A model with no ordinal-definable stationary, costationary subset of ω_1

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Abstract

It is shown that the existence of a measurable cardinal is equiconsistent to a model of ZFC in which there is no ordinal-definable, stationary, costationary subset of ω_1

1 Introduction

The following short note has its origin in a question of B. Farkas who asked the author whether there is a model of ZFC in which every ordinal definable subset of ω_1 is either club-containing or nonstationary. It is a trivial observation that for any regular $\kappa > \omega_1$ there are always ordinal-definable, stationary, co-stationary subsets of κ, e.g. the ordinals below κ of some fixed cofinality $< \kappa$. This does not apply to ω_1 of course, thus the question seems very natural.

By work of R. Solovay from the late 1960's (see [1] Theorem 33.12), the axiom of determinacy implies that the club filter on ω_1 is an ultrafilter. The axiom of determinacy is however inconsistent with the axiom of choice. M. Spector ([6]) showed that ZF + "the club filter on ω_1 is an ultrafilter" is in fact equiconsistent with ZF, but due to the lack of choice, the club filter is not σ-closed, even though the intersection of countably many club sets is still club. If one wants a model for ZF + "the club filter on ω_1 is an ultrafilter" + DC, it is proved by W. Mitchell (see [3]) that a measurable cardinal κ with Michell rank $o(\kappa) = \kappa^{++}$ is an upper bound in terms of consistency strength. Note that all these results talk about models in which choice fails.

On the other hand R. Solovay ([5]) showed that any stationary set on a regular cardinal κ can be partitioned into κ-many stationary sets, its proof however relies heavily on the axiom of choice and does not allow in general any ordinal definable description of this partition. It implies nevertheless that models of ZFC with a definable wellorder on $P(\omega_1)$ allow definable, stationary, costationary subsets of ω_1.

We make use of two wellknown Inner models of ZFC, HOD and $L[U]$ for U a normal measure on a measurable cardinal κ. Recall that a set X is ordinal definable if there is a formula $\varphi(v_0, ..., v_n)$ and ordinals $\alpha_1, ..., \alpha_n$ such that

$$x \in X \text{ iff } \varphi(x, \alpha_1, ..., \alpha_n) \text{ is true.}$$

Then HOD, the class of hereditarily ordinal-definable sets is defined to be the collection of all sets X which are ordinal-definable and all the elements of the transitive closure of X are ordinal-definable as well. It was Gödel who showed that HOD is a definable class (definable without parameters) and HOD satisfies ZFC.

The second model is the canonical Inner model for one measurable cardinal $L[U]$. Assume that κ is a measurable cardinal and let U be a normal, nonprincipal, κ-complete ultrafilter on κ witnessing the measurability of κ.

Theorem 1. *The inner model $L[U]$ has the following properties (due to Silver (see [4]) and Kunen ([2]) respectivley):*

- *in $L[U]$ the GCH does hold.*

- *there is exactly one normal, κ-complete ultrafilter on κ namely $U \cap L[U]$. Consequentially κ is measurable in $L[U]$ and U is ordinal definable over $L[U]$.*

Our notation will be completely standard. $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$ will denote partial orders. Two partial orders $(\mathbb{P}, <_{\mathbb{P}})$ and $(\mathbb{Q}, <_{\mathbb{Q}})$ are isomorphic if there is a bijection $\pi : \mathbb{P} \to \mathbb{Q}$ which respects $<_{\mathbb{P}}$, i.e. $p_1 <_{\mathbb{P}} p_2$ if and only if $\pi(p_1) <_{\mathbb{Q}} \pi(p_2)$. We start to recall a property which will be crucial for the rest of this paper.

Definition 2. *A partial order $\mathbb{P}$ is weakly homogeneous if for every pair $p, q \in \mathbb{P}$ there is an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ such that $\pi(p)$ and q are compatible.*

The reason of its importance stems from the following.

Fact 3. *Let $\mathbb{P}$ be a homogeneous partial order and $\varphi(v_0, v_1, ..., v_n)$ be a formula with free variables $v_0, ..., v_n$. Then for any ground model sets $x_0, ..., x_n \in V$*

$$\Vdash_{\mathbb{P}} \varphi(x_0, ..., x_n) \text{ or } \Vdash_{\mathbb{P}} \neg\varphi(x_0, ..., x_n)$$

holds true.

