General Relativity and the Analysis of Black Hole Spacetimes

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Abstract

Notes under construction. All comments and corrections welcome.

Contents

1 Introduction

The main goal of this course is to introduce the mathematical framework of Einstein's theory of general relativity. As we shall see soon, this will require ideas from **differential geometry**, **analysis** of PDEs and **physics**.

Perhaps you have already seen the Einstein equations written down

$$\mathrm{Ric}_g - \frac{1}{2} R_g g = 8\pi T \,. \tag{1}$$

These are equations imposed on a (4-dimensional) Lorentzian manifold $(\mathcal{M}, g)$, a manifold $\mathcal{M}$ equipped with a piece of additional structure on its tangent spaces, namely a Lorentzian metric (a bi-linear form of signature $(-,+,...+)$ on each tangent space that varies smoothly over $\mathcal{M}$). The tuple $(\mathcal{M}, g)$ is called a spacetime and supposed to represent the time-space continuum that we live in. The equations (1) relate the curvature of the spacetime to its matter content T (the energy momentum tensor). As we shall see, (1) abandon the idea of gravity as a "force" and instead incorporate it into the curvature of the spacetime.

The plan is to

- address the geometric part of (1): I will introduce the concepts of differential geometry (manifold, tangent space, Lie-derivative, connection, curvature tensors etc.) that allow us to formulate (1). This might be boring to those of you who have taken a differential geometry course but I will include it for completeness and the benefit of those who do not have this background. We will often take a pedestrian approach and computational point of view (in particular using indices in tensor computations) as we will often need to make very concrete computations to understand the physics!

- to understand (1) from the PDE point of view (analysis). Note that the equations (1) are coordinate invariant (unlike say, Newton's equation $m\ddot{x} = F$ whose formulation already involves a notion of (absolute) time). This is a manifestation of Einstein's idea of *general covariance*: The physical laws should take the same form in any coordinate system. However, in order to do analysis (or to measure actual physical quantities) we need to choose a coordinate system. In turns out that there are coordinates such that when we write (1) in these coordinates, we obtain a system of non-linear wave equations. We will spend some time understanding (at least toy-models for) linear and non-linear wave equations, to formulate the Cauchy problem for (1). The Einstein equations are evolution equations!

- to understand geometrically some exact solutions of (1): flat space (from a geometric point of view), the Schwarzschild metric (1916) and the Kerr metric (1963), the latter two being examples of black hole geometries. Here we will introduce Penrose diagrams and compute some of the geodesics on these spacetimes to gain further insight (geodesics on spacetime will correspond to observers (timelike) and light rays (null))

- to understand at least heuristically some of the physics: How does one "derive" the equations (1)? Why are the equations so successful? (classical tests of general relativity: perihelion of mercury, light bending; also modern GPS relies on GR corrections to SR)

Once we have understood the above, we will study the covariant wave equation $\Box_g \psi = 0$ on black holes as a toy-model for the Einstein equations to gain insight into the stability properties of black holes. As promised in the announcement we will also prove Penrose's incompleteness theorem.

1.1 Books

The following books are useful for complementary reading and contain many more details:

1. Robert Wald, "General Relativity", The University of Chicago Press

2. John Stewart, "Advanced General Relativity", Cambridge Monographs on Mathematical Physics

3. Hawking and Ellis, "The large scale structure of spacetime, Cambridge University Press,

4. Barrett O'Neill, "Riemannian Geometry", Academic Press

5. Hans Ringström, "The Cauchy Problem in General Relativity", EMS

For Chapter 5 I recommend also the (now a bit outdated but nicely written and providing a lot of background) 2008 lecture notes of Mihalis Dafermos and Igor Rodnianski and the (more recent) ETH lecture notes of Dafermos, both available from `https://web.math.princeton.edu/~dafermos/teaching.html`.

2 Review of differential geometry

2.1 Manifolds, smooth maps and tangent spaces

We start with the fundamental concept of a manifold which is of course central for stating (1).

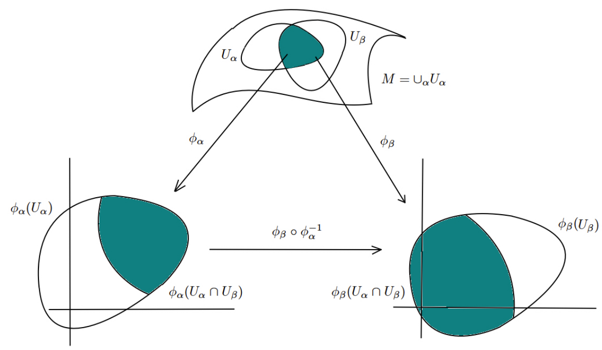
Definition 2.1. *An n-dimensional smooth manifold is a topological space M (Hausdorff, second countable) together with a collection of open sets U_α and homeomorphisms $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ with the following properties*

(1) $M = \cup_\alpha U_\alpha$.

(2) If $U_\alpha \cap U_\beta \neq \emptyset$ then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth diffeomorphism.

(3) The collection (U_α, ϕ_α) is maximal with respect to (1) and (2).

The (U_α, ϕ_α) are called (coordinate) charts, the $\phi_\beta \circ \phi_\alpha^{-1}$ transition maps. A collection of (U_α, ϕ_α) satisfying (1) *and* (2) *is called an atlas (or smooth structure) for M.*



Remark 2.2. *One may show that a given atlas on M (which for instance can contain only finitely many charts) is contained in a unique maximal atlas. Keep in mind, however, that there are inequivalent atlases (in the sense that their union is not an atlas) even on $\mathbb{R}$. See the Exercises.*

Remark 2.3. *The Hausdorff assumption is required for uniqueness of convergent sequences, the second countability assumption for the existence of a partition of unity. See the Exercises.*

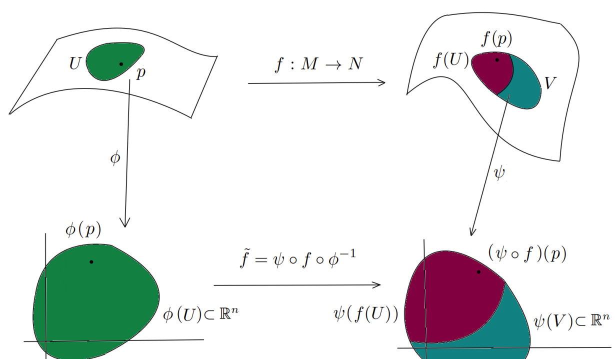
Exercise 2.4. *(Exercises for those who have never seen manifolds.) Show that* $\mathbb{R}^n$ *is a smooth manifold. Show that* $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$ *is a smooth manifold. Is the cone* $C = \{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{x^2 + y^2}\}$ *a smooth manifold? (The answer may depend on the smooth structure you want to install.)*
Show that an open subset $V \subset M$ *of a smooth manifold is a smooth manifold. Show that if* M *and* N *are smooth manifolds, then* $M \times N$ *is a smooth manifold.*

In order to do calculus on manifolds, we need to define functions on them and be able to differentiate them:

Definition 2.5. *A function* $f : M \to N$ *between two manifolds (of dimension* m *and* n*) is called smooth at* p *if it is smooth in coordinates, i.e. if given a chart* (V, ψ) *around* $f(p)$ *one can find a chart* (U, ϕ) *around* p *such that* $f(U) \subset V$ *and*

$$\tilde{f} = \psi \circ f \circ \phi^{-1}$$

is smooth at $\phi(p)$*. We say that* f *is smooth on an open set* U *in* M *if it is smooth at all points of* U*.*



Remark 2.6. *Note that this definition is independent of the charts by the smoothness of the transition functions (why?).*

We now recall the definition of the tangent space. To motivate it, it is useful to recall briefly the situation in $\mathbb{R}^n$. Given $\gamma : I \to \mathbb{R}^n$ a smooth curve $\gamma(t) = (x^1(t), \dots, x^n(t))$, $\gamma(0) = p$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we can consider the *rate of change of* f *along* γ:

$$\frac{d}{dt}(f \circ \gamma)|_{t=0} = \frac{d}{dt} f(x^1(t), \dots, x^n(t))|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_p \frac{dx^i}{dt}(0) = \sum_{i=1}^{n} \left[(x^i)'(0) \frac{\partial}{\partial x^i} \right] f.$$

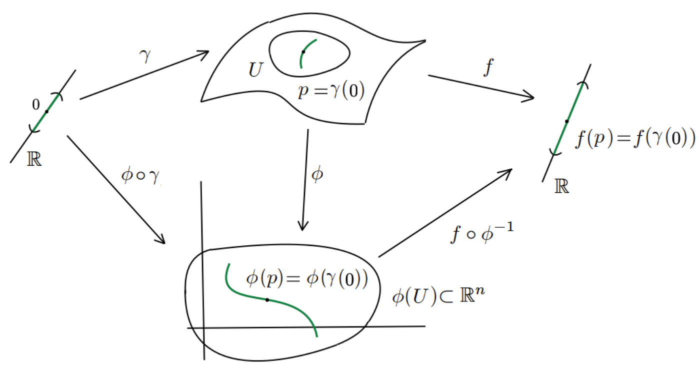
Conversely, given a (tangent) vector in $\mathbb{R}^n$ one may interpret it as an operator acting on real-valued functions and spitting out a number.

We let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve on a smooth manifold and denote by $C_p^\infty(M)$ the space of smooth functions at p.

Definition 2.7. *Given a smooth curve through p and $f \in C_p^\infty(M)$, we define the tangent vector to γ at p (denoted γ_p' or $\gamma'(0)$) as*

$$\begin{aligned} \gamma_p' : C_p^\infty(M) &\to \mathbb{R} \\ f &\mapsto \frac{d}{dt}(f \circ \gamma)|_{t=0} \,. \end{aligned} \tag{2}$$

The set of all tangent vectors at p that can be obtained from curves in this way is called the tangent space of M at p and denoted T_pM.



In a chart we have

$$\phi \circ \gamma = (x^1(t), \ldots, x^n(t))$$

and $f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma$ hence

$$\begin{aligned} \frac{d}{dt}(f \circ \gamma)|_{t=0} &= \sum_{i=1}^n \frac{\partial(f \circ \phi^{-1})}{\partial x^i}\Big|_{\phi(p)} \frac{d}{dt}(\phi \circ \gamma)^i\Big|_{t=0} \\ &= \sum_{i=1}^n \left[\frac{\partial}{\partial x^i}\right]_p f \cdot (x^i)'(0) \,, \end{aligned} \tag{3}$$

where we have defined

$$\left[\frac{\partial}{\partial x^i}\right]_p f := \frac{\partial(f \circ \phi^{-1})}{\partial x^i}\Big|_{\phi(p)} .$$

Note that if we choose γ_k such that $\phi \circ \gamma_k$ corresponds to the k^{th} coordinate curve, then

$$(\gamma_k')_p f = \left[\frac{\partial}{\partial x^k}\right]_p f \,,$$

so $\left[\frac{\partial}{\partial x^k}\right]_p$ is the tangent vector corresponding to the k^{th} coordinate curve. In the exercises you will show that T_pM is an n-dimensional vectorspace with basis $\left[\frac{\partial}{\partial x^1}\right]_p, \ldots, \left[\frac{\partial}{\partial x^n}\right]_p$. Hence from (3) we see that $\frac{d}{dt}(\phi \circ \gamma)^i\Big|_{t=0}$ are then the components of the tangent vector in this basis.

Remark 2.8. *Note that the above implies that two curves $\gamma, \tilde{\gamma} : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = \tilde{\gamma}(0) = p$ define the same tangent vector provided they satisfy $\frac{d}{dt}(\phi \circ \gamma)\big|_{t=0} = \frac{d}{dt}(\phi \circ \tilde{\gamma})\big|_{t=0}$ in some (hence any) chart ϕ at p, where the identity is one of vectors in $\mathbb{R}^n$. We can therefore associate a tangent vector with an equivalence class of curves, which is sometimes used as the definition of the tangent space. In particular, given $v \in T_pM$ we can unambiguously write $v(f) = \gamma'_p f$.*

We finally define the differential (or push-forward) of a function at p:

Proposition 2.9. *Let $f : M \to N$ be smooth, $\dim M = m$, $\dim N = n$. For every $p \in M$, $v \in T_pM$ choose*

$$\gamma : (-\epsilon, \epsilon) \to M$$

with $\gamma(0) = p$, $\gamma'_p = v$ and define $\beta = f \circ \gamma$, which is a smooth curve in N through $f(p)$ with tangent $\beta'_{f(p)}$. Then the map

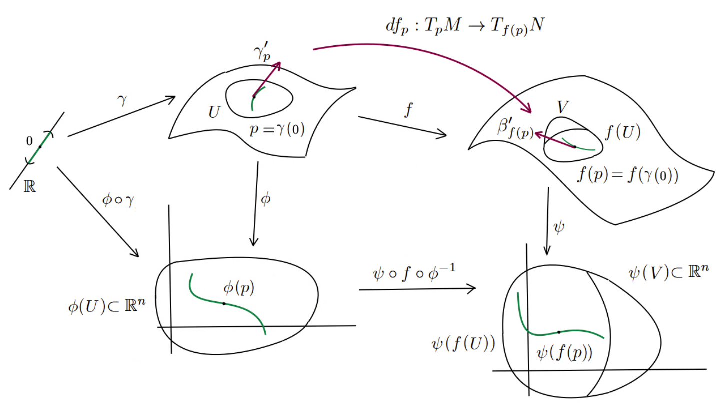
$$df_p : T_pM \to T_{f(p)}N \tag{4}$$

$$\gamma'_p \mapsto \beta'_{f(p)} \tag{5}$$

is linear and does not depend on the choice of γ.

Definition 2.10. *The map df_p defined in Proposition 2.9 is called the differential (of f at p) or the pushforward (of tangent vectors at p by f).*

Proof. We first draw a picture of the situation:



We know

$$\psi \circ f \circ \phi^{-1} = \big(y^1(x^1, \ldots, x^n), \ldots, y^m(x^1, \ldots, x^n)\big)$$

and

$$\phi \circ \gamma = \big(x^1(t), \ldots, x^m(t)\big)$$

From $\psi \circ \beta = \psi \circ f \circ \phi^{-1} \circ \phi \circ \gamma$ we compute what we know to be the components of $\beta'_{f(p)}$ in the basis $\left[\frac{\partial}{\partial y^i}\right]_{f(p)}$, namely

$$\frac{d}{dt}(\psi \circ \beta)\big|_{t=0} = \left(\sum_{i=1}^{m} \frac{\partial y^1}{\partial x^i}\Big|_{\phi(p)} (x^i)'(0), \ldots, \sum_{i=1}^{m} \frac{\partial y^n}{\partial x^i}\Big|_{\phi(p)} (x^i)'(0)\right). \tag{6}$$

But we also know that $(x^i)'(0)$ are the components of γ'_p in the basis $\left[\frac{\partial}{\partial x^i}\right]_p$. Therefore, expressed in these bases, the map df_p is multiplication by the matrix $\frac{\partial y^i}{\partial x^k}\big|_{\phi(p)}$ hence a linear map which also clearly does not depend on the choice of γ. □

Exercise 2.11. *Let $f : M \to N$ be smooth, $v \in T_pM$ and $g : N \to \mathbb{R}$ differentiable. Prove that*

$$(df_p v)g|_{f(p)} = v(g \circ f)|_p .$$

Definition 2.12. *Let M and N be smooth manifolds. A map $f : M \to N$ is a diffeomorphism if it is smooth, bijective and its inverse f^{-1} is smooth. The map f is called a local diffeomorphism at $p \in M$ if there exists neighbourhoods U of p and V of $f(p)$ such that $f : U \to V$ is a diffeomorphism.*

From the point of view of smooth manifolds, manifolds which are diffeomorphic to one another should be considered “the same”.

Note that if $f : M \to N$ is a diffeomorphism, then df_p is an isomorphism for all $p \in M$ (why?). Conversely, if df_p is an isomorphism, then f is a local diffeomorphism by the implicit function theorem.

2.2 Tensor algebra (tensors at a point)

We collect the transformation formulae for vectors and covectors. Recall that we can write a $X \in T_pM$ as $X = X^i[\frac{\partial}{\partial x^i}]_p$, where the X^i are the components of the vectorfield X in the basis $[\frac{\partial}{\partial x^i}]_p$ (which is induced by a coordinate chart (U, ϕ) at p as seen above). What happens with the coordinate basis if we change the chart? Let (W, ψ) be another chart at p with y^i coordinates. We compute from the definition

$$\begin{aligned}\left[\frac{\partial}{\partial x^j}\right]_p f = \frac{\partial}{\partial x^j}(f \circ \psi^{-1} \circ (\psi \circ \phi^{-1}))\big|_{\phi(p)} &= \frac{\partial}{\partial y^i}(f \circ \psi^{-1})\big|_{\phi(p)} \cdot \frac{\partial(\psi \circ \phi^{-1})^i}{\partial x^j}(\phi(p)) \\ &= \left[\frac{\partial}{\partial y^i}\right]_p f \cdot \frac{\partial y^i}{\partial x^j}(\phi(p)),\end{aligned} \tag{7}$$

which provides the transformation rule

$$\left[\frac{\partial}{\partial x^j}\right]_p = \frac{\partial y^i}{\partial x^j}\left[\frac{\partial}{\partial y^i}\right]_p , \tag{8}$$

hence the basis vectors transform with the Jacobian of the coordinate change. We have used in (7) and (8) the *Einstein summation convention* stating that repeated indices are always summed over allowing us to skip a $\sum_{i=1}^n$.

We define the cotangent space $T_p^\star M$ at the dual space of T_pM equipped with coordinate basis $[dx^1]_p, \ldots, [dx^n]_p$ defined by $[dx^i]_p\left(\left[\frac{\partial}{\partial x^j}\right]_p\right) = \delta^i_j$. Given $\alpha \in T_p^\star M$ we can write $\alpha = \alpha_i dx^i$, where the α_i are the components of the co-vector α in the (basis induced by the) chart ϕ. It is an easy exercise to compute the transformation rule

$$\left[dx^j\right]_p = \frac{\partial x^j}{\partial y^i}\left[dy^i\right]_p ,$$

that is the dual basis of co-vectors transforms by the inverse of the Jacobian of the coordinate change. Note that this also tells one how the *components* of a vector and the *components* of a co-vector transform under a change of basis (why?).

We now define the (k,l)-tensor space $T_p^{(k,l)}M$ to consist of all multi-linear maps

$$T : T_p^\star M \times \ldots \times T_p^\star M \times T_pM \times \ldots \times T_pM \to \mathbb{R} , \tag{9}$$

where we have k copies of $T_p^\star M$ and l copies of T_pM on the left. Note that a $(0,1)$-tensor is just a co-vector and a $(1,0)$-tensor is a vector (why?).

