bratteli-Elliott-Krieger theorem

Date: 02/06/2022
Speaker: Grigoris Kopsacheilis

We begin by stating the Effros-Handelman-Shen theorem (prop. 8.4). We omit the proof, but might return to it at some point during the seminar, if time permits. Next, we introduce the concept of intertwining (def. 9.1) and prove the Bratteli-Elliott-Krieger theorem (prop. 9.2), which tells us that AF-equivalence relations on two Bratteli diagrams are isomorphic if and only if there exists an intertwining between the two diagrams. This is an early result towards our main goal: the classification of all minimal Cantor dynamics up to orbit equivalence.

Talk 8: Strong orbit equivalence

Date: 23/06/2022
Speaker: Shirly Geffen

In this talk we return to studying abstract $\mathbb{Z}$-actions on the Cantor space. We define orbit cocycles and strong orbit equivalence (prop 10.1, def. 10.5). Using the result of Boyle and Tomiyama (prop. 10.4), the proof of which will most likely be omitted, we show that two minimal $\mathbb{Z}$-actions on the Cantor space are strongly orbit equivalent if and only if the D-invariants of their orbit relations (in the sense of def. 7.10) are isomorphic as ordered abelian groups (prop 10.7).

Talk 9: The D_m-invariant (part I)

Date: 30/06/2022
Speaker: Diego Martínez

We define states on ordered abelian groups (def. 11.8) and R-invariant Borel probability measures on compact totally disconnected spaces equipped with an étale equivalence relation R (def. 11.15). We identify a state on the D-invariant of an étale equivalence relation as integration w.r.t. an R-invariant measure (prop. 11.20) and introduce the D_m-invariant for étale equivalence relations (def. 11.21).

Talk 10: The D_m-invariant (part II)

Date: 07/07/2022
Speaker: James O'Quinn

We show that the D_m-invariant is preserved under orbit equivalence (prop. 11.29) and study its features. In particular, we consider the special cases of AF-equivalence relations of Bratteli diagrams (prop. 11.31) and orbit relations of free $\mathbb{Z}$-actions on the Cantor space (prop. 11.35, prop 11.36). As an application, we classify all odometer actions up to orbit equivalence (prop 11.38).

Talk 11: Matui's absorption theorem

Date: 14/07/2022
Speaker: James O'Quinn

In this talk we prove a special case of H. Matui's absorption theorem (prop. 12.1), which tells us that a fairly small "extension" of an AF-equivalence relation is orbit equivalent to the initial relation. We also state the theorem in full generality (prop. 12.12) but omit the proof.

Talk 12: Classification of minimal Cantor systems

Date: 21/07/2022
Speaker: Rafaela Gesing

In this talk we combine the tools that we have developed in the previous talks, namely the Effros-Handelman-Shen theorem, the Bratteli-Elliott-Krieger theorem and Matui's absorption theorem, to classify AF-equivalence relations and orbit relations of minimal $\mathbb{Z}$-actions on the Cantor space up to orbit equivalence (prop. 13.1, corollary 13.2, prop. 14.1, prop. 14.2). We do not present full proofs but rather a very detailed treatment for a specific example, which will be enough to shed some light on the main ideas behind this classification results (chapters 13, 14).

Talk 13: Orbit equivalence, crossed product C*-algebras of minimal Cantor systems and K-theory

Date: 28/07/2022
Speaker: Sam Evington

In this concluding talk we draw some connections between the topic of the seminar and the classification theory of AF C*-algebras. The main part of the talk is an exposition of the theorems in the seminal paper by Giordano, Putnam and Skau ([5], theorems 2.1, 2.2), which characterize (strong and plain) orbit equivalence by use of the associated crossed product and relate the dynamical invariants that we have developed here to the K-theoretic data of the associated crossed product C*-algebras. Since we have already developed the necessary techniques, we also take a small detour and see how AF C*-algebras are classified by K-theoretic data.

References: [5], [2] (pre-print), [1], [4] (chapter 7)

References

[1] G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *Journal of Algebra*, 38(1):29–44, 1976.

[2] J. Melleray and S. Robert. From invariant measures to orbit equivalence, via locally finite groups: a new proof of a theorem of giordano, putnam, and skau. *arXiv preprint arXiv:2109.04701*, 2021.

[3] I. F. Putnam. *Cantor minimal systems*, volume 70. American Mathematical Soc., 2018.

[4] M. Rørdam, F. Larsen, F. Larsen, and N. Laustsen. *An introduction to K-theory for C*-algebras*. Number 49. Cambridge University Press, 2000.

[5] V. . C. F. Skau, I. F. Putnam, and T. Giordano. Topological orbit equivalence and c*-crossed products. 1995.