Consequentially, if $\mathbb{P}$ is a homogeneous notion of forcing with generic filter G and A is an ordinal-definable set of ordinals in the extension $V[G]$, then A was already an element of V. Indeed if A is ordinal-definable over $V[G]$ using the formula $\varphi(v_0, ..., v_n)$ then $\alpha_0 \in A$ iff $\varphi^{V[G]}(\alpha_0, \alpha_1, ..., \alpha_n)$ iff there is a $p \in G \cap \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \varphi(\alpha_0, \alpha_1, ..., \alpha_n)$ and by weak homogeneity

this is equivalent to $1 \Vdash_\mathbb{P} \varphi(\alpha_0, \alpha_1, ..., \alpha_n)$. But the latter is an A-defining formula over V, thus $A \in V$ as desired. If the forcing notion $\mathbb{P}$ is additionally assumed to be ordinal-definable itself, then an ordinal-definable subset A of $V[G]$ is in fact ordinal-definable over the ground model V for the relation $\Vdash_\mathbb{P}$ and the weakest element $1_\mathbb{P}$ becomes ordinal-definable in that case. As $\alpha_0 \in A$ iff $1 \Vdash_\mathbb{P} \varphi(\alpha_0, ..., \alpha_n)$, the latter is an ordinal-definable formula in case that $\mathbb{P}$ is ordinal-definable, thus A, which was assumed to be ordinal-definable over $V[G]$, is ordinal-definable over V.

We recall another standard fact, namely the club shooting forcing for a stationary $S \subset \omega_1$ which adds a closed unbounded subset of S.

Definition 4. *Let $S \subset \omega_1$ be a stationary subset. The club shooting forcing $\mathbb{P}_S$ consists of closed, countable subsets of S as conditions, ordered by end extension.*

The definition of club shooting relies on the fact that stationary subsets S of ω_1 contain for every $\alpha < \omega_1$ a closed subset $C \subset S$ of ordertype α. This statement becomes wrong for regular cardinals $\kappa > \aleph_1$. The next fact is a wellknown folklore result and we state it without a proof:

Fact 5. *For a stationary set S, the club shooting $\mathbb{P}_S$ is ω-distributive and weakly homogeneous.*

2 The proof

Goal of this section is a proof of the following theorem.

Theorem 6. *The existence of measurable cardinal is equiconsistent to the existence of a model of* ZFC *in which there is no ordinal-definable, stationary, costationary subset of ω_1.*

Proof. One direction of the equiconsistency is observed easily. Assume that V is a model of ZFC in which there is no ordinal-definable, stationary, costationary subset of ω_1. Consider the club filter of V, $U := \{X \subset \omega_1 : X$ contains a club$\}$. It is clear that U is ordinal-definable and hence $U \cap HOD$ is an element of HOD, the class of hereditarily ordinal definable subsets of V. As any V-ordinal-definable subset of ω_1 is either club containing or nonstationary we see that $U \cap HOD$ is a nontrivial ultrafilter in HOD. As any subset $X \in HOD$ of ω_1 can easily be split into two proper subsets which are elements of HOD, it follows that $U \cap HOD$ is also nonprincipal. Finally, as the intersection of less than ω_1^V-many club sets are club again, we see that $U \cap HOD$ is an ω_1^V-closed ultrafilter on ω_1^V in HOD, hence $U \cap HOD$ witnesses that ω_1^V is measurable in HOD.

The other direction is more involved and we start to prove it now. Assume there is a measurable cardinal κ, and let the ultrafilter U witness this.

We let $V = L[U]$ be our ground model, which satisfies GCH and has κ as a measurable cardinal, with U being the unique normal κ-closed ultrafilter witnessing the measurability of κ in $L[U]$. Note that U is ordinal definable in $L[U]$ as it is the unique ultrafilter on κ which is κ-closed and normal. Further note that U is also ordinal definable in all set-sized generic forcing extensions $L[U][G]$ of $L[U]$, it is just the unique set X such that X is a normal, κ-complete ultrafilter on κ in $L[X]$.

In the next step we take $L[U]$ as the ground model and collapse the cardinal κ to ω_1 using the Levy collapse. More particularly we let $\mathbb{Q}$ consist of functions p on subsets of $\kappa \times \omega$ such that

1. $|dom(p)|$ is finite and

2. $p(\alpha, n) < \alpha$ for every $(\alpha, n) \in dom(p)$.

It is wellknown that the Levy collapse has the κ-cc, hence if G_0 denotes the generic filter for $\mathbb{Q}$ we obtain that $L[U][G_0] \models \kappa = \aleph_1$. As a consequence of the κ-cc of $\mathbb{Q}$, every stationary subset of ω_1 in $L[U][G_0]$ will contain an $S' \subset \kappa$ which is a stationary element of $L[U]$. The Levy collapse is a weakly homogeneous forcing, which yields that every ordinal-definable stationary, costationary $S \subset \omega_1$ is in fact an element of $L[U]$ already. Indeed if there is a formula $\varphi(x, \beta_0, ..., \beta_n)$ such that $\alpha \in S$ if and only if $L[U][G_0] \models \varphi(\alpha, \beta_0, ..., \beta_n)$ then the latter is equivalent to $L[U] \models \Vdash_{\mathbb{Q}} \varphi(\alpha, \beta_0, ..., \beta_n)$ witnessing the ordinal-definability of S over $L[U]$ already. Note however that S will be a stationary, costationary subset of κ in $L[U]$ as we have not collapsed κ in $L[U]$.