We define the elements (let us drop the $[...]_p$ from the notation for simplicity)

$$\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l}$$

where $1 \leq i_1, ..., i_k \leq n$ and $1 \leq j_1, ..., j_l \leq n$ by

$$\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l}\left(dx^{a_1}, \ldots, dx^{a_k}, \frac{\partial}{\partial x^{b_1}}, \ldots, \frac{\partial}{\partial x^{b_l}}\right) = \delta^{a_1}_{i_1} \cdot \ldots \cdot \delta^{a_k}_{i_k}\delta^{b_1}_{j_1} \cdot \ldots \cdot \delta^{b_l}_{j_l} . \tag{10}$$

They clearly form a basis of $T_p^{(k,l)}M$. Given $T \in T_p^{(k,l)}M$ we can thus write

$$\begin{aligned} T &= T^{i_1\ldots i_k}{}_{j_1\ldots j_l}\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l} \\ &= \tilde{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l}\frac{\partial}{\partial y^{a_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{a_k}} \otimes dy^{b_1} \otimes \ldots \otimes dy^{b_l} \end{aligned} \tag{11}$$

which expresses the tensor in two different bases. We have

$$T^{i_1\ldots i_k}{}_{j_1\ldots j_l} = T\left(dx^{i_1}, \ldots, dx^{i_k}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_l}}\right) \tag{12}$$

and

$$\tilde{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} = T\left(dy^{a_1},\ldots,dy^{a_k},\frac{\partial}{\partial y^{b_1}},\ldots,\frac{\partial}{\partial y^{b_l}}\right), \tag{13}$$

from which one easily deduces the transformation rule for the components of a $(k,l)-tensor$:

$$\tilde{T}^{a_1\ldots a_k}{}_{b_1\ldots b_l} = T^{i_1\ldots i_k}{}_{j_1\ldots j_l}\frac{\partial y^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{a_k}}{\partial x^{i_k}}\frac{\partial x^{j_1}}{\partial y^{b_1}}\cdots\frac{\partial x^{j_l}}{\partial y^{b_l}} \tag{14}$$

Note that this also implies that if two tensors agree in one coordinate system, they must agree in all others (why?).

Finally, we can define the operation of *contraction* of a tensor (over the i^{th} covariant and the j^{th} contravariant slot). More precisely, contraction is a map $C^i_j : T_p^{(k+1,l+1)}M \to T_p^{(k,l)}M$ defined by $C^i_j T = \sum_{k=1}^n T\left(\cdot,\cdot,dx^k,\cdot,...,\cdot,\frac{\partial}{\partial x^k},\cdot,\cdot\right)$. Note that this map does not depend on the choice of coordinates (why?).

2.3 Tensor-fields

So far we have been looking at tensors at a point and their transformation properties. We now want to define tensors which vary smoothly from point to point on the manifold (and eventually differentiate them etc.). We give a pedestrian definition:

Definition 2.13. *A smooth (k,l)-tensor field T on M is a map*

$$M \ni p \mapsto T(p) \in T_p^{(k,l)}M$$

for all $p \in M$ such that in local coordinates $\phi : M \supset U \to V \subset \mathbb{R}^n$ the components $T^{i_1\ldots i_k}{}_{j_1\ldots j_l} : V \to \mathbb{R}$ are smooth functions.

Remark 2.14. *More abstractly (but equivalently), we can define the tensor-bundle*

$$T^{(k,l)}M = \{(p,S) \mid p \in M\ ,\ S \in T_p^{(k,l)}M\}$$

which can be shown to be a smooth manifold[1] *and define a smooth (k,l)-tensor field T on M to be a smooth section of the tensor bundle $T^{(k,l)}M$, i.e. a smooth map $T : M \to T^{(k,l)}M$ such that $\pi \circ T = id$ where $\pi : TM \to M$ denotes the projection on M.*

We denote by $\mathcal{X}(M)$ the space of smooth vectorfields on M and by $\Omega^1(M)$ the space of smooth co-vector fields (1-forms) on M.

In the exercises you will prove

[1] This is an exercise. The set $T^{(k,l)}M$ inherits a natural differentiable structure from M: Use as coordinates the coordinates of p induced by the chart and the coordinates of S in the coordinate basis induced by the chart as seen above.

Exercise 2.15. *A map*

$$\tau : \Omega^1(M) \times \ldots \times \Omega^1(M) \times \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \to C^\infty(M) \tag{15}$$

with k copies of $\Omega^1(M)$ and l copies of $\mathcal{X}(M)$ is induced by a (k,l)-tensor field if and only if it is multi-linear over $C^\infty(M)$, so for each slot $\tau(fX+Y,\cdot,\ldots,\cdot) = f\tau(X,\cdot,\ldots,\cdot) + \tau(Y,\cdot,\cdot,\ldots,\cdot)$ holds for $f \in C^\infty(M)$.

The above exercise provides a convenient way to identify multilinear maps as tensorfields.

2.4 The Lorentzian metric and Lorentzian manifolds

Having defined tensorfields, we can finally introduce the fundamental object of Lorentzian geometry.

Definition 2.16. *A Lorentzian metric g on M is a smooth $(0,2)$-tensor field such that at every point $p \in M$*

$$g(p) : T_pM \times T_pM \to \mathbb{R}$$

is a Lorentzian inner product (defined immediately below). We call a pair (M,g),i.e. a manifold equipped with a Lorentzian metric a Lorentzian manifold.

In local coordinates we have $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$ (or more precisely $g(p) = g_{\mu\nu}(\phi(p))dx^\mu \otimes dx^\nu$ where the $g_{\mu\nu}$ are smooth function of the coordinates of the chart ϕ).

2.4.1 A brief intermezzo: Lorentz inner products

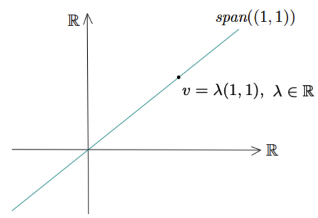
Definition 2.17. *Let V be an $(n+1)$-dimensional vector space.*
A Lorentzian inner product is a map

$$m : V \times V \to \mathbb{R}$$

which is

- *bi-linear*
- *symmetric ($m(v,w) = m(w,v)$)*
- *non-degenerate (if $v \in V$ satisfies $m(v,w) = 0$ for all $w \in V$, then $v = 0$)*
- *maximal dimension of any subspace $W \subset V$ such that $m|_W$ is positive definite is n.*

Example 2.18. *Let $m : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by $m(v,w) = v_1w_1 - v_2w_2$. Check that it is an example. Note that $v \in$ span$((1,1))$ has $m(v,v) = 0$, so span$((1,1))$ forms a degenerate subspace.*



Recall that a symmetric bi-linear form is completely determined by the associated quadratic form $q(v) = m(v,v)$. Indeed, by polarization we have $m(v,w) = \frac{1}{2}(q(v+w) - q(v) - q(w))$. Moreover, given a basis $e_0, e_1, \ldots, e_n$, $m_{ij} = m(e_i, e_j)$ are the components of m. The non-degeneracy condition implies that the symmetric matrix m_{ij} has full rank and is hence invertible.

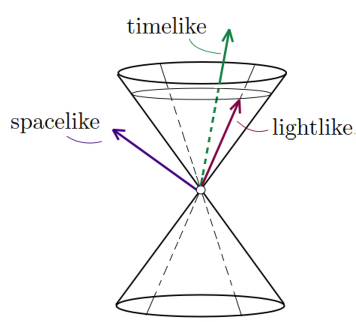
Proposition 2.19. *If V is a vector space and m a Lorentzian inner product, then there exists a basis $e_0, e_1, \ldots, e_n$ such that $m(e_0, e_0) = -1$ and $m(e_i, e_j) = \delta_{ij}$.*

Proof. Linear algebra exercise. □

In other words, any Lorentzian inner product space (V, m) is isometric with $(\mathbb{R}^4, \eta)$ where $\eta = \text{diag}(-1, 1, 1, 1)$.

Definition 2.20. *Let V be as above. A vector $v \neq 0$ is said to be timelike if $m(v,v) < 0$, null if $m(v,v) = 0$ and spacelike if $m(v,v) > 0$. The zero vector is by convention spacelike. Timelike and null vectors are collectively knows as causal.*

One easily sees that the set of null vectors forms a double null cone N in V (minus the origin).



$N \cup \{0\}$ then partitions the space into three connected components, two inside the cone (corresponding to the set of timelike vectors), which we call I, and one (unless $n = 1$ in which case two or $n = 0$ in which case none) outside (corresponding to the set of non-zero spacelike vectors), which we call S. The cone N is also known as the light cone and null vectors as lightlike.

More generally, we have

Definition 2.21. *A subspace $W \subset V$ is called*

- *spacelike if $m|_W$ defines a Euclidean inner product*
- *timelike if $m|_W$ defines a Lorentzian inner product*
- *null if $m|_W$ defines a degenerate inner product*

It is easy to see that if a vector v is timelike, then $\operatorname{span}(v)$ is timelike, etc. For us, besides the one-dimensional case, the most relevant will be the co-dimension 1 case of hyperplanes. (Again, the zero subspace is by convention spacelike.)

In the exercises you will prove

Exercise 2.22. *If $z \in V$ is timelike, then $z^{\perp} := \{y \in V \mid g(y, z) = 0\}$ is spacelike and V is the direct sum $V = \mathbb{R}z + z^{\perp}$.*

We may pick one of the two components of N and denote in N^+ and call it the future null cone. The other null cone will then be denoted N^- and called the past cone. This partitions the timelike vectors I into two connected components (timecones) $I = I^+ \cup I^-$ where I^+ is the component whose boundary is N^+ and I^- the component whose boundary is N^-.

Another way of thinking about this is that we fix a timelike vector $T \in V$ and define

$$\begin{aligned} N^+ &= \{v \in N \mid m(v, T) < 0\} \\ I^+ &= \{v \in I \mid m(v, T) < 0\} \end{aligned} \tag{16}$$

Lemma 2.23. *Timelike vectors $v, w \in V$ lie in the same time cone if and only if $m(v, w) < 0$.*

Proof. Let T define the time cone I^+ as above. We can assume wlog $g(T, T) = -1$. Write $v = aT + \tilde{v}$, $w = bT + \tilde{w}$ where $\tilde{v}$, $\tilde{w}$ are in $T^{\perp}$, hence spacelike (by Exercise 2.22). Then $m(v, T) = -a$, $m(w, T) = -b$ and also $|a| > |\tilde{v}|$ and $|b| > |\tilde{w}|$ since v, w are timelike. But $m(v, w) = -ab + \tilde{v}\tilde{w}$ and hence

$$-ab - |ab| < -ab - |\tilde{v}||\tilde{w}| \leq m(v, w) \leq -ab + |\tilde{v}||\tilde{w}| < -ab + |ab|$$

which it is obvious that $m(v, w) < 0$ if a and b have the same sign and $m(v, w) > 0$ if a and b have opposite sign. ∎

Note also that timecones are convex: If $v, w \in I^+$, we have that $av + bw \in I^+$ for $a \geq 0, b \geq 0$ (not both of them zero).

An interesting feature of Lorentzian inner product space is that the Cauchy-Schwarz and the triangle inequality are reversed. As we shall see, this is the source of the twin paradox in special relativity. Let us define the notation

$$|v| = \sqrt{|m(v, v)|}\,.$$

Proposition 2.24. *Let v, w be timelike vectors in (V, m). Then*

1. *$|m(v,w)| \geq |v||w|$ with equality if and only if v, w are collinear.*
2. *If v and w are in the same timecone of V, there is a unique number $\varphi \geq 0$ called the hyperbolic angle between v and w such that*

$$m(v,w) = -|v||w| \cosh \varphi \,. \tag{17}$$

Proof. Part (1): We decompose $w = av + \tilde{w}$ where $\tilde{w} \in v^{\perp}$ where necessarily $a \neq 0$ since w is timelike. More specifically, we have

$$0 > m(w,w) = a^2 m(v,v) + m(\tilde{w}, \tilde{w}) \,.$$

On the other hand,

$$(m(v,w))^2 = a^2 (m(v,v))^2 = m(v,v) \left(m(w,w) - m(\tilde{w},\tilde{w}) \right) \geq m(v,v) m(w,w)$$

where we have used the first identity in the second step. That step becomes an identity if $\tilde{w} = 0$ which is precisely the case when w and v are collinear.

For Part (2) recall that v, w in the same timecone is equivalent to $m(v,w) < 0$. By Part (1) we have $\frac{-m(v,w)}{|v||w|} \geq 1$ and the result follows from the property of cosh. □

Corollary 2.25. *If v and w are timelike vectors in the same timecone, then*

$$|v + w| \geq |v| + |w|$$

with equality if and only if v, w are collinear.

Proof. Exercise. □

Draw pictures!

2.4.2 Minkowski space

The simplest example of a Lorentian manifold is $\mathbb{R}^{1+n}$ equipped with the metric (expressed in standard (global) coordinates)

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \,. \tag{18}$$

This is called $(n+1)$-dimensional Minkowski space (or "flat space") and is the arena of special relativity. We will discuss Minkowski space in much more detail later. (Draw some pictures of lightcones at different points. Note the conceptual difference of $\mathbb{R}^{1+n}$ as a vector space and $\mathbb{R}^{1+n}$ as a manifold.)

2.4.3 Index Raising and Lowering

Since the metric g is non-degenerate, we have that $g(p)$ is invertible at every point, more specifically, the matrix $g_{\mu\nu}(\phi(p))$ (with $0 \leq \mu \leq n, 0 \leq \nu \leq n$) is invertible at every p (in any chart). We can hence consider the inverse matrix, which is by convention denoted $g^{\mu\nu}(\phi(p))$. Note that by the formula for the inverse, $g^{\mu\nu}(\phi(p))$ is a smooth function of the components $g_{\alpha\beta}(\phi(p))$. We can therefore define a $(2,0)$-tensor field g^{-1} which in local coordinates is given by $g^{-1} = g^{\mu\nu}(\phi(p))\partial_\mu \otimes \partial_\nu$.

This gives rise to raising and lowering indices with the metric. For instance, let $X = X^\mu \partial_\mu$ be a vector(field). Then we can define

$$X^\flat = g(X, \cdot)$$

which is a co-vector(field). In components we have $(X^\flat)_\mu = g_{\mu\nu}X^\nu$ and we often write simply X_μ instead of $(X^\flat)_\mu$. Similarly if $\alpha = \alpha_\mu dx^\mu$ is a co-vector (field), then

$$\alpha^\sharp = g^{-1}(\alpha, \cdot)$$

defines a vector(field). In components we have $(\alpha^\sharp)^\mu = g^{\mu\nu}\alpha_\nu$ and we often write simply α^μ instead of $(\alpha^\sharp)^\mu$. The maps $\flat_p : T_pM \to T_p^\star M$ and $\sharp_p : T_p^\star M \to T_pM$ are also known as the "musical isomorphisms".

2.4.4 Some more terminology

Given a Lorentzian manifold (M, g) we call a vectorfield V

- timelike/ spacelike/ null at p if $V(p)$ is timelike/ spacelike/ null.
- timelike/ spacelike/ null if $V(p)$ is timelike/ spacelike/ null for all p.

A curve γ in M is called timelike/ spacelike/ null if the tangent vector to γ is everywhere timelike/ spacelike/ null. The curve is called causal if its tangent vector is everywhere timelike or null.

A hypersurface $N \subset M$ is called timelike/ spacelike/ null if $T_pN \subset T_pM$ is timelike/ spacelike/ degenerate for all $p \in N$.

Exercise 2.26. *Find timelike/ spacelike/ null hypersurfaces in Minkowski space. Can you find examples which are not hyperplanes?*

2.4.5 The length of curves

Like in Riemannian geometry, we can discuss the length of curves in Lorentzian geometry. For a causal curve $\gamma : [0, T] \to M$, we define

$$L(\gamma) = \int_0^T \sqrt{-g(\dot{\gamma}, \dot{\gamma})}dt\,,$$

while in the spacelike case

$$L(\gamma) = \int_0^T \sqrt{g(\dot{\gamma}, \dot{\gamma})}dt\,.$$

Note the length is independent of the parametrization of the curve. For causal curve the length has an important physical interpretation as the *proper time* felt by the observer moving along the curve γ in spacetime (i.e. the time elapsed by a stopwatch carried by the observer).

2.4.6 Time orientation

Given a Lorentzian manifold (M, g) we now have two time cones at each point $p \in M$. (Fixing an arbitrary timelike vector $T \in T_pM$ these are $I^+ = \{v \in T_pM \text{ timelike} \mid g(v, T) < 0\}$ and $I^- = \{v \in T_pM \text{ timelike} \mid g(v, T) > 0\}$, i.e. the timelike vectors in each T_pM fall into two equivalence classes and if v is in one cone, then $-v$ must be in the other.) There is no way to intrinsically distinguish one cone from the other unless we make a further choice (for instance of a timelike vectorfield). The question arises whether we can consistently time-orient each tangent space over the manifold, i.e. make a consistent choice of what is future and past over the entire manifold.

For Minkowski space one easily sees that if we declare the global vectorfield $\frac{\partial}{\partial x^0}$ to be *future*-directed then we can globally distinguish the past from the future in a consistent way over the manifold. However looking at the following picture

insert picture of tilting light cones on the cylinder

we see that on this Lorentzian manifold one will not be able to distinguish the future from the past consistently.

We now make the above ideas precise.

Definition 2.27. *A <u>time orientation</u> of (M, g) is a function τ on M which assigns to each $p \in M$ a time cone τ_p in T_pM and which is smooth in the following sense: For each $p \in M$ there is a smooth vectorfield V on some neighbourhood U of p such that $V_q \in \tau_q$ for each $q \in U$.*

If (M, g) admits a time orientation, (M, g) is called time-orientable and choosing a function τ as in the definition is to time-orient (M, g).

Note that the above definition formalises the idea that the local choices of time-cones can be consistently put together. The following Lemma is elucidating.

Lemma 2.28. *A Lorentzian manifold (M, g) is time-orientable iff there exists a timelike vectorfield $X \in \mathcal{X}(M)$.*

Proof. If X exists, then we can assign at each $p \in M$ the time cone containing X_p and this produces a time orientation (choose $V = X$).

Conversely, let τ be a time-orientation of M. Then by definition each $p \in M$ has a neighbourhood U for which we can find a vectorfield X_U which lies in τ_p for all $p \in U$. Let $\{f_\alpha \mid \alpha \in A\}$ be a smooth partition of unity subordinate to the covering of M by such neighbourhoods U_α. (Recall this means in particular that each $\{\text{supp} f_\alpha \mid \alpha \in A\}$ is contained in some U_α and that the $\{\text{supp} f_\alpha \mid \alpha \in A\}$

are locally finite, i.e. each $p \in M$ has a neighbourhood on which only finitely many of the f_α are supported.) We can then choose

$$X = \sum_\alpha f_\alpha X_{U_\alpha}$$

At each point this is a finite sum. Moreover since all summands point in the same light cone at any given point and since light cones are convex (and $0 \leq f_\alpha \leq 1$), we conclude that X is a timelike vectorfield defined on all of M. $\square$

Remark 2.29. *The notion of time-orientability of (M, g) is completely independent of the standard notion of orientability of the manifold M. (Note for instance that the manifold in the example above is orientable.) Recall that a manifold is orientable if it admits an orientable atlas. An atlas is orientable if the Jacobian of all its transition maps has positive determinant.*

2.4.7 Existence of Lorentzian metrics

One can show that one can equip any smooth manifold M with a *Riemannian* metric (Sheet 2). For the existence of a Lorentzian metric there are in fact topological obstructions as we shall see now.

We shall need the following Lemma whose (easy) proof is left to you:

Lemma 2.30. *Suppose U is a smooth unit vectorfield on a Riemannian manifold (M, g). Then*

$$\tilde{g} = g - 2U^\flat \otimes U^\flat \qquad U^\flat = g(U, \cdot)$$

defines a Lorentzian metric on M. Furthermore, U becomes timelike, so the resulting Lorentzian manifold $(M, \tilde{g})$ is time-orientable.

Proposition 2.31. *For a smooth manifold M the following are equivalent*

(1) There exists a Lorentzian metric on M.

(2) There exists a time-orientable Lorentzian metric on M.

(3) There is a non-vanishing vectorfield on M.

(4) Either M is non-compact, or M is compact and has Euler characteristic $\chi(M) = 0$.

Proof. (3) $\Leftrightarrow$ (4) is a standard result in topology (Poincare-Hopf theorem + degree theory) and will not be discussed further. (2) $\Rightarrow$ (1) is trivial. For (3) $\Rightarrow$ (2) equip M with a Riemannian metric g, normalise the non-vanishing vectorfield to 1 with g and apply Lemma 2.30. (2) $\Rightarrow$ (3) is immediate from Lemma 2.28. For (1) $\Rightarrow$ (4) one again needs some topology: If M is time-orientable the conclusion (4) follows from the previous. If M is not time-orientable it has a double cover $\tilde{M}$ which is time-orientable (see Sheet 2). By the previous, $\tilde{M}$ is either non-compact or compact with $\chi(\tilde{M}) = 0$. But the covering map being two to one this holds iff M is non-compact or compact with $\chi(M) = 0$. $\square$

Example 2.32. *The spheres S^{n+1} admit a Lorentzian metric iff $n+1 \geq 2$ is odd, since $\chi(S^{n+1}) = 1 + (-1)^{n+1}$.*

Definition 2.33. *A time-oriented $(n+1)$-dimensional Lorentzian manifold is called a spacetime.*

We will exclusively deal with spacetimes (for physical reasons it is important to distinguish between future and past!). We shall also see that compact Lorentzian manifolds have little relevance in general relativity as they necessarily admit closed timelike curves.

2.5 Vectorfields and Flows

We now turn back to tensor fields and study explicitly the special case of $(0,1)$-tensor fields, i.e. vectorfields. We can think of a smooth vectorfield on M as a map $X : C^\infty(M) \to C^\infty(M)$. In particular, we can consider iterates of vectorfields $XY : f \mapsto X(Y(f))$. They will not be vectorfields as the computation

$$\begin{aligned} X_p Y(f) &= X_p^i \left[\frac{\partial}{\partial x^i}\right]_p \left(Y^m \frac{\partial (f \circ \phi^{-1})}{\partial x^m}\right) \\ &= X_p^i \left(\left[\frac{\partial}{\partial x^i}\right]_p Y^m\right) \left[\frac{\partial}{\partial x^m}\right]_p f + X_p^i Y_p^m \frac{\partial^2 (f \circ \phi^{-1})}{\partial x^m \partial x^i} \end{aligned} \tag{19}$$

shows. However, if we consider the commutator

$$X_p(Yf) - Y_p(Xf) = \left(X_p^i \left[\frac{\partial}{\partial x^i}\right]_p Y^m - Y_p^i \left[\frac{\partial}{\partial x^i}\right]_p X^m\right) \left[\frac{\partial}{\partial x^m}\right]_p f \tag{20}$$

we see that it *is* a vectorfield.

Definition 2.34. *Given smooth vectorfields X, Y near p, we call the vectorfield $[X,Y]$ defined above the commutator of X and Y.*

Note that the (components of the) commutator at p depend on the behaviour of (the components of) both X and Y in a neighbourhood of p. In the exercises you will study properties of the commutator

Exercise 2.35. *Let X, Y, Z be smooth vectorfields on M, a and b be real numbers and f and g smooth functions on M. Then*

- $[X,Y] = -[Y,X]$ *(anticommutativity)*
- $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ *(linearity)*
- $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ *(Jacobi identity)*
- $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$

We can also interpret the commutator as a derivative along an integral curve of a vectorfield (Lie derivative). To make this precise we need the concept of integral curves of vectorfields.

Definition 2.36. *Given a smooth vectorfield X on a smooth manifold M we call $\gamma : \mathbb{R} \supset I \to M$ an integral curve of X if at each point p along γ the tangent vector to γ is X_p.*

Note that in a chart (coordinate system) this translates into the first order ODE

$$\frac{d}{dt}(x^i \circ \gamma)(t) = X^i(x^1 \circ \gamma(t), \ldots, x^n \circ \gamma(t)) \tag{21}$$

for the curve. The existence, uniqueness and continuous dependence on data theory for ODEs gives

Theorem 2.37. *Let M be a smooth manifold and X a smooth vectorfield on M near p. Then there exists an open set $U \subset M$ containing p, a $\delta > 0$ and a smooth map $\varphi : (-\delta, \delta) \times U \to M$ such that the following holds: For any $q \in U$, the curve $t \mapsto \varphi(t, q)$ is the unique curve which satisfies*

$$\frac{\partial}{\partial t}\varphi(t,q) = X(\varphi(t,q)) \quad , \quad \varphi(0,q) = q\,.$$

Remark 2.38. *The notation on the left hand side denotes the tangent vector to the curve $\varphi(\cdot, q)$. It is common to write $\varphi_t(q) = \varphi(t,q)$ and call this the local flow of X. The notation suggests that q is flown forwards by time t along the integral curve of X.*

Proof. See the exercises. Note that in a chart near $p \in M$ the equation becomes

$$\frac{\partial}{\partial t}(x^i \circ \varphi)(t,q) = X^i(x^1 \circ \varphi(t,q), \ldots, x^n \circ \varphi(t,q)) \quad x^i \circ \varphi(0,q) = x^i(q)\,. \tag{22}$$

We can interpret this as an ODE in t which depends smoothly on a parameter q. □

Note that integral curves emanating from different points of U cannot intersect as this would contradict uniqueness. However, they might not be defined for all times (take the vectorfield $u^2 \frac{d}{du}$ on $\mathbb{R}$). (If they are, i.e. if all its integral curves are defined on the entire real line, then the vectorfield is called complete.) Note also that to define (maximal) integral curves globally on M one may need to solve the ODE system in several patches and glue the solutions together.

Here is the promised interpretation of the commutator as a derivative, the *Lie derivative*. Note that to differentiate we will need to compare vectors in different tangent spaces. The idea is that the notion of pushforward provides a way to achieve this along curves.

Proposition 2.39. *Let X, Y be vectorfields on M, $p \in M$ and φ_t be the local flow of X near p. Then*

$$[X, Y]_p = (\mathcal{L}_X Y)_p \tag{23}$$

where

$$(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{1}{t}\left[d\varphi_{-t} Y_{\varphi_t(p)} - Y_p\right] .$$

is called the Lie-derivative of the vectorfield Y along X.

Proof. Both sides of (23) are vectors at p. We can therefore establish the identity in any coordinate system.

Let us assume first that $X \neq 0$ at p. We can choose coordinates such that $X = \frac{\partial}{\partial x^1}$ holds in a small neighbourhood around p (this is an exercise on Example Sheet 1). Hence the flow of X only changes the x^1 coordinate of points q near p:

$$x^1(\varphi_t(p)) = x^1(p) + t \quad , \quad x^i(\varphi_t(p)) = x^i(p) \text{ for } i = 2, 3, \dots n.$$

Hence in these coordinates

$$\begin{aligned}(\mathcal{L}_X Y)^i_p &:= \lim_{t\to 0} \frac{1}{t}\left[Y^i\left(x^1(p) + t, ..., x^n(p)\right) - Y^i(x^1(p), \dots, x^n(p))\right] \\ &= \frac{\partial Y^i}{\partial x^1}\Big|_p = X_p(Y^i) = X_p(Y^i) - Y_p(X^i) = [X, Y]^i_p ,\end{aligned} \tag{24}$$

where the penultimate step follows because $X^i = \delta^i_1$ is constant in our chosen coordinate system.

Now if $X = 0$ at p and in a neighbourhood of p, then the formula trivially holds. If $X = 0$ at p but there is a sequence (p_i) with $p_i \to p$ and $X(p_i) \neq 0$, then we can argue that the formula holds for each p_i and that both sides depend continuously on p. ☐

2.6 The connection

We now introduce an additional piece of structure on the manifold which will allows us to differentiate vectorfields and, more generally, tensorfields. Note that we already have one such derivative, the Lie-derivative. However, as we have seen, $(\mathcal{L}_X Y)_p$ depends on X in an entire neighbourhood of p. Another way of saying this is that the Lie-derivative is *not* tensorial in X as $\mathcal{L}_{fX} Y \neq f\mathcal{L}_X Y$ for general functions $f \in C^\infty(M)$, cf. Exercise 2.15. The covariant derivative that we will define next, has this property.

Note also that the connection is a priori entirely independent of having a metric (Riemannian or Lorentzian) on the manifold. Later we will see that a natural compatibility condition singles out a unique connection, the Levi-Civita connection, which is compatible with the metric.

Definition 2.40. *A linear connection ∇ on M is a map*

$$\mathcal{X}(M) \times \mathcal{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathcal{X}(M)$$

such that for $f \in C^\infty(M)$ and $X, Y, Z \in \mathcal{X}(M)$ we have

- $\nabla_{X+fY} Z = \nabla_X Z + f \nabla_Y Z$ *($C^\infty(M)$-linear in the first argument)*
- $\nabla_X (aY + bZ) = a\nabla_X Y + b\nabla_X Z$ *$\mathbb{R}$-linear in the second argument*
- $\nabla_X (fY) = f\nabla_X Y + X(f)Y$ *(Leibniz rule)*

The vectorfield $\nabla_X Y$ is called the co-variant derivative of Y with respect to X. The $(1,1)$-tensor field ∇Y is called the covariant derivative of Y.

Note that ∇Y is indeed a $(1,1)$-tensor field by Exercise 2.15 and the C^∞-linearity in the first argument. We can write in local coordinates

$$\left(\nabla \frac{\partial}{\partial x^\nu}\right)\left(dx^\mu, \frac{\partial}{\partial x^\alpha}\right) = dx^\mu \left(\nabla_{\partial_\alpha} \partial_\nu\right) =: \Gamma^\mu{}_{\alpha\nu} \tag{25}$$

which are called the Christoffel symbols of the connection. From the defining properties we compute

$$\nabla_X Y = \nabla_{X^\mu \partial_\mu}(Y^\nu \partial_\nu) = X^\mu \nabla_{\partial_\mu}(Y^\nu \partial_\nu) = (X^\mu \partial_\mu Y^\nu)\partial_\nu + X^\mu Y^\nu \Gamma^\alpha{}_{\mu\nu} \partial_\alpha \tag{26}$$

so

$$(\nabla_X Y)^\alpha = X^\mu \partial_\mu Y^\alpha + \Gamma^\alpha{}_{\mu\nu} X^\mu Y^\nu \,. \tag{27}$$

For the components of the tensor $\nabla Y = \nabla_\mu Y^\nu dx^\mu \otimes \partial_\nu$ (the notation is unfortunately slightly ambiguous but standard) we hence obtain

$$\nabla_\mu Y^\alpha = \partial_\mu Y^\alpha + \Gamma^\alpha{}_{\mu\nu} Y^\nu \,.$$

The linear connection gives rise to a notion of *parallel transport* defined as follows

Definition 2.41. *We say that a vectorfield Y is parallely transported along X if $\nabla_X Y = 0$.*

To understand that definition we consider as an example the flat $\mathbb{R}^2$. The vectorfield ∂_x and ∂_y form a basis and if we interpret "parallel transport" in the usual way, namely that a vector doesn't change its length or direction when being parallely transported, we have

$$\nabla_{\partial_x} \partial_x = \nabla_{\partial_x} \partial_y = \nabla_{\partial_y} \partial_x = \nabla_{\partial_y} \partial_y = 0 \tag{28}$$

hence $\Gamma^\alpha{}_{\mu\nu} = 0$ for all indices. Indeed, this means $(\nabla_X Y)^i = X^j \partial_j Y^i$, consistent with this being zero if the components of Y do not change along X. Now consider $\mathbb{R}^2$ equipped with polar coordinates and consider the unit vectorfields $\hat{e}_r = \partial_r$, $\hat{e}_\theta = \frac{1}{r}\partial_\theta$. Then one computes (or sees)

$$\nabla_{\hat{e}_r}\hat{e}_r = \nabla_{\hat{e}_r}\hat{e}_\theta = 0 \quad \text{but} \quad \nabla_{\hat{e}_\theta}\hat{e}_r = -\frac{1}{r}\hat{e}_\theta \quad , \quad \nabla_{\hat{e}_\theta}\hat{e}_\theta = -\frac{1}{r}\hat{e}_r \tag{29}$$

hence $\Gamma^\theta{}_{r\theta} = \frac{1}{r}$ and $\Gamma^r{}_{\theta\theta} = -\frac{1}{r}$. We see immediately that the Christoffel symbols Γ are NOT tensors.

Exercise 2.42. *Compute the transformation law of the Christoffel symbols under a change of basis in the tangent space.*

Exercise 2.43. *Show that the difference of two linear connections is a tensor. (Hint: Observe that $D(X,Y) = \nabla_X Y - \tilde{\nabla}_X Y$ is $C^\infty(M)$-linear in Y!)*

We can extend the definition of the connection to act on arbitrary tensorfields (to produce another such tensorfield) by requiring the Leibniz rule and that it commutes with all contractions (which makes the extension unique).[2] More precisely, we impose that

$$\nabla_X f = X(f) \quad \text{for } f \in C^\infty(M), \tag{30}$$

$$\nabla_X(\alpha \otimes \beta) = (\nabla_X \alpha) \otimes \beta + \alpha \otimes (\nabla_X \beta) \quad \text{for } \alpha,\, \beta \text{ arbitrary tensor fields} \tag{31}$$

and

$$\nabla_X \text{ commutes with all contractions: } tr\nabla_X \alpha = \nabla_X(tr\alpha). \tag{32}$$

For instance, if α is a 1-form, we compute

$$\begin{aligned}(\nabla_X \alpha)Y = tr(\nabla_X \alpha \otimes Y) &= tr\left(\nabla_X(\alpha \otimes Y) - \alpha \otimes \nabla_X Y\right) \\ &= X(\alpha(Y)) - \alpha(\nabla_X Y)\end{aligned} \tag{33}$$

In components, we compute

$$\begin{aligned}(\nabla_X \alpha)_a Y^a &= X^b \partial_b(\alpha_a Y^a) - \alpha_b(X^a \partial_a Y^b + \Gamma^b{}_{ac} X^a Y^c) \\ &= \left(X^b \partial_b \alpha_a - \Gamma^b{}_{ca} \alpha_b X^c\right) Y^a\,,\end{aligned} \tag{34}$$

hence

$$(\nabla_b \alpha)_a = \partial_b \alpha_a - \Gamma^c{}_{ba} \alpha_c\,.$$

Exercise 2.44. *Derive the component formula for the covariant derivative of an arbitrary $T^{(k,l)}$-tensor field:*

$$\begin{aligned}\nabla_c T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = \partial_a T^{a_1\ldots a_k}{}_{b_1\ldots b_l} &+ \Gamma^{a_1}{}_{cd} T^{da_2\ldots a_k}{}_{b_1\ldots b_l} + \ldots + \Gamma^{a_k}{}_{cd} T^{a_1\ldots a_{k-1}d}{}_{b_1\ldots b_l} \\ &- \Gamma^d{}_{cb_1} T^{a_1\ldots a_k}{}_{db_2\ldots b_l} - \ldots - \Gamma^d{}_{cb_l} T^{a_1\ldots a_k}{}_{b_1 b_2\ldots b_{l-1}d}.\end{aligned}$$

[2]The same strategy can be followed for the Lie derivative. See Exercise Sheet 2.

2.7 Parallel Transport and Geodesics

One can consider the parallel transport of a vector Y given at p along a curve γ through p. Let $X = X_{\gamma(t)}$ denote the tangent vector along γ. Recall that in components $X^b(t) = \frac{d}{dt}\left(x^b \circ \gamma(t)\right)$ are the components of X along γ in the chart ϕ with coordinates x. Therefore, parallel transport along X becomes

$$\frac{d}{dt}Y^c(x\circ\gamma(t)) + \Gamma^c{}_{ab}(x\circ\gamma(t))X^a(x\circ\gamma(t))Y^b(x\circ\gamma(t)) = 0\,. \tag{35}$$

Note that the first term is indeed $X^a \partial_a Y^b$. This is a first order *linear* ODE along γ, hence solutions exist along the entire curve.

A vectorfield that is parallely transported along itself is called *geodesic*:

Definition 2.45. *Let X be a vectorfield such that $\nabla_X X = 0$. Then the integral curves of X are called geodesics.*

Remark 2.46. *Note this makes sense in flat $\mathbb{R}^n$ where geodesics are straight lines. The tangent vector of a straight line is parallely transported along itself.*

Theorem 2.47. *There is precisely one geodesic through each point p whose tangent at p is X_p.*

Proof. Note that the equation $\nabla_X X = 0$ when expressed in a local chart ϕ (with coordinates x) near p becomes a system of second order non-linear ODEs for the components of γ in the chart:

$$\frac{d}{dt}\frac{d}{dt}(x^b\circ\gamma(t)) + \Gamma^b{}_{ac}(x\circ\gamma(t))\frac{d}{dt}(x^a\circ\gamma(t))\frac{d}{dt}(x^c\circ\gamma(t)) = 0 \tag{36}$$

that is typically written shorthand as

$$\ddot{x}^b + \Gamma^b{}_{ac}(x(t))\dot{x}^a\dot{x}^c = 0\,. \tag{37}$$

Standard ODE theory gives existence of a unique local solution $x : (-\delta, \delta) \to \mathbb{R}^n$ with data

$$x(0) = \text{coordinates of } p \text{ in the chart } \phi, \tag{38}$$

$$\dot{x}(0) = \text{components of } X_p \text{ in the coordinate chart } \phi. \tag{39}$$

□

We will talk more about the local and global properties of (timelike/ spacelike/ null) geodesics and their physical interpretation soon. Suffice it to say here that timelike geodesics (of a particular connection to be defined below) will correspond to freely falling (i.e. moving without any external force) particles in spacetime. In a non-flat geometry these will, in general, not be straight lines.

We note however already the following global uniqueness statement (which can in turn be used to define the notion of a *maximal* (or geodesically inextendible) geodesic):

Lemma 2.48. *Let $\gamma_1, \gamma_2 : I \to M$ be geodesics. If there is a number $T \in I$ such that $\gamma_1(T) = \gamma_2(T)$ and $\gamma_1'(T) = \gamma_2'(T)$, then $\gamma_1 = \gamma_2$.*

Proof. Exercise. □

2.8 Torsion tensor

Definition 2.49. *The torsion tensor is a* $(1,2)$*-tensorfield* T *defined by*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

Exercise 2.50. *Check that this is indeed a tensorfield!*

In the following, **we will consider connections with vanishing torsion** $T = 0$ **only**. These are also called symmetric since in a coordinate basis ∂_i the vanishing torsion implies $\Gamma^\mu{}_{\sigma\tau} = \Gamma^\mu{}_{\tau\sigma}$, i.e. the Christoffel symbols are symmetric. Note that the geodesic equation (37) only involves the symmetric part of the connection so the torsion does not influence the geodesics.