We are now in the position to define the final iteration. Let $L[U][G_0]$ be the ground model. In $L[U][G_0]$, list the ordinal-definable, stationary, co-stationary subsets of ω_1, write the list as $(S_i : i < \omega_2)$. We can assume that this enumeration is ordinal definable over $L[U][G_0]$ as every ordinal-definable, stationary, costationary $S \subset \omega_1$ in $L[U][G_0]$ is, by the homogeneity of the Levy collapse already an element of $L[U]$ and hence we can use the canonical wellorder $<_{L[U]}$, which is ordinal definable over $L[U][G_0]$, to wellorder the S_i's. Let $\mathbb{R} := (\mathbb{R}_\alpha : \alpha < \omega_2)$ be an ω_2-length product defined inductively as follows: suppose we are at stage $\alpha < \omega_2$, and $\mathbb{R}_\alpha$ has already been defined. Suppose further that S_α is an ordinal-definable, stationary, costationary subset of ω_1 in $L[U][G_0]$. Note here again that even though κ has been collapsed to ω_1 in $L[U][G_0]$, by the homogeneity of the Levy collapse, if S_α is an ordinal-definable set in $L[U][G_0]$ it is also an element of $L[U]$. As U is ordinal-definable over $L[U][G_0]$, the question whether S_α is an element of U is therefore definable in $L[U][G_0]$, hence we can define the α-th forcing of the product as follows:

- If S_α is stationary in $L[U][G_0]$ and an element of U we force with the club shooting forcing $\mathbb{R}_\alpha := \mathbb{P}_{S_\alpha}$ which shoots a club set through S_α.

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- If S_α is not an element of U we do nothing.

We use countable support for the product, which is ordinal definable over $L[U][G_0]$. Let G_1 be a generic filter for the product $(\mathbb{R}_\alpha : \alpha < \omega_2)$. To finish the proof we need to show two things, namely that $\omega_1^{L[U][G_0][G_1]} = \omega_1^{L[U][G_0]}$ and that in $L[U][G_0][G_1]$ there are no ordinal-definable, stationary, costationary subsets of ω_1.

Claim 7. *The product* $(\mathbb{R}_\alpha : \alpha < \omega_2)$ *is ω-distributive and therefore preserves $\aleph_1$.*