Example 2.51. *Consider* $\mathbb{R}^3$ *with the connection* $\nabla_{\partial_i}\partial_j = \sum_{k=1}^{3} \epsilon_{ijk}\partial_k$ *for* $i,j \in \{1,2,3\}$*. What are the geodesics? Geometric interpretation?*

2.9 Riemann curvature tensor

The Riemann curvature tensor is a $(1,3)$ tensorfield R defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

Exercise 2.52. *Check that this is indeed a tensorfield!*

Exercise 2.53. *Define the components of the Riemann tensor in a coordinate chart by* $R(\partial_c,\partial_d)\partial_b = R^a{}_{bcd}\partial_a$*. Obtain the following expression for the components in a coordinate basis:*

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\,.$$

where $\Gamma^a{}_{bc,d} = \partial_d\Gamma^a{}_{bc}$ *etc.*

Note that the Riemann tensor measures the failure of second covariant derivatives to commute. More interpretations of the Riemann curvature tensor will be discussed in the exercises.

2.10 The Levi-Civita connection

We now come to the miracle of Riemannian (Lorentzian) geometry, namely that there is a unique connection which is compatible with the structure of a Riemannian (Lorentzian) metric in a suitable sense:

Proposition 2.54. *A Lorentzian (or more generally Riemannian or pseudo-Riemannian) manifold has a unique symmetric (=torsion free) connection which is compatible with the metric in that* $\nabla g = 0$*. It is called the* **Levi-Civita connection***. In a coordinate induced basis, the connection coefficients are given by*

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(g_{bd,c} + g_{dc,b} - g_{bc,d}\right)\,. \tag{40}$$

Proof. See the exercise class. Idea of proof is the following. Such a connection needs to satisfy

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\,, \tag{41}$$
$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)\,, \tag{42}$$
$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\,. \tag{43}$$

Consider now $X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))$ and solve this for $2g(\nabla_X Y, Z)$ using the vanishing torsion condition. □

Proposition 2.55. *Parallel transport with respect to the Levi-Civita connection is a linear isometry.*

Proof. If $P_{pq}(\gamma) : T_pM \to T_qM$ denotes the parallel transport from p to q along a curve γ (an integral curve of a vectorfield Z, say), then it is easy to see that $P_{pq}(\gamma)$ is linear and injective (why? – recall the linear ODE governing parallel transport!) and since the tangent spaces have the same dimension, a linear isomorphism of the tangent spaces. Finally, if X, Y are vectorfields which are parallely transported from p to q along an integral curve γ, then $\nabla_Z X = 0$ and $\nabla_Z Y = 0$ hence $\nabla_Z(g(X,Y)) = 0$ by the Leibniz rule and metric compatibility. Hence $g_q(P_{pq}(\gamma)X_p, P_{pq}(\gamma)Y_p) = g_p(X_p, Y_p)$, which shows that it is a linear isometry. □

Note however, that parallel transport will depend on the curve from p to q. In fact, the curvature tensor measures precisely this dependence (see Sheet 2).

Observe as an easy corollary of the above that geodesics preserve the causal nature of their tangent vectors, i.e. geodesics are necessarily either timelike, spacelike or null. (Of course this can also be seen directly from the formula $\nabla_{\dot\gamma} g(\dot\gamma, \dot\gamma) = 0$ along a geodesic, which implies that if the tangent vector is time-like/ spacelike/ null at one point, then it will be so along the entire geodesic.)

2.11 Symmetries of the Riemann-tensor

Theorem 2.56. *The Riemann tensor of the Levi-Civita connection has the following symmetries:*

$$R_{abcd} = -R_{abdc} \tag{44}$$
$$R_{abcd} = -R_{bacd} \tag{45}$$
$$R_{abcd} = R_{cdab} \tag{46}$$
$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \quad \textit{(first Bianchi identity)} \tag{47}$$
$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0 \quad \textit{(second Bianchi identity)} \tag{48}$$

Proof. Exercise. You might want to wait with the proof of these tensor equations until we have introduced normal coordinates. □

2.12 The Ricci tensor

Definition 2.57. *The* <u>*Ricci curvature tensor*</u> *is the* $(0,2)$*-tensor field obtained by contracting the Riemann tensor in such a way that the components of the Ricci tensor relative to a coordinate system are*

$$R_{ij} = \sum_m R^m{}_{imj} \,. \tag{49}$$

The <u>*Ricci scalar*</u> *is obtained as the trace of the Ricci-curvature tensor:* $R = g^{ij} R_{ij}$*.*

Recall that the operation of contraction does not depend on the coordinate system that is being used, so the Ricci tensor this defined is indeed a $(0,2)$ tensor. Also, in view of the symmetries of the Riemann tensor any contraction of its indices will produce either zero or $\pm R_{ij}$. The remaining part of the curvature tensor is the Weyl tensor (see Sheet 3).

2.13 The Einstein equations

We can finally understand the (vacuum) Einstein equation, which states that a spacetime (M, g) should satisfy

$$R_{ij} - \frac{1}{2} R g_{ij} = 0 \,, \tag{50}$$

which is equivalent to (why?)

$$R_{ij} = 0 \,. \tag{51}$$

You might recall from (1) that the full Einstein equations read $R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}$, where T_{ij} is the energy momentum tensor of the matter present in spacetime. Einstein called the left hand side of this equation the "marble" (note that to define the vacuum equations we did not need any assumptions or approximations coming from physics!) and the right hand side the "wood". The right hand side typically involves models for the matter that can be derived from physics. As mentioned earlier, the vacuum equations do have a very rich structure already, so we will start by focussing on these. Natural questions are

- Are there any non-trivial examples of explicit solutions to (50)?
- Can we construct large classes of solutions to (50)? What are their geometric properties?
- Why does (50), or more generally (1), describe gravity? How to we derive, say, the planetary orbits (and in what limit can we see the Newtonian description?).

Before we turn to the first point and discuss the geometry of the Schwarzschild black hole, we need one more notion, that of symmetries of the metric, which plays a fundamental role.

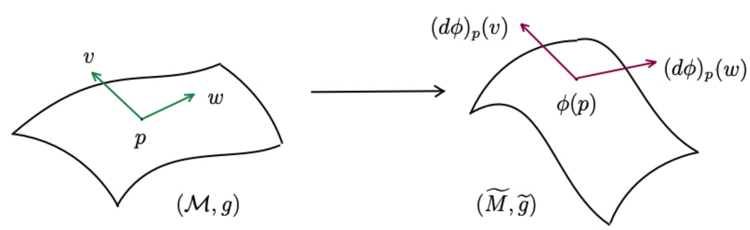
2.14 Isometries

In mathematics it is quite typical that one studies a set with a certain structure (and maps between such sets) and defines an appropriate notion of equivalence between these sets, i.e. a notion of when two objects should be considered "the same". A topologist studies topological spaces and the natural equivalence is that of homeomorphism. A differential topologist studies smooth manifolds and the natural equivalence is that of diffeomorphism. For a Lorentzian (Riemannian) geometer, who studies Lorentzian manifolds (M, g), the appropriate notion of equivalence is that of isometry:

Definition 2.58. *A diffeomorphism $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is called an isometry if*

$$\phi^\star \tilde{g} = g \,,$$

i.e. for all $p \in M$ we have $g_p(v, w) = \tilde{g}_{\phi(p)}(d\phi_p v, d\phi_p w)$ for all $v, w \in T_pM$.



Remark 2.59. *For spacetimes, we may want the isometries to preserve also the time orientation, i.e. impose that $[\phi_\star T] = [\tilde{T}]$ holds (in other words the future cone at p gets mapped to the future cone at $\phi(p)$ for all $p \in M$).*

We will be particularly interested in the set of isometries from (M, g) to itself. To define the relevant notions, we first relate one-parameter groups of diffeomorphisms to their infinitesimal generators (vectorfields). Logically, this discussion should have been taken place after Section 2.5.

Definition 2.60. *A one-parameter group of diffeomorphisms on M is a smooth map*

$$\begin{aligned} F : \mathbb{R} \times M &\to M \\ (t, x) &\mapsto F_t(x) \end{aligned}$$

such that

1. *$\forall t \in \mathbb{R}$ the map $F_t : M \to M$ is a diffeomorphism*
2. *$F_0 = id_M$*
3. *$F_s \circ F_t = F_{s+t} \quad \forall s, t \in \mathbb{R} \qquad$ (group action)*

Example 2.61. *Consider the sphere with standard θ, ϕ coordinates. Define $F_t : \mathbb{S}^2 \to \mathbb{S}^2$ to be a rotation by $\Delta\phi = t$ (north pole and south pole are fixed points). For fixed (θ, ϕ) the map $t \mapsto F_t(\theta, \phi) = (\theta, \phi + t)$ defines a curve with tangent $F_t'(\theta, \phi) = \partial_\phi$.*

Given a one parameter group of diffeomorphisms we obtain smooth curves $\gamma(t) = F_t(p)$ for each $p \in M$. We define a smooth vectorfield

$$V_p = \gamma'(0) = \frac{d}{dt} F_t(p)\Big|_{t=0}$$

where the right hand side is a short hand notation capturing that the components of V in a chart with coordinates x are obtained from $(V_p)^i = \frac{d}{dt}(x \circ F_t(p))^i\Big|_{t=0}$. The vectorfield V is called the infinitesimal generator of F because $t \mapsto F_t(p)$ are integral curves of V. This follows from the computation

$$\frac{d}{dt}\Big|_{t=t_0} F_t(p) = \frac{d}{dt}\Big|_{t=t_0} F_{t-t_0}(F_{t_0}(p)) = \frac{d}{ds}\Big|_{s=0} F_s(F_{t_0}(p)) = V_{F_{t_0}(p)} \, .$$

In other words, there is a one-to-one correspondence in that we can associate one-parameter groups of diffeomorphisms with smooth vectorfields whose integral curves are complete and conversely, given a complete vectorfield $V \in \mathcal{X}(M)$, we can associate with it a one-parameter group of diffeomorphisms. (The latter claim is an exercise. Use that the integral curves exist for all times by assumption and that integral curves cannot intersect.)

We can now return to the discussion of isometries. Given a one-parameter group of *isometries*, i.e. $F_t : M \to M$ is an isometry for each $t \in \mathbb{R}$, we let V be its infinitesimal generator. Then, using a result from Sheet 2 we have

$$\mathcal{L}_V g = \lim_{t \to 0} \frac{F_t^\star g - g}{t} = 0 \tag{52}$$

Definition 2.62. *A vectorfield on M satisfying $\mathcal{L}_V g = 0$ is called a Killing vectorfield.*

Note that if V is Killing, then in a chart where $V = \partial_{x_0}$ (the adapted coordinates of Sheet 2) the Killing condition becomes $\partial_{x_0} g_{\mu\nu} = 0$ so the metric components do not depend on the x^0 variable in these coordinates. Conversely, if $\partial_y g_{\mu\nu} = 0$ holds in some coordinate chart, then the vectorfield Y defined by ∂_y in these coordinates satisfies $\mathcal{L}_Y g = 0$. Revisit Example 2.61 and note that ∂_ϕ is indeed a Killing vectorfield.

The following proposition collects fundamental properties of Killing vectorfields, some of which you will proof on Example Sheet 3.

Proposition 2.63. *Let (M, g) be a Lorentzian (Riemannian) manifold equipped with the Levi-Civita connection.*

1. *The Killing vectorfields form a Lie algebra.*

2. *V is a Killing field if and only if $\nabla_a V_b + \nabla_b V_a = 0$.*

3. *If V is a Killing vectorfield, then $\nabla_a \nabla_b V_c = -R_{adbc} V^d$.*

4. *If V is a Killing vectorfield and $\gamma : I \to M$ a geodesic ($\nabla_{\dot\gamma}\dot\gamma = 0$). Then $g(V, \dot\gamma)$ is constant along γ.*

Proof. The first item is proven on Sheet 3. The second item is a computation. Starting from the Leibniz rule and using the fact that the connection is symmetric and metric ($\nabla g = 0$) we obtain

$$\begin{aligned}\mathcal{L}_V g(\partial_\mu, \partial_\nu) &= V(g_{\mu\nu}) - g([V, \partial_\mu], \partial_\nu) - g(\partial_\mu, [V, \partial_\nu]) \\ &= (\nabla_V g)_{\mu\nu} + g(\nabla_V \partial_\mu, \partial_\nu) + g(\partial_\mu, \nabla_V \partial_\nu) \\ &\quad - g(\nabla_V \partial_\mu - \nabla_{\partial_\mu} V, \partial_\nu) - g(\partial_\mu, \nabla_V \partial_\nu - \nabla_{\partial_\nu} V) \\ &= g(\nabla_{\partial_\mu} V, \partial_\nu) + g(\partial_\mu, \nabla_{\partial_\nu} V) \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu \,. \end{aligned} \tag{53}$$

□

We combine the formula from Sheet 2,

$$\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{cdab} V^d = -R_{dcab} V^d \,, \tag{54}$$

with the first Bianchi identity $R_{dcab} + R_{dabc} + R_{dbca} = 0$ as follows:

$$\begin{aligned} 0 &= (\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c) + (\nabla_b \nabla_c V_a - \nabla_c \nabla_b V_a) + (\nabla_c \nabla_a V_b - \nabla_a \nabla_c V_b) \\ &= 2(\nabla_a \nabla_b V_c + \nabla_b \nabla_c V_a - \nabla_c \nabla_b V_a) \\ &= 2(\nabla_a \nabla_b V_c - R_{dabc} V^d) \,. \end{aligned} \tag{55}$$

For the final item we compute

$$\dot\gamma(g(V, \dot\gamma)) = \dot\gamma^\mu \nabla_\mu (g_{\alpha\beta} V^\alpha \dot\gamma^\beta) = \dot\gamma^\mu \dot\gamma^\beta g_{\alpha\beta} \nabla_\mu V^\alpha = \dot\gamma^\mu \dot\gamma^\beta \nabla_\mu V_\beta = 0 \tag{56}$$

where we have used that γ is a geodesic and that the connection is metric. The last identity follows since $\nabla_\mu V_\beta$ is antisymmetric in β and μ by item 2.

Example 2.64. *Consider $\mathbb{R}^{1+n}$ equipped with canonical coordinates $(x^0, x^1, ..., x^n)$ and metric*

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \,.$$

Check that the following are Killing vectorfields:

- *$\partial_{x^0}, \partial_{x^1}, \ldots, \partial_{x^n}$ (translations)*
- *$x^i \partial_{x^j} - x^j \partial_{x_i}$ for $i, j \in \{1, 2, \ldots, n\}$ (spatial rotations)*
- *$x^0 \partial_i + x^i \partial_{x_0}$ for $i \in \{1, 2, \ldots, n\}$ (Lorentz boosts)*

We will interpret them below and on Example Sheet 3 you will be showing this is the maximal number of linearly independent Killing vectorfields a manifold can hold in general.

2.15 The exponential map and normal neighbourhoods

We collect all geodesics emanating from a point $p \in M$ into a single map, the exponential map.[3] For $p \in M$ we let

$$\mathcal{D}_p := \{v \in T_pM \mid \text{the geodesic } \gamma_v \text{ is defined at least on } [0,1]\},$$

where γ_v is the unique geodesic starting from p with velocity v. Note that $\mathcal{D}_p$ is non-empty by Theorem 2.47 (and the scaling properties of the time of existence with the scaling of v).

Definition 2.65. *The exponential map at p is the function*

$$\exp_p : \mathcal{D}_p \to M \quad \textit{defined by} \quad \exp_p(v) = \gamma_v(1) \tag{57}$$

Note that M is complete iff $\mathcal{D}_p = T_pM$ for every $p \in M$.

Now fix $v \in T_pM$ and $t \in \mathbb{R}$. The gedoesic $s \mapsto \gamma_v(ts)$ has $\frac{d}{ds}\gamma(s=0) = tv$ hence $\gamma_{tv}(s) = \gamma_v(ts)$ (for all s, t such that this is defined). We conclude that for $v \in \mathcal{D}_p$ we have $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$. Therefore, the exponential map maps lines through the origin of T_pM to geodesics through p in M.

Proposition 2.66. *For each point $p \in M$ there exists a neighbourhood U of 0 in T_pM on which the exponential map $\exp_p$ is a diffeomorphism onto a neighbourhood U of p in M.*

Proof. We have $\exp_p : T_pM \supset \mathcal{D}_pM \to M$ hence $(d\exp_p)_0 : T_0(T_pM) \to T_pM$. For $v \in T_pM$ arbitrary, $\rho(t) = tv$ is a curve in T_pM through the origin with tangent vector v. We compute (recall Proposition 2.9)

$$(d\exp_p)_0 v = \frac{d}{dt}\exp_p(tv)\Big|_{t=0} = \frac{d}{dt}\gamma_v(t)\Big|_{t=0} = v\,.$$

It follows that $(d\exp_p)_0$ is the identity and the claim of the proposition follows directly from the implicit function theorem. □

A neighbourhood U of p in M for which $\exp_p$ is a diffeomorphism is called a normal neighbourhood. We will actually follow the convention of [O'Neill] and demand in addition that any normal neighbourhood U is the image under the exponential map of a union of radial line segments through the origin in T_pM. (Clearly any normal neighbourhood without that additional stipulation can be shrunk to satisfy it.) With this definition it is not hard to show (cf. page 72 (Proposition 31) of O'Neill's book) that given any $q \in U$ there is a unique (radial) geodesic from p to q which remains in U.

Remark 2.67. *Proposition 2.66 ensures the existence of normal neighbourhoods around each point $p \in M$. One can refine the analysis to prove the existence of a convex normal neighbourhood U of p. This is a neighbourhood U of p that is normal for each of its points. In particular, any two points of a convex normal neighbourhood can be connected by a geodesic and the geodesic remains entirely inside that neighbourhood.*

[3] Recall that the notion of geodesic is defined as soon as we have a connection, a metric is not required.

2.16 Normal coordinates

To define normal coordinates on a Lorentzian manifold[4], we consider a normal neighbourhood $U \subset M$ with diffeomorphism $\exp_p : T_pM \supset \tilde{U} \to U$. We choose an orthonormal basis $e_0, e_1, \dots, e_n$ of T_pM, i.e. $g(e_i, e_j) = \eta_{ij}$. This choice determines a normal coordinate system on U as follows. We associate with $q \in U$ the vector coordinates relative to $e_0, e_1, \dots, e_n$ of the point $\exp_p^{-1}(q) \in \tilde{U} \subset T_pM$. In other words, $x^i(q)$ is defined by

$$\exp_p^{-1}(q) = \sum_{i=0}^{n} x^i(q) e_i \,.$$

Note that if $\omega^0, \omega^1, \dots, \omega^n$ is the dual basis to $e_0, e_1, \dots, e_n$ then

$$x^i \circ \exp_p = \omega^i \quad \text{on } \tilde{U} \tag{58}$$

as is easily verified by contracting with e_j.

Proposition 2.68. *Let (M, g) be a Lorentzian manifold. If $x^0, x^1, \dots, x^n$ is a normal coordinate system at $p \in M$, then for all $i, j, k \in \{0, 1, \dots, n\}$ we have*

$$g_{ij}(p) = \eta_{ij} \qquad \text{and} \qquad {\Gamma^k}_{ij}(p) = 0 \,. \tag{59}$$

Proof. Given $v \in \tilde{U} \subset T_pM$, write $v = \sum a^i e_i$, where the e_i are the ONB determining the normal coordinate system. Since $\exp_p(tv) = \gamma_v(t)$ we have (using (58))

$$x^i(\gamma_v(t)) = x^i \circ \exp_p(tv) = t\omega^i(v) = ta^i \,.$$

It follows that $v = a^i \partial_i\big|_p$ because the coordinates of the vectorfield v in the coordinate basis $\partial_i|_p$ induced by x^i are given by $\frac{d}{dt} x^i(\gamma_v(t))\big|_{t=0}$ (recall the considerations preceding Remark 2.8 if that is not clear). Hence $e_i = \partial_i|_p$ and it follows that indeed $g_{ij}(p) = g(\partial_i|_p, \partial_j|_p) = g(e_i, e_j) = \eta_{ij}$ as claimed. For the second identity note that the geodesic equation reduces in the coordinates x^i to

$${\Gamma^k}_{ij}(\gamma_v(t)) a^i a^j = 0 \;\text{ for all } k. \tag{60}$$

In particular, we have that

$${\Gamma^k}_{ij}(p) a^i a^j = 0 \;\text{ for all } k \tag{61}$$

holds for all directions $(a^0, a^1, ..., a^n) \in \mathbb{R}^{n+1}$.[5] For fixed k (61) states that a certain quadratic form is identically zero. By polarisation the associated bilinear form is identically zero and that proves the second claim. □

Remark 2.69. *Normal coordinates are extremely useful when establishing tensor identities. As these can be established in any coordinate system choosing normal coordinates often simplifies the algebra considerably.*

[4] The procedure we are about to present works for any pseudo-Riemannian manifold. In particular, for a Riemannian manifold, replace η_{ij} by δ_{ij} everywhere.

[5] Note that at some $q \neq p$ on $\gamma_v(t)$, the identity (60) only holds for a *specific* a, namely the tangent vector to the geodesic at that point (preventing the conclusion ${\Gamma^k}_{ij}(q) = 0$, which indeed does not hold in general). On the other hand, at p, (60) holds for any direction a.

2.17 Local Lorentz geometry

We now establish that the *local* causal structure of a Lorentzian spacetime is similar to that of Minkowski space. As we shall see soon, *globally* the causality can be very different and the curvature of the spacetime can have a strong influence on the global causal structure.

We recall the notational convention that $I^+(p)$ lives in M while I_p^+ lives in T_pM.

Proposition 2.70. *Let (M, g) be a time-oriented Lorentzian manifold, $p \in M$ and $U_p \subset M$ a normal neighbourhood such that $\exp_p : T_pM \supset V_p \to U_p \subset M$ is a diffeomorphism. Then*

$$I^+_{U_p}(p) = \exp_p(I^+_p \cap V_p)\,, \tag{62}$$

where $I^+_{U_p}(p)$ denotes the chronological future of p in the Lorentzian manifold (U_p, g). In particular, if $q \in I^+_{U_p}(p)$ then q can be connected with p through a timelike geodesic.

Remark 2.71. *Note that in general $I^+_{U_p}(p) \subset I^+(p) \cap U_p$. One may refine the neighbourhood further such that equality holds but we will prove (62)*

Proof. One direction is simple: Clearly $\exp_p(I^+_p \cap V_p) \subset I^+_{U_p}(p)$ as given $Y \in I^+_p \cap V_p$ we have $\exp_p(Y) = \gamma_Y(1)$, where γ_Y is the timelike geodesic with $\gamma_Y(0) = p$, $\dot\gamma_Y(0) = Y$ (recall that length is preserved along a geodesic so the geodesic is indeed timelike).

Suppose now $q \in I^+_{U_p}(p)$ and we have a timelike curve $\alpha : [0, 1] \to U_p \subset M$ with $\alpha(0) = p$, $\alpha(1) = q$. We will show that $\exp_p^{-1}(\alpha)$ initially enters the interior of the future null cone at p and that it then remain within that cone (note that $\exp_p^{-1}(q) \notin I^+_p \cap V_p$ is possible only if $\exp_p^{-1}(\alpha)$ leaves the future null cone).

We choose normal coordinates at p. Define for $\tilde q \in I^+_{U_p}(p)$ the function

$$W_p(\tilde q) = -(x^0(\tilde q))^2 + (x^1(\tilde q))^2 + \ldots + (x^n(\tilde q))^2 = \eta_{\mu\nu} x^\mu(\tilde q) x^\nu(\tilde q)\,,$$

which is the Lorentzian length of the position vector of $\exp_p^{-1}(\tilde q)$ in T_pM. Note that we have $W_p(0) = 0$. We want to show $W_p(t) := W_p(\alpha(t)) < 0$ for $t > 0$ which implies that $\exp_p^{-1}(\alpha(t))$ stays in the future cone.

We compute

$$\dot W_p(t) = 2\eta_{\mu\nu}\dot x^\mu(t) x^\nu(t)\,, \tag{63}$$

$$\ddot W_p(t) = 2\eta_{\mu\nu}\dot x^\mu(t)\dot x^\nu(t) + 2\eta_{\mu\nu}\ddot x^\mu(t) x^\nu(t)\,, \tag{64}$$

where $x^\mu(t) := x^\mu(\alpha(t))$ are the (normal) coordinates of the position vector along the curve and hence $\dot x^\mu(t)$ are the (normal) coordinates of the vector tangent to $\alpha(t)$.

We have $\dot W_p(0) = 0$ and $\ddot W_p(0) = 2g(\dot\alpha(0), \dot\alpha(0)) < 0$. Hence there exists $\epsilon > 0$ such that $W_p(t) < 0$ for $t \in (0, \epsilon)$, so the curve $\exp_p^{-1}(\alpha(t))$ in the tangent space moves initially into the future cone.

Consider now an arbitrary point $(\tilde{x}^0, \tilde{x}^1, \dots, \tilde{x}^n) = \exp_p^{-1}(\alpha(\tilde{t}))$ on the curve $\exp_p^{-1}(\alpha(t))$ which lies in $I_p^+ \cap V_p$, i.e. $W_p(\alpha(\tilde{t})) < 0$ (note we just established the existence of such points). We claim that $\dot{W}_p(\tilde{t}) < 0$, which shows that $\exp_p^{-1}(\alpha(t))$ remains in $I_p^+ \cap V_p$ in a neighbourhood of $\tilde{t}$, which by continuity means that $\exp_p^{-1}(\alpha(t))$ remains in $I_p^+ \cap V_p$ for all $t \in [0,1]$ (and we are done). To establish the claim $\dot{W}_p(\tilde{t}) < 0$ we recall that by the Gauss Lemma

$$\begin{aligned} \eta_{\mu\nu} \dot{x}^\mu(\tilde{t}) x^\nu(\tilde{t}) &= g_p\left((d\exp_p^{-1})_{\exp_p^{-1}(\alpha(\tilde{t}))}(\dot{\alpha}(\tilde{t})), \exp_p^{-1}(\alpha(\tilde{t})) \right) \\ &= g_{\alpha(\tilde{t})}\left(\dot{\alpha}(\tilde{t}), (d\exp_p)_{\exp_p^{-1}(\alpha(\tilde{t}))} \exp_p^{-1}(\alpha(\tilde{t})) \right) \\ &= g_{\alpha(\tilde{t})}(\dot{\alpha}(\tilde{t}), P(\tilde{t})) \,, \end{aligned} \tag{65}$$

where $P(\tilde{t}) = (d\exp_p)_v v$ for $v = \exp_p^{-1}(\alpha(\tilde{t}))$ equals $\frac{d}{dt} \exp_p(t \exp_p^{-1}(\alpha(\tilde{t})))|_{t=1}$ which is the tangent vector at the point $\alpha(\tilde{t})$ to the radial geodesic γ which starts from p with (timelike) tangent vector $\exp_p^{-1}(\alpha(\tilde{t}))$. Since the right hand side of (65) is the product of two future directed timelike vectors it is negative and $\dot{W}_p(\tilde{t}) < 0$ follows from (63). □

Corollary 2.72. *We have* $\overline{I^+_{U_p}(p)} = J^+_{U_p}(p)$.

Proof. Exercise. □

Corollary 2.73. *Let* $S \subset M$ *for* (M,g) *a spacetime. Then the sets* $I^\pm(S)$ *are open.*

Proof. It suffices to show $I^\pm(p)$ is open since $I^+(S) = \bigcup_{p \in S} I^+(p)$. Let $q \in I^+(p)$ and γ a curve with $\gamma(0) = p$, $\gamma(1) = q$. Pick a point r on γ close to q such that a normal neighbourhood U_r of r contains the segment of the curve γ from r to q. Applying $\exp_r^{-1}$ we can pass to $T_r M$ and obtain a neighbourhood V_q of $\exp_r^{-1}(q)$ lying in $I_r^+ \subset T_r M$. By Proposition 2.70 we have that $\exp_r(V_q)$ lies in $I^+(r)$ and hence in $I^+(p)$. □

We close with a formal version of the twin paradox.

Proposition 2.74. *Let* U_p *be a normal neighbourhood of* p *in a spacetime* (M,g). *If there exists a timelike curve* γ *in* U_p *from* p *to* q*, then the radial geodesic segment* σ *from* p *to* q *satisfies* $\tau(\gamma) \leq \tau(\sigma)$ *with equality iff* γ *is a reparametrisation of* σ.

Proof. Parametrise γ as $\gamma : [0,1] \to U_p$, $\gamma(t) = \exp_p(r(t)n(t))$ for $g(n(t), n(t)) = -1$ (hence $g(n(t), \dot{n}(t)) = 0$) and $r(t) \geq 0$ with $\gamma(0) = p$ and $\gamma(1) = q$. We compute

$$\dot{\gamma}(t) = (d\exp_p)_{r(t)n(t)}(\dot{r}(t)n(t) + r(t)\dot{n}(t)) = \dot{r}(t) X_{\gamma(t)} + Y(t) \tag{66}$$

where $X_{\gamma(t)}$ denotes the tangent vector to the timelike geodesic $\tilde{\sigma}(\tau) = \exp_p(\tau n(t))$ (emanating from p with tangent vector $n(t)$) at $\tau = r(t)$ and the vector $Y(t) =$

$(d\exp_p)_{r(t)n(t)}(r(t)\dot{n}(t))$ is orthogonal to $X_{\gamma(t)}$ (hence spacelike) by the Gauss Lemma (and the aforementioned $g(n(t), \dot{n}(t)) = 0$). It follows that

$$\tau(\gamma) = \int_0^1 \sqrt{-g(\dot{\gamma}, \dot{\gamma})}dt = \int_0^1 \left(\dot{r}^2 - |Y(t)|^2\right)^{\frac{1}{2}} dt \leq \int_0^1 \dot{r}(t)dt = r(1) = \tau(\sigma).$$

where we have used $\dot{r} \geq 0$ and $r(1) = \tau(\sigma)$ following from $q = \exp_p(r(1)n(1))$. The "iff" follows since equality holds iff $|Y(t)| = 0$ for all $t \in [0, 1]$ hence iff $Y(t) \equiv 0$ since $Y(t)$ is spacelike. Now $Y(t) \equiv 0$ iff $\dot{n}(t) \equiv 0$ hence iff n is constant. But $\gamma(t) = \exp_p(r(t)n)$ is then indeed (a reparametrisation of) the radial geodesic connecting p and q. □

Exercise 2.75. *Generalise Proposition 2.74 to include the case where γ is merely piecewise smooth (i.e. the tangent vector to γ may be discontinuous at finitely many points but the jump has to be within the same future lightcone).*

3 Examples

We discuss the geometry of flat space and that of the simplest non-trivial solution of the vacuum Einstein equations, the Schwarzschild black hole.

3.1 Minkowski space

We consider $\mathbb{R}^{1+n}$ equipped with canonical coordinates $(x^0, x^1, ..., x^n)$ and metric

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$$

The case of interest is of course $n = 3$. We note that geodesics are straight lines in these coordinates since ${\Gamma^i}_{jk} = 0$ for all indices.

3.1.1 Hypersurfaces

Note that for fixed $\tau \in \mathbb{R}$, the hypersurface $\{x^0 = \tau\}$ is an example of a spacelike hypersurface. A more complicated example is the hyperboloid

$$-(x^0)^2 + (x^1)^2 + \ldots (x^n)^2 = -\tau^2 \, .$$

You can compute the induced metric and see that it is Riemannian or compute the normal to the hypersurface as the vector $n = x^0\partial_{x_0} + x^1\partial_{x^1} + ... + x^n\partial_{x^n}$ which has norm $-\tau^2 < 0$ by the above and is hence timelike.

An example for a timelike hypersurface is $\{x^1 = \tau\}$.

Concerning null hypersurfaces, we can define null hyperplanes by choosing a null vector n, i.e. $g(n, n) = 0$ and considering

$$N_n = \{v \in \mathbb{R}^{1+n} \mid g(v, n) = 0\} \, .$$

that is the set of all vectors "normal" to n. Note that the normal n is also tangent to the hypersurface! We will talk more about the geometry of general null hypersurfaces in a spacetime due course. Note as a second fundamental example the future light cone emanating from the origin

$$-(x^0)^2 + (x^1)^2 + \dots (x^n)^2 = 0 \quad \text{with} \quad \{x^0 > 0\} .$$

3.1.2 Causal structure

We recall the following (general) definitions for a spacetime: The set

$$J^+(p) = \{p\} \cup \{q \mid \text{there exists a future directed causal curve from } p \text{ to } q.\}$$

is called **the causal future** of $p \in M$. The set

$$I^+(p) = \{q \mid \text{there exists a future directed timelike curve from } p \text{ to } q.\}$$

is called the **timelike (or chronological) future** of $p \in M$. In the above, we will typically allow curves which are piecewise smooth (where at the breakpoints the respective limits of the tangent vectors lie in the same lightcone). We will make that more precise as we need it.

For now we note (draw a picture) that $I^+(p)$ is open (why?) and that $J^+(p)$ is closed and that the boundary of $\partial I^+(p) = J^+(p) \setminus I^+(p)$ is the future null cone at p. (While we will soon prove that $I^+(p)$ is open also in a general spacetime (intuitively this is quite clear), $J^+(p)$ is not always closed as can be seen by removing a point from Minkowski space.) Note that the null cone itself is generated by null geodesics emanating from p. Finally, given $q \in I^+(p)$ there exists a (unique) timelike geodesic connecting p and q. This is also not true in a general spacetime.[6]

Broadly speaking, we shall see soon that the curvature and the topology of the manifold can drastically change the causal structure!

3.1.3 Some words about special relativity

We now briefly discuss the physical relevance of the Minkowski metric.[7]

Let us begin with the principle of covariance, a first formulation of which was given by Galileo '1632 (*Galilean invariance*): The physical laws should take the same form in any (Galilean) inertial system of coordinates, i.e. in any system attached to an observer who moves freely, i.e. not subject to any external forces. Two Galilean inertial systems O and O' of $\mathbb{R}^4$ are related by the transformation

$$t' = at + b \quad , \quad \vec{x}' = R\vec{x} + \vec{v}t + \vec{c} . \tag{67}$$

[6] You might recall from the proof of the Hopf-Rinow theorem in Riemannian geometry that for a geodesically complete *Riemannian* manifold, any two points can be connected by a length minimsing geodesic. You will meet on Sheet 4 a Lorentzian manifold which is geodesically complete but there are $p, q \in M$ with $q \in I^+(p)$ that cannot be connected by a timelike geodesic. The condition of geodesic completeness in Riemannian geometry has to be replaced by the condition of "global hyperbolicity" in Lorentzian geometry in order for the above to hold. We will discuss this in more detail later.

[7] The geometric formulation of Einstein's special relativity in terms of a Lorentzian metric on $\mathbb{R}^4$ is due to Hermann Minkowski, hence the name.

Here $R \in SO(3)$ is a constant rotation matrix, $\vec{v}, \vec{c} \in \mathbb{R}^3$ and $a, b \in \mathbb{R}$. In other words, an observer moving with constant velocity $\vec{v}$ with respect to another observer should come up with the same physical laws, no matter how she orients her coordinate axes (the meaning of R) and not matter how she sets the units of time and the origin of the coordinates.

Exercise 3.1. *Recall that the two body problem in Newtonian mechanics is given by the two coupled ODEs for point masses m_1, m_2 at the points $\vec{x}_1(t), \vec{x}_2(t)$ respectively.*

$$m_1 \ddot{\vec{x}}_1 = -G \frac{m_1 m_2 (\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3} \quad , \quad m_2 \ddot{\vec{x}}_2 = G \frac{m_1 m_2 (\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3} \, . \tag{68}$$

Check the above is indeed invariant (transforms covariantly) under Galilei transformations.

The famous (and extremely successful) Maxwell's equations describing electromagnetism are *not* invariant under spacetime transformations of the form (67), which constituted a mystery before special relativity was discovered in (1905). Maxwell's equations remain invariant under a different group of transformations, an example of which is given by[8]

$$\tilde{t} = \frac{t + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \quad \tilde{x} = \frac{x + vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \quad \tilde{y} = y \quad , \quad \tilde{z} = z \, , \tag{69}$$

where c is the speed of light.[9] Special relativity, in the formulation of Minkowski, gives (among other things) a beautiful geometric explanation of this algebraic fact.

To illustrate this, we recall that in relativity we have a new, more geometric notion of inertial observer: Such observers move along timelike geodesics in the spacetime, which in the Minkowski case are just lines inside the light cone. How are two such inertial observers related?

Consider one such observer O moving along the curve $\alpha(t) = (t, 0, 0, 0)$. Note this curve has $|\dot{\alpha}| = 1$ and is hence parametrised with respect to proper time (length). Consider a second observer, moving in a different inertial system that is moving at speed v with respect to O along the x-axis

$$\tilde{\alpha}(t) = (t, vt, 0, 0) \, .$$

This curve is not parametrised with respect to proper time but an easy computation yields that

$$\tilde{\alpha}(\tau) = (\gamma\tau, \gamma v \tau, 0, 0) \qquad \text{with } \gamma = \frac{1}{\sqrt{1 - v^2}} .$$

[8]Such special Lorentz transformation were known to Lorentz before special relativity had been formulated.

[9]Note that in the limit $c \to \infty$ (or better $|\frac{v}{c}| \ll 1$, i.e. small velocities) the transformation becomes the Galilean one $\tilde{t} = t$, $\tilde{x} = x + vt$.

represents the same curve now parametrised with respect to proper time τ. Note that $\gamma = \cosh\phi$ where ϕ is the hyperbolic angle between the two observers.

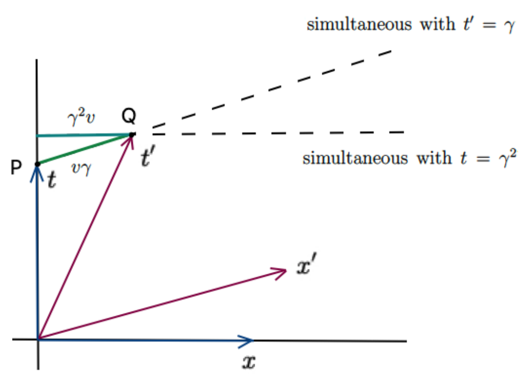
Now the fundamental point about special relativity is that there exists an isometry, a Lorentz boost, that maps observer O into observer O'. It is given by a map $F_v : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$,

$$F_v(x^0, x^1, x^2, \ldots, x^n) = \big(\gamma(x^0 + vx^1), \gamma(x^1 + vx^0), x^2, \ldots, x^n\big) \ , \tag{70}$$

which is of course nothing but (69)! Note that this is indeed a one-parameter family (in v) of (linear) isometries with infinitesimal generator being the Killing vectorfield $x^1\partial_{x^0} + x^0\partial_{x^1}$ introduced in Example 2.64. It maps

$$F_v(O(\lambda)) = F_v(\lambda, 0, \ldots, 0) \mapsto (\gamma\lambda, \gamma v\lambda, 0, \ldots, 0) = O'(\lambda)$$

the position of the observer O after proper time λ to the position of the observer O' after proper time λ. The following picture is illuminating. The isometry maps the inertial system $(x^0, x^1, \ldots, x^n)$ of observer O to the inertial system $((x^0)', (x^1)', \ldots, (x^n)')$ of observer O'. Note that since it is an isometry, the metric for O' is still given by $g' = -(d(x^0)')^2 + (d(x^1)')^2 + \ldots + (d(x^n)')^2$.