Proof. This is proved by induction on $\alpha < \omega_2$. First note that for a stationary set $S \subset \omega_1$, club shooting itself is an ω-distributive forcing which immediately gives the successor case. For limit ordinals α, note that $\mathbb{R}_\alpha$ is the countably supported product of certain club shooting forcings $\prod_{\beta<\alpha} \mathbb{P}_{S_\beta}$. We can assume without loss of generality that $\alpha = \omega_1$ as we can pick a bijection $f : \alpha \to \omega_1$ and use the fact that $\prod_{\beta<\alpha} \mathbb{P}_{S_\beta}$ is isomorphic to $\prod_{f(\beta)<f(\alpha)} \mathbb{P}_{S_{f(\beta)}}$. By definition, we only used a club shooting forcing $\mathbb{P}_{S_\beta}$ for an $S_\beta, \beta < \omega_1$ if S_β was an element of the ultrafilter U. As U is normal and hence closed under diagonal intersections, $\Delta_{\beta<\omega_1} S_\beta$ is an element of U again. Moreover elements of U are stationary subsets of κ in $L[U]$, and by the κ-cc of the Levy collapse they remain stationary in $L[U][G_0]$, so $\Delta_{\beta<\omega_1} S_\beta$ is a stationary subset of ω_1 in $L[U][G_0]$. Let $\dot{r}$ be the $\mathbb{R}_{\omega_1}$-name of a real in $L[U][G_0]$. We shall show that there is a condition $q \in \mathbb{R}_{\omega_1}$ and a real $r \in L[U][G_0]$ such that $q \Vdash \dot{r} = r$. Fix a countable model $M \prec H_\theta$ for θ sufficiently large and which contains $\dot{r}$. As $\Delta_{\beta<\omega_1} S_\beta$ is stationary in $L[U][G_0]$ we can assume that $M \cap \omega_1 \in \Delta_{\beta<\omega_1} S_\beta$. We list the dense subsets $(D_n : n \in \omega)$ of $\mathbb{R}_{\omega_1}$ which are in M and recursively construct a descending sequence of elements of $\mathbb{R}_{\omega_1} \cap M$, $(p_n : n \in \omega)$, such that $p_n \in D_n$ and p_{n+1} decides the value of $\dot{r}(n)$ and such that for every coordinate $i \in \omega_1$ such that there is an $n \in \omega$ for which $p_n(i) \neq 1$, the value $sup_{n\in\omega} p_n(i) \cap \omega_1 = M \cap \omega_1$. We claim that the lower bound for the sequence $(p_n : n \in \omega)$, taken coordinatewise, i.e. $q := \prod_{\beta\in\omega_1} \bigcup_{n\in\omega} p_n(\beta)$ is a condition in $\mathbb{R}_{\omega_1}$. Indeed every nontrivial coordinate i of a condition p_n in the descending sequence is a countable, closed subset of the ordinal-definable, stationary, costationary set S_i. Thus it is sufficient to show that for every coordinate i in the support of q, the set $\bigcup_{n\in\omega} p_n(i)$ is a closed subset of S_i, which boils down to ensure that $sup_{n\in\omega}(p_n(i) \cap \omega_1)$ is an element of S_i. But $(p_n : n \in \omega) \subset M$ and $M \cap \omega_1 \in \Delta_{\beta<\omega_1} S_\beta$, and as $supp(q) \subset M \cap \omega_1$ we know that for every $i \in supp(q)$ it must hold that $sup_{n\in\omega}(p_n(i) \cap \omega_1) = M \cap \omega_1$ is an element of S_i. So q is a condition in $\mathbb{R}_{\omega_1}$ which witnesses that the name of a real $\dot{r}$ is already an element of $L[U][G_0]$.

To show that the full product $\mathbb{R} = \prod_{\alpha<\omega_2} \mathbb{R}_\alpha$ is ω-distributive note that in our ground model $L[U][G_0]$ the club shooting forcings have size $\aleph_1$. As a consequence every $\mathbb{R}$-name of a real $\dot{r}$ is in fact a name in some initial

segment $\prod_{\beta<\alpha} \mathbb{R}_\beta$ for $\alpha < \omega_2$. But we have seen already that $\prod_{\beta<\alpha} \mathbb{R}_\beta$ is ω-distributive, so we are finished.

$\square$

What is left is to show that in $L[U][G_0][G_1]$ there are no ordinal-definable, stationary, costationary subsets of ω_1. Assume for a contradiction that S would be such a set in $L[U][G_0][G_1]$. Then as $\mathbb{R}$ is the product of homogeneous forcings it is homogeneous itself and so S is an element of $L[U][G_0]$. The forcing $\mathbb{R}$ is ordinal-definable over $L[U][G_0]$ as the sequence of the ordinal-definable, stationary, costationary subsets of ω_1 in $L[U][G_0]$ is itself ordinal definable there, using the canonical (ordinal-definable) wellorder of $L[U]$. Indeed, working in $L[U][G_0]$ we can first define U to be the unique κ-closed and normal ultrafilter X on κ which is an element of $L[X]$, and having defined U we can also define the canonical $L[U]$-wellorder in an ordinal-definable way, which we use to define (with $\omega_1 = \kappa$ as the parameter) the sequence of ordinal-definable, stationary, costationary subsets of ω_1 in $L[U][G_0][G_1]$. But as $\mathbb{R}$ is a homgeneous, ordinal-definable forcing over $L[U][G_0]$ we see that any ordinal definable set S over $L[U][G_0][G_1]$ is in fact ordinal definable over $L[U][G_0]$: if $\varphi(x, \alpha, ..)$ is such that $\beta \in S$ if and only if $L[U][G_0][G_1] \models \varphi(\beta, \alpha, ...)$ then the latter is equivalent to $L[U][G_0] \models 1 \Vdash_{\mathbb{R}} \varphi(\check{\beta}, \check{\alpha}, ...)$ and the relation $\Vdash_{\mathbb{R}}$ and the weakest condition 1 are ordinal definable. To summarize, an ordinal-definable, stationary, costationary element S of $L[U][G_0][G_1]$ is ordinal-definable, stationary, co-stationary already in $L[U][G_0]$. But we have killed the stationarity or co-stationarity of every such set with the forcing $\mathbb{R}$. So there is no S which is ordinal-definable and stationary and costationary in $L[U][G_0][G_1]$ as desired.

$\square$

Vielleicht noch ein paar Fragen, schreibe auch etwas zu PMax?

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