Through the picture we can illustrate several features and phenomena of special relativity:

1. Because different observers are related by an isometry, all observers agree on the spacetime difference $-(x_0 - \tilde{x}_0)^2 + \sum_{i=1}^{3}(x_i - \tilde{x}_i)^2$ between two points $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ and (x_0, x_1, x_2, x_3). In particular, the light cone (the set of vectors of "norm" 0) is invariant under the transformation: For instance, the vector $(1,1,0,0)$ gets mapped to $\gamma(1+v)\,(1,1,0,0)$. So all observers actually agree on the path of light rays and light has the same velocity ($= 1$ in our normalisation) in all inertial systems.

2. While observers agree on the spacetime difference between two points, the notion of "two events happening at the same time" has no invariant meaning. Instead each observer has their own hypersurface of simultaneity as indicated in the picture. (The invariance of the light cone ensures that events that are causally (un)related remain so in any inertial system.)

3. Let us illustrate the two phenomena of time dilation and length contraction with a concrete computation. Consider the point $P = (1, 0, 0, 0)_O$ (we use the subscript O to denote the coordinate that O will give to the space time point P). In the coordinate system O', with orthonormal basis $e'_0 = (\gamma, \gamma v, 0, 0)_O$, $e'_1 = (\gamma v, \gamma, 0, 0)_O$, $e'_2 = e_2$, $e'_3 = e_3$, we compute $P = (\gamma, -\gamma v, 0, 0)_{O'}$, which is how O' will refer to the point P. Note that the two observers agree on the spacetime distance from 0 to P to be 1. We can do the same computation for the point Q which is defined as the point which lies – for O' – on the same surface of simultaneity as P. Here one computes $Q = (\gamma, 0, 0, 0)_{O'} = (\gamma^2, \gamma^2 v, 0, 0)_O$. We can explain the phenomenon of time dilation as follows. In the clock carried by O, one unit of t-time has passed when O reaches P. However, for O' actually γ units of t'-time have passed (Q is simultaneous with P at $t' = \gamma$). In other words, to O', the clock of O will appear to tick slower than it does for O. We can use the same picture to explain the phenomenon of length contraction: When observer O' is at Q, he will say that O is at distance γv. Observer O will say that Q is distance $\gamma^2 v$ away on her surface of simultaneity. So the distance seems smaller (length contracted) for the moving observer. Finally, note that in the picture it seems that γv^2 is smaller than γv. This is of course due to the underlying Lorentzian geometry: The Pythagorean theorem holds in the form $-(\gamma^2 - 1)^2 + (\gamma^2 v)^2 = \gamma^2 v^2$!

How could one have come up with the Lorentz transformations (other than purely algebraically from Maxwell's equations)? At the heart of the theory is the postulate (that can be experimentally verified) that the speed of light is measured to be the same in all inertial frames. This means that even if you are moving with half the speed of light and a light ray overtakes you from behind, you would still measure the speed of the light ray as c, not $\frac{c}{2}$. Once you accept this, you notice the following: If a flash of light emanates from a spacetime point labeled $(t_A, \vec{x}_A)$ with respect to the coordinates of an observer O, then after a certain amount of time a point on the spherical wave front will be labelled $(t_B, \vec{x}_B)$. Another observer might label these same spacetime points as $(\tilde{t}_A, \tilde{\vec{x}}_A)$ and $(\tilde{t}_B, \tilde{\vec{x}}_B)$. However, since the speed of light is the same for both observers we must have

$$c^2|t_A - t_B|^2 - |\vec{x}_A - \vec{x}_B|^2 = 0 = c^2|\tilde{t}_A - \tilde{t}_B|^2 - |\tilde{\vec{x}}'_A - \tilde{\vec{x}}'_B|^2 \,.$$

From here it is not far to write down (linear) transformations of $\mathbb{R}^4$ that keep the null cones invariant. Of course it is still some way from writing down the transformations (and connecting them to a new structure of space and time) to the geometric formulation in terms of a Lorentzian metric. This was the achievement of Hermann Minkowski.

Finally, the great success of special relativity also created a new fundamental problem. The Newtonian equation are *not* invariant under the Lorentz transformations. This was the beginning of Einstein's struggle to formulate a theory of gravity consistent with special relativity. Here an important conceptual step was to generalise the principle of special covariance to the principle of *general* *covariance*: The physical laws should take the same form in *any* coordinate system (not only inertial ones), in other words, the physical laws have to be described by tensor equations.

3.1.4 The Penrose diagram of Minkowski space

Definition 3.2. *Let (M, g) be a Lorentzian manifold. We say that another metric $\tilde{g}$ on M is* <u>*conformal*</u> *to g if $\tilde{g} = \Omega^2 g$ for some smooth (positive) function $\Omega \in C^\infty(M)$.*

Note that if $\tilde{g}$ is conformal to g, then $\tilde{g}(X, X) = \Omega^2 g(X, X)$ so

$$X \;\; \tilde{g}\text{-timelike/ null/ causal/ spacelike} \;\Leftrightarrow\; X \;\; g\text{-timelike/ null/ causal/ spacelike}$$

hence $J^\pm_{\tilde{g}}(S) = J^\pm_g(S)$ so the metrics have the same causal structure.

The main idea of Penrose diagrams is the following: We want to understand the global structure of a spacetime (M, g). To achieve this, we do a coordinate transformation that maps the spacetime to a bounded region (whose boundary now corresponds to the asymptotic infinities of the spacetime). The components of the metric in the new coordinates will blow up at the asymptotic boundaries but we will multiply the metric with a conformal factor such that $\tilde{g} = \Omega^2 g$ extends regularly to the boundary (or at least part of the boundary!). We can then add the infinities as the boundary obtaining a conformal compactification of the spacetime. The key is that in this process, because we have kept the conformal class of the metric, we can read off the causal structure from the bounded region equipped with the metric $\tilde{g}$, which is often easier. Let us see some of the details.

Penrose diagram of $1 + 1$ dimensional Minkowski space

Consider $M = \mathbb{R}^{1+1}$ with metric $g = -dt^2 + dx^2$. Define null coordinates $u = t - x$ and $v = t + x$ with coordinate range $u, v \in \mathbb{R}$. We obtain

$$g = -\frac{1}{2}(dv \otimes du + du \otimes dv)$$

We now define $\tilde{u} = \arctan u$, $\tilde{v} = \arctan v$ so $\tilde{u}, \tilde{v} \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Note $u = \tan \tilde{u}$ and $du = \frac{1}{\cos^2 \tilde{u}} d\tilde{u}$ and $dv = \frac{1}{\cos^2 \tilde{v}} d\tilde{v}$. We obtain

$$g = -\frac{1}{2} \frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}} \left(d\tilde{v} \otimes d\tilde{u} + d\tilde{u} \otimes d\tilde{v} \right) .$$

We now define $\Omega^2 = \cos^2 \tilde{u} \cdot \cos^2 \tilde{v}$ and set $\tilde{g} = \Omega^2 g$, hence

$$\tilde{g} = -\frac{1}{2} \left(d\tilde{v} \otimes d\tilde{u} + d\tilde{u} \otimes d\tilde{v} \right)$$

and $\tilde{g}$ is regular for $\tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$. What have we done abstractly? We have defined a diffeomorphism

$$\Phi : (\mathbb{R}^{1+1}_{t,x}, g) \to (\tilde{M}, \tilde{g})$$

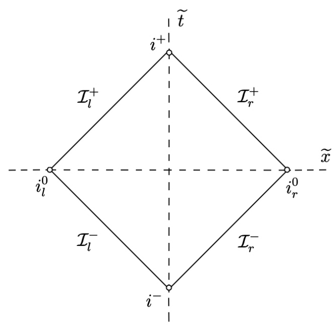
where $\tilde{M} = (-\frac{\pi}{2}, \frac{\pi}{2})_{\tilde{u}} \times (-\frac{\pi}{2}, \frac{\pi}{2})_{\tilde{v}}$. The diffeomorphism Φ satisfies

$$(\Phi^{-1})^\star g = \Omega^{-2}\tilde{g}$$

i.e. Φ is a conformal isometry. In summary, we have found a conformal isometry from $\mathbb{R}^{1+1}$ to a bounded subset of $\mathbb{R}^{1+1}$ (equipped with the standard metric).

We can finally attach the boundary to $\tilde{M}$. Set $\tilde{t} = \frac{1}{2}(\tilde{u}+\tilde{v})$ and $\tilde{x} = \frac{1}{2}(\tilde{v}-\tilde{u})$ and

$$\overline{\tilde{M}} = \{(\tilde{t},\tilde{x}) \in \mathbb{R}^2 \mid -\frac{\pi}{2} \leq \tilde{t}+\tilde{x} \leq \frac{\pi}{2} \quad , \quad -\frac{\pi}{2} \leq \tilde{t}-\tilde{x} \leq \frac{\pi}{2}\}$$



Note that for $p \in \tilde{M}$ we have

$$J^+_{\tilde{g}}(p) = J^+_{\Omega^{-2}\tilde{g}}(p) = J^+_{(\Phi^{-1})^\star g}(p)$$

so we can read off the global causality of M from $\tilde{M}$, i.e. the Penrose diagram. We have given names to the boundary components

- $\mathcal{I}_r^+/\mathcal{I}_l^+$ called right/left future null infinity (This set corresponds to the asymptotic endpoints of all future directed right/left going null geodesics.)
- $\mathcal{I}_r^-/\mathcal{I}_l^-$ called right/left past null infinity (This set corresponds to the asymptotic endpoints of all past directed right/left going null geodesics.)
- i^+ called future timelike infinity (This is the asymptotic endpoint of all future directed timelike geodesics of (M, g).[10])
- i^- called past timelike infinity (This is the asymptotic endpoint of all past directed timelike geodesics of (M, g).
- i_r^0 / i_l^0 called right/ left spacelike infinity (This is the asymptotic endpoint of all right/left going spacelike geodesics.

Note that there are of course future directed timelike curves in $\tilde{M}$ going to $\mathcal{I}_r^+/\mathcal{I}_l^+$ but they are not timelike geodesics in M.

[10]Indeed, note that $t+x \to \infty$ and $t-x \to \infty$ holds along a future directed timelike geodesic in M.

Penrose diagram of $3+1$ dimensional Minkowski space We now consider $M = \mathbb{R}^{1+3}$ with metric $g = -dt^2 + dr^2 + r^2 d\sigma^2$ where $d\sigma^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the round metric on the unit sphere. Note that this way of writing the Minkowski metric corresponds to a choice of $SO(3)$ action.

As before we introduce null coordinates $u = t - r$ and $v = t + r$. We note $r \geq 0$ and $r = 0$ iff $u = v$. The coordinate range is $-\infty < u \leq v < \infty$ and the metric reads

$$g = -\frac{1}{2}(dv \otimes du + du \otimes dv) + \frac{1}{4}(v-u)^2 d\sigma^2 \,.$$

Note that constant u hypersurfaces correspond to outgoing cones with vertex on the axis $r = 0$ and constant v hypersurfaces to ingoing cones with vertex on that axis. This is the simplest case of a double null foliation of spacetime (away from the axis). We again set $\tilde{u} = \arctan u$ and $\tilde{v} = \arctan v$ to obtain

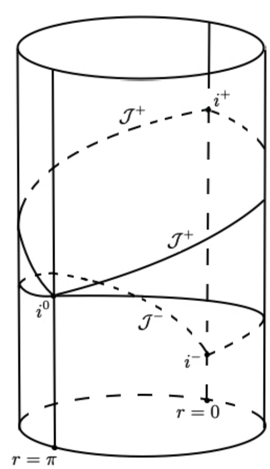
$$g = \frac{1}{\cos^2\tilde{u}\cos^2\tilde{v}}\left(-\frac{1}{2}(d\tilde{v} \otimes d\tilde{u} + d\tilde{u} \otimes d\tilde{v}) + \frac{1}{4}\sin^2(\tilde{v}-\tilde{u})d\sigma^2\right)$$

with coordinate range $-\frac{\pi}{2} < \tilde{u} \leq \tilde{v} < \frac{\pi}{2}$. Here the form of the metric follows from $\frac{\sin(\tilde{v}-\tilde{u})}{\cos\tilde{u}\cos\tilde{v}} = \frac{\sin\tilde{v}\cos\tilde{u} - \sin\tilde{u}\cos\tilde{v}}{\cos\tilde{v}\cos\tilde{u}} = \tan\tilde{v} - \tan\tilde{u} = v - u$.

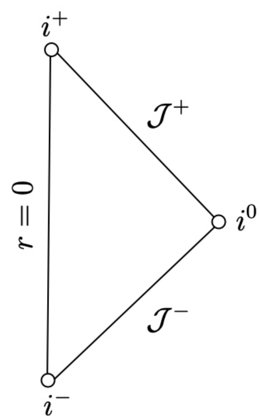
We choose $\Omega^2 = 4\cos^2\tilde{u}\cos^2\tilde{v}$ and $\tilde{t} = \tilde{v} + \tilde{u}$ and $\tilde{x} = \tilde{v} - \tilde{u}$ and set

$$\tilde{g} = \Omega^2 g := -d\tilde{t}^2 + d\tilde{x}^2 + \sin^2\tilde{x} d\sigma^2$$

where the coordinate range is now $-\frac{\pi}{2} < \frac{1}{2}(\tilde{t}+\tilde{x}) < \frac{\pi}{2}$ and $0 \leq \tilde{x} < \pi$. We recognise $\tilde{g}$ as the natural metric on (a subset of) $\mathbb{R} \times S^3$, the Einstein static universe.



In more elaborate language, we have constructed a conformal isometry from $(\mathbb{R}^{1+3}, g)$ to a bounded subset $\tilde{M}$ of the Einstein static universe. We can peel off the region from the cylinder and draw the Penrose diagram in the plane. Geometrically, we are considering the quotient with respect to $SO(3)$.



Every point in the picture corresponds to a 2-sphere except points on $\{r = 0\}$, and the points i^+, i^- and i^0, which are points. The set $i^\pm$ is the future/ past endpoint of all future/past directed timelike geodesics. The set $\mathcal{I}^\pm$ is the future/ past endpoint of all future/ past directed null geodesics and i^0 is the endpoint of all spacelike geodesics.

Note the difference between radial null geodesics in M (which correspond to lines at 45 degrees in the Penrose diagram) and non-radial null geodesics (which will look like *timelike* curves in the Penrose diagram).

We can now start to see some of the usefulness of the Penrose diagram: (1) It allows for a direct assessment of causal relations between points. (2) It allows to make precise sense of the notion of "fields at infinity" by giving a precise meaning to the asymptotic structure of spacetime. Of course these things can only be appreciated in more complicated examples. One might already hope that "asymptotically flat spacetimes" (we have to give meaning to this!) should have a similar asymptotic structure at infinity.

3.2 The Schwarzschild geometry

We now turn to the most important example of this course which is the simplest example of a black hole solution. While the most natural way to introduce the Schwarzschild manifold is perhaps the one in Section 3.2.1 below, I will actually start with the form of the metric as first discovered by Schwarzschild in 1916 as it is quite instructive to appreciate some of the struggles to understand the spacetime geometry.

Let $M > 0$ and define the manifold $\mathcal{M} = (-\infty, \infty)_t \times (2M, \infty)_r \times S^2$ equipped with the metric

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{71}$$

We note a few things directly:

- g satisfies $Ric[g] = 0$ (Exercise Sheet 4 or 5).
- $T = \partial_t$ is Killing, i.e. $(\mathcal{L}_T g)_{\mu\nu} = \partial_t g_{\mu\nu} = 0$. (The metric is called stationary (in fact static).) Note that ∂_t is timelike and has norm $-(1 - \frac{2M}{r})$. We can time-orient $(\mathcal{M}, g)$ by declaring ∂_t to be future directed.
- $g \to -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ as $r \to \infty$. (The metric is called asymptotically flat.)
- The group $SO(3)$ acts by isometries (i.e. the metric is spherically symmetric). Another way of saying this is that the vectorfields

  $$K_1 = \partial_\phi \ , \ K_2 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi \ , \ K_3 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$$

  are all Killing and generate the Lie algebra of SO(3).

The first big question we have to address is what happens at $r = 2M$. In many papers until the 1950s, the set $r = 2M$ is referred to as the Schwarzschild singularity but if there is one thing you should take away from this course, then that the metric is NOT singular at $r = 2M$!

Let us explain. First a cautionary (trivial) example that illustrates the issue that finding a solution to Einstein's equations does not only consist in writing down a metric but also involves specifying the manifold and its differentiable structure on which the metric is defined. The question what this manifold should be will only be satisfactorily be answered once we discuss the initial value problem!

Consider the metric $g = -t^2dt^2 + dx^2$ on $M = (0, \infty)_t \times (-\infty, \infty)_x$. A priori this metric looks singular at $t = 0$. However consider the metric $\tilde{g} = -d\tilde{t}^2 + dx^2$ on $\mathbb{R}^{1+1}$ and consider the coordinate transformation $\tilde{t} = \frac{1}{2}t^2$ for $t > 0$. In other words, (M, g) is isometric to the flat metric on the upper half plane $t > 0$. Nothing singular happens at $t = 0$. In fact, we could extend the metric past $\{t = 0\}$ to the flat metric defined on all of $\mathbb{R}^2$. How would we notice this (if the example wasn't so trivial)? Well, we could compute timelike past directed geodesics from $t = 1$ and realise that they reach $\{t = 0\}$ in finite affine parameter time (exercise!). Hence (M, g) is geodesically incomplete but we can isometrically embed it into a larger manifold which is geodesically complete.

For now we will take the tentative point of view that the manifold on which to define the metric should be as large as possible in the sense that geodesics should be either complete (i.e. defined on all of $\mathbb{R}$) or run into a singularity.

Back to the Schwarzschild metric, let us send a future directed null geodesics from some point $(t_0, r_0 > 2M, \theta_0, \phi_0)$ inwards towards $r = 2M$.[11] We recall from Sheet 4 that the geodesics $x^\mu(\tau)$ can be obtained by varying the functional (Lagrangian)

$$L = g_{\mu\nu}(x(\tau))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -(1 - \frac{2M}{r})\dot{t}^2 + (1 - \frac{2M}{r})^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta d\dot{\phi}^2\right)$$

[11] The same considerations could be done for timelike geodesics.

Proposition 2.63 (or the Euler Lagrange equations directly) tell us that

$$E := g_{\mu\nu}\,(\partial_t)^\mu\,\frac{dx^\nu}{d\tau} = \left(1-\frac{2M}{r}\right)\dot{t} \quad \text{and} \quad L := g_{\mu\nu}\,(\partial_\phi)^\mu\,\frac{dx^\nu}{d\tau} = r^2\sin^2\theta\dot{\phi}$$

are conserved. We may also deduce from the Euler Lagrange equation for θ that we can set wlog $\theta = \frac{\pi}{2}$ (Exercise. Hint: Derive the ODE for θ. Deduce that setting $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ initially implies $\theta = \frac{\pi}{2}$ along the geodesic. Use the spherical symmetry of the background to achieve the initial conditions.) All this implies that a null geodesic in Schwarzschild satisfies

$$-\frac{E^2}{1-\frac{2M}{r}} + \frac{\dot{r}^2}{1-\frac{2M}{r}} + \frac{L^2}{r^2} = 0\,.$$

Note that had we considered timelike geodesics the only difference would be the right hand side being equal to -1. If we look at radially ingoing future directed (i.e. $\dot{t} > 0$) null geodesics ($L = 0$, $\dot{\phi} = 0$) we see that

$$\frac{dr}{d\tau} = -E$$

This shows that we can reach $r = 2M$ from a point $(t_0, r_0, \theta_0, \phi_0)$ in $\mathcal{M}$ in finite affine parameter time τ. On the other hand, we have that

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\frac{1}{1-\frac{2M}{r}}$$

so as $r \to 2M$ we have $t \to \infty$ so t becomes infinite indicating that t is not a good coordinate at $r = 2M$.

We now introduce a coordinate $r^\star = r + 2M\log(r-2M)$ and define $v = t + r^\star$. Then the v-coordinate is constant along the radially ingoing null geodesics since $\frac{dv}{d\tau} = \dot{t} + \frac{dr^\star}{dr}\dot{r} = \frac{E}{1-\frac{2M}{r}} + \frac{\dot{r}}{1-\frac{2M}{r}} = 0$. We compute $dt = dv - dr^\star = dv - \frac{dr}{1-\frac{2M}{r}}$. The metric in these coordinates (called ingoing Eddington-Finkelstein coordinates) becomes

$$\tilde{g} = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{72}$$

This metric is completely regular at $r = 2M$ and we can consider it on the manifold $\tilde{\mathcal{M}} = (-\infty,\infty)_v \times (0,\infty)_r \times S^2$ with time orientation such that $-\partial_r|_{(v,\theta,\phi)}$ is future directed. Note that $\tilde{g}(\partial_r, \partial_r) = 0$ so the vectorfield $\partial_r|_{(v,\theta,\phi)}$ is null.[12] The old $(\mathcal{M}, g)$ embeds isometrically (and time-orientation preserving) to the new manifold $(\tilde{\mathcal{M}}, \tilde{g})$. We also observe that $r = 2M$ is in fact a null hypersurface in $(\mathcal{M}, g)$.

Exercise 3.3. *Check that hypersurfaces of constant r are spacelike for $r < 2M$ and timelike for $r > 2M$.*

[12]Beware of the fundamental confusion of calculus that $\partial_r|_{(v,\theta,\phi)} \neq \partial_r|_{(t,\theta,\phi)}$, the latter vectorfield being spacelike for $r > 2M$.

It turns out that the new manifold is still geodesically incomplete! To see this we again compute timelike geodesics. Imagine you are at a point for which $r < 2M$. Suppose also you want to move towards the future, i.e. $\frac{dv}{d\tau} > 0$. Then along any timelike causal curve $x^\mu = (v(\tau), r(\tau), \theta(\tau), \phi(\tau))$ normalised with proper time we have

$$2\dot{v}\dot{r} = -1 + (1 - \frac{2M}{r})\dot{v}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) < 0$$

We see that necessarily $\dot{r} < 0$ along the curve so such an observer necessarily moves towards smaller r.

Exercise 3.4. *Show that any future directed causal geodesic starting from a point in $r < 2M$ reaches $r = 0$ in finite affine parameter time.*

What happens at $r = 0$? It turns out there is ferocious and genuine singularity:

Lemma 3.5. *The Kretschmann scalar satisfies $R_{abcd}R^{abcd} = \frac{C_M}{r^6}$.*

The proof of the Lemma is a (long) computation. The statement implies that we cannot extend the metric passed $r = 0$ as a C^2 metric. It turns out that actually one cannot extend even as a C^0 metric, this being a relatively recent result (Sbierski, 2018).

3.2.1 The Kruskal manifold and the Penrose diagram

One could continue the above explorations of the geometry to find further analytic extensions of the $(\tilde{\mathcal{M}}, g)$ by studying spacelike and past directed null geodesics towards $2M$. We cut the story short and present directly the maximally extended Schwarzschild spacetime. We define

$$\mathcal{M}_{Kruskal} := (-\infty, \infty)_U \times (-\infty, \infty)_V \times S^2 \cap \{UV < 1\} \tag{73}$$

equipped with metric

$$g_M = -4\Omega_K^2 dUdV + r^2(U,V)d\sigma^2 \,,$$

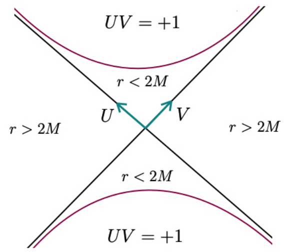
where

$$\Omega_K^2 = \frac{8M^3}{r}\exp\left(-\frac{r}{2M}\right)$$

and $r : (-\infty, \infty)_U \times (-\infty, \infty)_V \cap \{UV < 1\} \to \mathbb{R}^+$ is defined implicitly by the relation

$$\left(\frac{r(U,V)}{2M} - 1\right)\exp\left(\frac{r(U,V)}{2M}\right) = -UV \,.$$

We time orient the Lorentzian metric $(\mathcal{M}_{Kruskal}, g_M)$ by declaring $\partial_U + \partial_V$ to be future directed. We see that hypersurfaces of constant U and hypersurfaces of constant V are null hypersurfaces as the induced metric generates. We hence depict the UV coordinate system at 45 degrees to capture the causal geometry of the spacetime:



We can now appreciate the black hole aspect of this metric. Being in the region $\{U > 0\} \cap \{r < 2M\}$ we are causally disconnected from the region $r > 2M$ as not future directed causal curve can reach the region $r > 2M$. In other words, not even light can escape the region $r < 2M$. The hypersurface $\{V > 0\} \cap \{r = 2M\}$ is called the future event horizon of the black hole. Note that it separates the region of the manifold that can causally communicate with observers at large r from those which cannot.

To relate $(\mathcal{M}_{Krusal}, g_M)$ to the old Schwarzschild manifold $(\mathcal{M}, g)$ note that for $U < 0$ and $V > 0$ we can define coordinates (t, r, θ, ϕ) by the relations

$$\frac{V}{U} = -e^{-\frac{t}{2M}} \quad , \quad UV = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}} \, .$$

A straightforward computation shows that g_M in (t, r, θ, ϕ) coordinates takes the form of the metric g, establishing that $(\mathcal{M}, g)$ embeds isometrically as the region $U < 0, V > 0$ of the Kruskal manifold.

As for Minkowski, we can compactify the U and V directions to obtain the Penrose diagram of Schwarzschild. We set

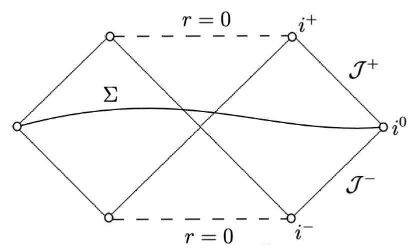
$$\tilde{u} = \arctan U \quad , \quad \tilde{v} = \arctan V$$

and note that the range of the new coordinates is

$$\{(\tilde{u}, \tilde{v}) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \mid |\tilde{u} + \tilde{v}| < \frac{\pi}{2}\} \, .$$

The latter restriction follows from $\tan \tilde{u} \cdot \tan \tilde{v} < 1$ which implies $\cos(\tilde{u} + \tilde{v}) > 0$, hence $|\tilde{u} + \tilde{v}| < \frac{\pi}{2}$. You can write down the conformally rescaled metric $\tilde{g} = \Omega^2 g_M$

for $\Omega^2 = \frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}}$ to deduce the Penrose diagram:



One observes that $\tilde{g}$ extends continuously to $\mathcal{J}^+$, $\mathcal{J}^-$ and i_0 but not to $i^\pm$ (why not?). Timelike geodesics either asymptote to $\{r = 0\}$ or to $i^\pm$. Null geodesics asymptote to $\{r = 0\}$ or to $\mathcal{J}^+$, $\mathcal{J}^-$ or to $i^\pm$. All of these claims will follow easily once you have solved Sheet 5 and know how to compute geodesics on the Schwarzschild manifold!

A famous example of null geodesics on the Schwarzschild manifold are the so-called trapped geodesics: On Sheet 5 you will show that there are null geodesics that remain tangent to the hypersurface $r = 3M$ for all times. This is very different from Minkowski space where all light rays disperse to infinity, i.e. leave any bounded region!

While the above trapped null geodesics are unstable (i.e. if you slightly perturb the initial conditions the geodesics either enter the black hole or disperse to infinity), there are also trapped *timelike* geodesics and these are stable. In fact, they describe the planetary orbits that we observe. The prediction of the (exact correction to the) perihelion of Mercury was one of the great early successes of Einstein's theory.

Back to the black hole nature of the solution we could define the black hole region more abstractly (not making reference to the Schwarzschild metric) as

$$\mathcal{M}_{Kruskal} \setminus J^- \left(\mathcal{J}^+\right) .$$

and the event horizon $\mathcal{H}^+$ as the future boundary of that region. This definition requires the notion of the asymptotic set $\mathcal{J}^+$ whose existence one may hope to establish for any asymptotically flat spacetime (more on this later). It also expresses the idea of the black hole region as the region of the manifold which is not in the causal past of observers at infinity.

A final remark: There is a more complicated explicit black hole geometry, the famous **Kerr family of rotating black hole solutions**. However, we postpone discussing their geometry to a later point.

This concludes our discussion of the Schwarzschild geometry. As mentioned it took (unfortunately) decades to reach the conclusion of this geometric picture. Important questions now arise: How physically realistic is the black hole character of the solution? Are there other solutions of the Einstein vacuum equations describing black holes? Will they have the same type of singularity in the interior? How can we potentially find or construct these solution? The answer to these questions lies in the PDE nature of the Einstein equations to which we now turn.

4 Geometric (non)linear wave equations and the hyperbolic character of the Einstein equations

The goal of this section is to review some basic material on linear and non-linear wave equations. Clearly, this cannot replace a proper course on this subject and is merely to give you some intuition for the main ideas involved and a working knowledge of the key existence and uniqueness theorems, which we then want to apply to establish local existence for the Einstein vacuum equations.

We'll begin with a discussion of wave equations on $\mathbb{R}^{1+n}$ (so you can forget for the moment everything we did about geometry!) and then connect to the Lorentzian geometric picture we established in previous chapters.

4.1 The wave equation on Minkowski space

Let us start with the most basic wave equation, the linear wave equation on Minkowski space. We want to solve the Cauchy problem

$$\Box\phi = -\partial_t^2\phi + \partial_{x^1}^2\phi + \dots \partial_{x^n}^2\phi = 0 \ , \ \phi(t=0,x) = f(x) \ , \ \partial_t\phi(t=0,x) = g(x) \, . \tag{74}$$

Here $f, g \in C^\infty(\mathbb{R}^n)$ for simplicity and we will be mostly interested in the case $n = 3$. (Of course you can already note $\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$ for η being the Minkowski metric connecting to our geometric picture.) We have

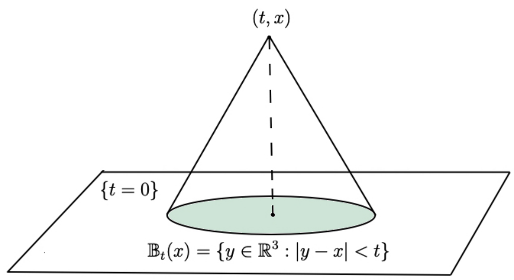
Theorem 4.1. *There exists a unique smooth solution $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ of the problem (74).*

Proof. For $n = 3$ a solution is given by the formula

$$\phi(t,x) = \frac{1}{4\pi t^2}\int_{|y-x|=|t|}\left(tg(y) + f(y) + \sum_i (\partial_{y^i} f)(y^i - x^i)\right) dS_y \, , \tag{75}$$

where dS_y is the induced surface measure on the sphere $|y-x| = |t|$, i.e. the sphere with center x and radius $|t|$. Similar explicit representation formulae exist for all dimensions. The uniqueness will be established further below. □

While we will not derive the above solution formula you should check that it indeeds solves (74) and understand the geometry of how the solution is obtained



We immediately see the domain of dependence property of the wave equation and how the finite speed of propagation (light) is built into the theory!

4.2 The energy estimate

Let us assume that we have a classical C^2 solution of (74). Multiplying the wave equation by $-\partial_t \phi$ yields (we denote by ∇_x the spatial gradient)

$$0 = -\Box\phi \cdot \partial_t \phi = \frac{1}{2}\partial_t \left(\partial_t \phi\right)^2 - div_x \left(\partial_t \phi \nabla_x \phi\right) + \nabla_x \partial_t \phi \cdot \nabla_x \phi = 0 \,, \tag{76}$$

which can be rearranged to

$$0 = \frac{1}{2}\partial_t \left[\left(\partial_t \phi\right)^2 + |\nabla_x \phi|^2\right] - div_x \left(\partial_t \phi \nabla_x \phi\right) \,. \tag{77}$$

If we integrate this over the spacetime slab $[0, T] \times \mathbb{R}^n$, then assuming that ϕ is of compact spatial support for all times (we'll justify this in a second) we would obtain the energy conservation law

$$\int_{t=T} d^n x \left[\left(\partial_t \phi\right)^2 + |\nabla_x \phi|^2\right] = \int_{t=0} d^n x \left[\left(\partial_t \phi\right)^2 + |\nabla_x \phi|^2\right] \,.$$

As this works for any $\tau \leq T$ we obtain for all $t \in \mathbb{R}$

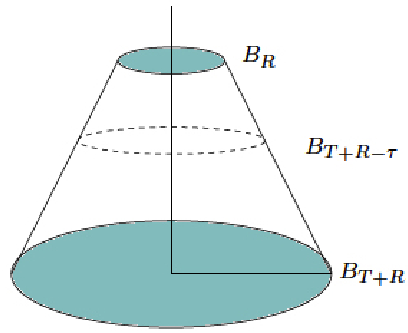
$$\|\partial_t \phi\left(t, \cdot\right)\|^2_{L^2_x} + \|\phi\left(t, \cdot\right)\|^2_{\mathring{H}^1_x} = \|g\|^2_{L^2} + \|f\|^2_{\mathring{H}^1} \,. \tag{78}$$

There is in fact a much better way to formulate this conservation law if we suitably localize the estimate.

Fix $T > 0$, $R > 0$ and consider a region

$$K = \bigcup_{\tau \in [0,T]} \{\tau\} \times B_{R+T-\tau} \tag{79}$$

where $B_{R+T-\tau}$ is the ball of radius $R+T-\tau$ centred at the origin.



You may think of this region as a cut-off (at $t=0$ and $t=T$) past light cone with tip at $\left(T+R,\vec{0}\right)$. We will denote the boundary of $B_{R+T-\tau}$ in $\mathbb{R}^n$ by $S_{R+T-\tau}$ and the unit outward normal to this boundary by N.

Integrating (77) over the region K then yields[13]

$$\begin{aligned} &\frac{1}{2}\int_{\{t=T\}\times B_R} d^n x\left[(\partial_t\phi)^2+|\nabla_x\phi|^2\right] \\ +&\int_0^T dt\int_{\{\tau\}\times S_{R+T-\tau}}\left[\frac{1}{2}(\partial_t\phi)^2+\frac{1}{2}|\nabla_x\phi|^2-\partial_t\phi\cdot N\phi\right]d\sigma_{S_{R+T-\tau}} \\ &=\frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^n x\left[(\partial_t\phi)^2+|\nabla_x\phi|^2\right]. \end{aligned} \tag{80}$$

It is not hard to see using Cauchy-Schwarz that the integrand in the second line is non-negative. We can actually obtain something more quantitative. Let us denote the induced gradient on the spheres $S_{R+T-\tau}$ by $\nabla\!\!\!\!/$ (i.e. the derivatives *tangent* to these $n-1$ dimensional spheres; explicitly $\nabla\!\!\!\!/_i\phi=\partial_i\phi-\frac{x_i}{r}\partial_r\phi$). We may decompose

$$\partial_t = N + V\,,$$

where V is a derivative tangent to the wall of the cone[14] Then, from the easily verified identities

$$\begin{aligned} -\partial_t\phi N\phi &= -(N\phi)^2 - N\phi\cdot V\phi\,, \\ \frac{1}{2}\partial_t\phi\partial_t\phi &= \frac{1}{2}(N\phi)^2 + N\phi\cdot V\phi + \frac{1}{2}(V\phi)^2\,, \\ \frac{1}{2}|\nabla_x\phi|^2 &= \frac{1}{2}(N\phi)^2 + \frac{1}{2}|\nabla\!\!\!\!/\phi|^2\,, \end{aligned}$$

[13] Parametrise the integration over K by $\int_0^T dt\int_0^{T+R-t} dr r^2\int_{S^2}\sin\theta d\theta d\phi$.

[14] In polar coordinates $\partial_t=\partial_r+(\partial_t-\partial_r)$ since the wall of the cone is given by zero set of $H(t,x_1,\ldots,x_n)=t+\sqrt{x_1^2+\ldots x_n^2}-R-T=t-T+r-R$, so that indeed $(\partial_t-\partial_r)H=0$.

we see that (80) becomes

$$\begin{aligned} &\frac{1}{2}\int_{\{t=T\}\times B_R} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \\ &+\int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} \left[\frac{1}{2}\left(V\phi\right)^2 + \frac{1}{2}|\nabla\!\!\!/\phi|^2\right] d\sigma_{S_{R+T-\tau}} \\ &= \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] . \end{aligned} \tag{81}$$

This identity is truly remarkable and illustrates the domain of dependence property of the wave equation. Indeed, we certainly have

$$\int_{\{t=T\}\times B_R} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \leq \int_{\{t=0\}\times B_{R+T}} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \tag{82}$$

and hence

Corollary 4.2. *Suppose $\phi = 0 = \partial_t\phi$ in $\{t=0\}\times B_{R+T}$. Then $\phi = 0$ in $\bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau}$.*

Corollary 4.3. *Two C^2 solutions ϕ and ψ in $K = \bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau}$ that satisfy $\phi = \psi$ and $\partial_t\phi = \partial_t\psi$ on $\{t=0\}\times B_{R+T}$ have to agree in all of K.*

This in particular proves the uniqueness part of Theorem 4.1.

Exercise 4.4. *Prove Corollary 4.3 for the more general linear wave equations $\Box_\eta\psi + b^\mu\partial_\mu\psi + c\psi = 0$ for b^μ and c smooth functions. HINT: Gronwall's inequality. What happens for semi-linear equations?*

Let us understand a bit better the underlying geometry of this computation. The expression (77) is apparently a boundary term and it will induce different expressions dependent on the geometry of the boundary hypersurfaces. What is useful in the estimates is if the expressions induced are non-negative, as it was the case for the hypersurfaces of constant t and the null hypersurfaces (the wall of the truncated cone; see the remark below) discussed above. It turns out that the expression obtained is always positive definite in $d\psi$ for a spacelike hypersurface:

Exercise 4.5. *Consider two smooth connected spacelike homologous hypersurfaces Σ_1, Σ_2 in $\mathbb{R}^{1+n}$ enclosing a bounded spacetime region and such that Σ_2 is in the future of Σ_1. Prove that $\int_{\Sigma_2}|\partial\phi|^2 \leq C_{\Sigma_1,\Sigma_2}\int_{\Sigma_1}|\partial\phi|^2$.*

The exercise will be much easier after the next section!

4.3 A geometric framework

All of the above can be cast into the geometric framework we have been setting up earlier. Indeed, we already observed that we can write the wave equation as $\eta^{\mu\nu}\partial_\mu\partial_\nu\psi = 0$. More generally, if (M, g) is a Lorentzian manifold, we can consider the *covariant wave equation*

$$\Box_g\psi = g^{\mu\nu}\nabla_\mu\partial_\nu\psi = 0 \tag{83}$$

which for (M, g) the Minkowski spacetime reduces to the standard wave equation. Note the covariant wave equation does not make use of any coordinates on the manifold, hence the name. We can define the energy momentum tensor associated with ψ,

$$T_{\mu\nu} := \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi \, .$$

Note that this is indeed a tensor! (If you don't like indices, you will prefer to write $T = d\psi \otimes d\psi - \frac{1}{2}g^{-1}(d\psi, d\psi) \cdot g$.) Check that if ψ satisfies the wave equation, then

$$\nabla^\mu T_{\mu\nu} = 0$$

so the energy momentum tensor is divergence free. Hence if X is a spacetime vectorfield on (M, g) we have the identity

$$\nabla^\mu(T_{\mu\nu}X^\nu) = T_{\mu\nu}{}^{(X)}\pi^{\mu\nu} \tag{84}$$

where

$${}^{(X)}\pi_{\mu\nu} = \frac{1}{2}\left(\mathcal{L}_X g\right)_{\mu\nu}$$

is the so-called deformation tensor of the vectorfield X. Note that the deformation tensor vanishes for X a Killing field.

We would like to integrate the identity (84) over a spacetime region. For this we establish two ingredients. One is the divergence theorem in the Lorentzian setting, the other is a positivity property of the energy momentum tensor.

Lemma 4.6. *Let (M, g) be an $(n+1)$-dimensional spacetime and T defined as above. For X, Y future directed causal vectors we have $T(X, Y) \geq 0$. For X, Y future directed timelike (at p) we have*

$$T(X, Y) \geq c\sum_{i=1}^{n+1}|\partial_i\psi|^2$$

where the constant c depends on X, Y and the choice of coordinates at p.

Proof. If X and Y are collinear, the proof is left as an exercise, so assume they are not. Then X and Y span a timelike plane in the tangent space. We can choose future directed null vectors L and $\underline{L}$ such that $\text{span}\{L, \underline{L}\} = \text{span}\{X, Y\}$ and $g(L, \underline{L}) = -2$ (how?). We have

$$X = a_1 L + a_2\underline{L} \quad , \quad Y = \tilde{a}_1 L + \tilde{a}_2\underline{L}$$

with $a_1, a_2, \tilde{a}_1, \tilde{a}_2$ all strictly bigger than zero if X and Y are timelike (why?) and non-negative if X or Y (or both) are causal. We now complement L and

$\underline{L}$ to a null frame in T_pM by picking an orthonormal basis $e_1, e_2, \ldots e_{n-1}$ in the orthogonal complement to span$\{L, \underline{L}\}$. In particular $g(L, e_1) = 0$, $g(L, e_i) = 0$, $g(\underline{L}, e_i) = 0$ for $i = 1, \ldots, n-1$. We compute now

$$T(L,L) = |L\psi|^2 \quad , \quad T(\underline{L},\underline{L}) = |\underline{L}\psi|^2 \quad , \quad T(L,\underline{L}) = |e_1(\psi)|^2 + \ldots + |e_{n-1}(\psi)|^2$$

where only the last identity requires some thought. The result now follows from the tensorial nature of T. □

4.3.1 The divergence identity in Lorentzian geometry

This Section follows Ringstroems book (p.98). First recall Stokes' theorem:

Theorem 4.7. *Let M be an oriented n-manifold with boundary ∂M. Let ω be a smooth $(n-1)$-form on M having compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega \, .$$

If (M, g) is an $(n+1)$-dimensional Lorentzian manifold (with boundary) we can define the volume form ϵ by

$$\epsilon = \sqrt{-\det g} dx^0 \wedge dx^1 \wedge \ldots \wedge dx^n \, ,$$

where $\det g$ is the determinant of the matrix $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$. Note that this definition is independent of the coordinates. Next, given a smooth k-form ω and a smooth vectorfield X we define

$$i_X\omega(v_1, ..., v_{k-1}) = \omega(X, v_1, ..., v_{k-1}) \quad \text{for } v_1, ..., v_{k-1} \in T_pM,$$

so $i_X\omega$ is a smooth $(k-1)$-form. We next claim

Lemma 4.8. *Let (M, g) be an $(n+1)$-dimensional oriented Lorentzian manifold with volume form ϵ as above. If X is a smooth vector field, then*

$$d(i_X\epsilon) = (divX)\epsilon$$

Proof. Compute for $\mu = 0, 1, ..., n$ and NO sum over μ below:

$$\begin{aligned} i_X\epsilon(\partial_0, \ldots, \widehat{\partial_\mu}, \ldots, \partial_n) &= \epsilon(X, \partial_0, \ldots, \widehat{\partial_\mu}, \ldots, \partial_n) \\ &= X^\mu \epsilon(\partial_\mu, \partial_0, \ldots, \widehat{\partial_\mu}, \ldots, \partial_n) \\ &= (-1)^\mu X^\mu \epsilon(\partial_0, \partial_1, \ldots, \partial_n) \\ &= (-1)^\mu X^\mu \sqrt{-\det g} \, . \end{aligned} \tag{85}$$

We conclude that

$$i_X\epsilon = \sum_{\mu=0}^{n} (-1)^\mu X^\mu \sqrt{-\det g} \, dx^0 \wedge \ldots \wedge \widehat{dx^\mu} \wedge \ldots \wedge dx^n \, .$$

We can now take the differential of this expression and observe the identities $\partial_\mu\sqrt{-\det g} = \Gamma^\nu{}_{\nu\mu}\sqrt{-\det g}$ as well as $divX = \nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^\nu{}_{\nu\mu} X^\nu$ to complete the proof. □

We finally want to apply Stokes' theorem to the Lemma. Assume for simplicity that (M, g) is an $(n+1)$-dimensional manifold with boundary and that the boundary ∂M is either spacelike or timelike (it is easy to generalise to several boundary components with finitely many timelike and spacelike pieces). Then the outward normal to ∂M is either timelike or spacelike. Then if $e_1, ..., e_n$ is an oriented orthonormal basis of $T_p(\partial M)$. Then $N, e_1, ..., e_n$ is an orthonormal basis of T_pM and

$$i_X\epsilon(e_1, ..., e_n) = \frac{g(X,N)}{g(N,N)}\epsilon(N, e_1, ..., e_n) = \frac{g(X,N)}{g(N,N)} .$$

We conclude

$$i_X\epsilon = \frac{g(X,N)}{g(N,N)}\epsilon_{\partial M}$$

and hence, applying Stokes' theorem to the Lemma, the formula

$$\int_M divX\epsilon = \int_{\partial M} \frac{g(X,N)}{g(N,N)}\epsilon_{\partial M} .$$

4.4 A priori estimates for the covariant wave equation

We now return to the covariant wave equation (83). More generally, we will consider the inhomogeneous equation

$$\Box_g \psi = F \tag{86}$$

on an $(n+1)$-dimensional spacetime (M, g), with F a smooth function on M.

We would like to prove estimates for ψ in the following geometric setting. Let us assume there exists a "time function" on M, i.e. a smooth function $\mathcal{T} : M \to \mathbb{R}$ such that each $\widetilde{\Sigma}_t := \mathcal{T}^{-1}(t)$ is a spacelike hypersurface, $M = \cup_{t\in\mathbb{R}}\widetilde{\Sigma}_t$, and the gradient $\nabla\mathcal{T}$ is past direct timelike everywhere.[15] In M we consider the family of compact spacetime regions $\mathcal{R}(0,\tau)$ for $0 < \tau \leq 1$, defined by being bounded by $\widetilde{\Sigma}_0$, $\widetilde{\Sigma}_\tau$ and an additional spacelike hypersurface Σ', which intersects $\widetilde{\Sigma}_0$ transversally and touches $\widetilde{\Sigma}_1$ as shown in the picture below. We will set

$$\Sigma_\tau = \widetilde{\Sigma}_\tau \cup J^-(\Sigma') \quad , \quad \Sigma'_\tau = \Sigma' \cap \{0 \leq \mathcal{T}(\Sigma') \leq \tau\}$$

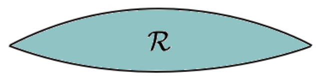
so that we have

$$\mathcal{R}(0,\tau) = \bigcup_{t\in[0,\tau]} \Sigma_t \cup \Sigma'_\tau .$$

We will write $\mathcal{R} = \mathcal{R}(0,1)$. Our goal is to estimate the solution ψ of (86) on Σ'_1 from "data" on Σ_0 and we will achieve this by controlling the solution on all Σ_τ.

[15] The existence of such a time function follows for the class of globally hyperbolic spacetimes to be considered below. See Theorem 4.17. For a small enough neighbourhood one can easily show the existence of such a function using normal coordinates. See the exercise.

picture needs to change



Exercise 4.9. *Realise the above geometric setting concretely for Minkowski space (i.e. give examples of the relevant hypersurfaces and regions). Use the exponential map and normal coordinates to construct the above geometric setting in a neighbourhood of a point $p \in M$ for an arbitrary (M, g).*

To obtain a priori estimates we assume that ψ is smooth and satisfies (86) on (the interior of) $(\mathcal{R}, g)$. Let V be a timelike vectorfield (for instance, we can choose the future directed unit normals to the hypersurfaces Σ_τ, denoted $n_{\Sigma_\tau} = \frac{-\nabla \mathcal{T}}{|\nabla \mathcal{T}|}$; note there is both an upper and a lower bound on $|\nabla \mathcal{T}|$ in the compact region $\mathcal{R}$). Define

$$f(t) := \int_{\Sigma_t} T_{\mu\nu} V^\mu (n_{\Sigma_t})^\nu \sim \int_{\Sigma_t} \sum_{\alpha=0}^{n} (\partial_\alpha \psi)^2 \,.$$

Here $A \sim B$ means that both $A \leq C\, B$ and $A \geq cB$ hold for constants C and c depending only on the background geometry, i.e. $\mathcal{R}, V, \Sigma$.[16] Note that the upper bound follows from the form of the energy momentum tensor while the lower bound was shown in Lemma 4.6. Integrating the divergence identity over the region $\mathcal{R}$ (note the minus signs from the timelike normal) yields for any $\tau \in [0, 1]$ the identity[17]

$$f(\tau) + \int_{\Sigma'_\tau} T_{\mu\nu} V^\mu (n_{\Sigma'_\tau})^\nu + \int_{\mathcal{R}(0,\tau)} F(V\psi) + \int_{\mathcal{R}(0,\tau)} T_{\mu\nu} {}^{(V)}\pi^{\mu\nu} = f(0)\,. \quad (87)$$

Note that $\nabla^\mu T_{\mu\nu} = \partial_\nu \psi \cdot F$ holds for a solution to the inhomogeneous equation (86) which account for the second term. We now use the (smooth) co-area formula which states that

$$\int_{\mathcal{R}(0,\tau)} g |\nabla \mathcal{T}| = \int_0^\tau dt \int_{\Sigma_t} g \quad \text{for } g \text{ a smooth function on } \mathcal{R},$$

[16] If $\mathcal{R}$ does not lie in a single coordinate patch, one has to understand the last integrals (and the coordinate derivatives appearing) by a partition of unity. Note that the closure of Σ_t is compact.

[17] Observe that we are cheating here slightly as $\mathcal{R}$ is not a smooth manifold with boundary but is only piecewise smooth. Generalising the identity to apply in the case with finitely many "corners" is mostly technical. See for instance Ringstroems book, which deals carefully with this issue.

as well as the estimate

$$|T_{\mu\nu}{}^{(V)}\pi^{\mu\nu}| \leq C\sum_{\alpha}(\partial_\alpha\psi)^2$$

to produce

$$f(\tau) + c\int_{\Sigma'_\tau}\sum_{\alpha}(\partial_\alpha\psi)^2 \leq f(0) + C\int_0^\tau dt\int_{\Sigma_t}F^2 + C\int_0^\tau dt f(t)\,. \tag{88}$$

We would like to control also the L^2 norm (on Σ_t) of ψ itself. For this we integrate the easily verified identity

$$\int_{\mathcal{R}(0,\tau)}\nabla_\alpha(\phi^2V^\alpha) = \int_{\mathcal{R}(0,\tau)}\left(2\phi V(\phi) + \phi^2\nabla^\alpha V_\alpha\right)\,. \tag{89}$$

Using the divergence theorem on the right and Cauchy-Schwarz and the co-area formula on the right, we deduce

$$\int_{\Sigma_\tau}\phi^2 + \int_{\Sigma'_\tau}\phi^2 \leq C\int_{\Sigma_0}\phi^2 + C\int_0^\tau dt\int_{\Sigma_t}\phi^2 + C\int_0^\tau dt f(t)\,. \tag{90}$$

Combining (88) and (90) we obtain for any $\tau\in[0,1]$

$$\|\psi\|^2_{H^1(\Sigma_\tau)} \leq C\left(\|\psi\|^2_{H^1(\Sigma_0)} + \|F\|^2_{L^2(\mathcal{R})} + \int_0^\tau dt\|\psi\|^2_{H^1(\Sigma_t)}\right)\,. \tag{91}$$

We can finally apply Gronwall's inequality to the above to deduce for all $\tau\in[0,1]$

$$\|\psi\|^2_{H^1(\Sigma_\tau)} \leq Ce^{C\tau}\left(\|\psi\|^2_{H^1(\Sigma_0)} + \|F\|^2_{L^2(\mathcal{R})}\right)\,. \tag{92}$$

Now, a posteriori, we can revisit (88) and (90) for $\tau = 1$ to also get the statement (92) with Σ'_1 replacing Σ_τ on the left.

Exercise 4.10. *Repeat this calculation assuming that V is Killing to deduce the stronger estimate*

$$\sup_{t\in[0,\tau]}\|\psi\|^2_{H^1(\Sigma_t)} \leq C\tau\left(\|\psi\|^2_{H^1(\Sigma_0)} + \left(\int_0^\tau\|F\|_{L^2(\Sigma_t)}dt\right)^2\right)\,. \tag{93}$$

HINTS: Estimate $\int_0^\tau dt\int_{\Sigma_t}|FV\psi| \leq \int_0^\tau dt\|F\|_{L^2(\Sigma_t)}\|V\psi\|_{L^2(\Sigma_t)}$ and the latter by $\sup_{t\in[0,\tau]}\|V\psi\|_{L^2(\Sigma_t)}\int_0^\tau dt\|F\|_{L^2(\Sigma_t)}$. Then apply Cauchy's inequality with ϵ. Now revisit (89) and do a similar trick to improve (90).

4.5 Global hyperbolicity

Our intention is to study the covariant wave equation (or more general non-linear geometric equations) on "general" Lorentzian manifolds. However, general Lorentzian manifold can exhibit many pathologies such as closed timelike curves or punctures ("holes") in the manifold. We will restrict to a class of spacetimes, called *globally hyperbolic*, that excludes most of the worst of these a priori. You should think of global hyperbolicity of (M, g) as the property that allows to sensibly address the global initial value problem for wave equations geometrically associated with (M, g) (i.e. the principal part of the wave operator being given by $g^{\mu\nu}\partial_\mu\partial_\nu$).

Remark 4.11. *While restricting to globally hyperbolic spacetimes when studying the Einstein equations themselves cannot be justified on a priori grounds, there is an important conjecture in the field (Penrose's strong cosmic censorship conjecture) that, if true, would justify restricting to this class. This conjecture, in turn, can be studied by restricting to this class!*[18]

Let (M, g) be a spacetime. We recall

Definition 4.12. *A smooth curve $\gamma : I \to M$ (where $I \subset \mathbb{R}$ is a finite interval) is a called inextendible if it is not a proper subset of another smooth curve, i.e. for all $\tilde{\gamma} : \tilde{I} \to M$ with $I \subset \tilde{I}$ and $\tilde{\gamma}|_I = \gamma$, we have $\tilde{I} = I$.*

Definition 4.13. *A Cauchy hypersurface is a spacelike hypersurface such that all inextendible causal curves $\gamma : I \to M$ intersect Σ once and only once.*

Definition 4.14. *A spacetime (M, g) is said to be globally hyperbolic if it admits a Cauchy hypersurface.*

Here are some examples:

- In Minkowski space $(\mathbb{R}^{1+n}, \eta)$ the hypersurface $\{t = 0\}$ is a Cauchy hypersurface. The hyperboloid $-t^2 + \sum_{i=1}^{n}(x^i)^2 = -1$ is not.
- The subset $\mathcal{U} = \mathrm{int}(J^-(1,0,0,0) \cap J^+(-1,0,0,0))$ of 4-dimensional Minkowski space $\mathbb{R}^{1+3}$ (equipped with the Minkowski metric) is globally hyperbolic, a Cauchy hypersurface being $\{t = 0\} \cap \mathcal{U}$.
- For the maximal analytic extension of the Schwarzschild spacetime
  - the exterior region $(\{U < 0\} \cap \{V > 0\})$ is globally hyperbolic.
  - the interior region $(\{U > 0\} \cap \{V > 0\})$ is globally hyperbolic.
  - the region $(\{V > 0\})$ is globally hyperbolic.

[18] The conjecture states that the maximum globally hyperbolic development of compact or asymptotically flat initial data is generically inextendible as a Lorentzian manifold. – In other words, studying the maximum globally hyperbolic development of initial data should be the largest set associated with data on which one can make sense of the Einstein equations.

Note that a spacetime may not be globally hyperbolic but there will exist subsets which are globally hyperbolic. For instance, if A is a spacelike acausal (no points can be connected by a causal curve) hypersurface in (M, g), then the *domain of dependence* (or *Cauchy development*) of A,

$$D(A) = \{x \in M \mid \text{every inextendible causal curve through } x \text{ intersects } A\},$$

is the biggest globally hyperbolic subset of M which admits A as a Cauchy hypersurface. We can distinguish the *future/ past Cauchy development* $D^{\pm}(A) = J^{\pm}(A) \cap D(A)$ and define the *future/ past Cauchy horizon* as the boundary of $D^{\pm}(A)$ in M.

Note that global hyperbolicity excludes closed timelike (in fact closed causal) curves: A closed timelike curve γ would have to intersect the Cauchy hypersurface Σ in a point $p \in \Sigma$ by global hyperbolicity. But then we could run around γ infinitely many times to produce an inextendible causal curve that intersects Σ more than once.

Let us briefly describe (without proof) some implications of global hyperbolicity. If (M, g) is globally hyperbolic, then

(1) $C(p, q)$, the space of continuous future directed causal curves from p to q is compact (in a natural topology).

(2) For any $p, q \in M$ the set $J^+(p) \cap J^-(q)$ is compact ("spacetime has no punctures")

(3) Given $p, q \in M$ with $p \in I^-(q)$ there exists a timelike geodesic from p to q that maximises the proper time.

In fact, Lerray (1952) originally defined global hyperbolicity by 1. (plus a mild causality condition excluding closed timelike curves). Also 2. (again plus a mild causality condition) is equivalent to our definition using a Cauchy hypersurface. See Appendix B for more on global hyperbolicity and a detailed outline of the proof of the statements (1)–(3) above.

Global hyperbolicity implies that the topology of the Cauchy hypersurface is unique:

Proposition 4.15. *If a globally hyperbolic spacetime (M, g) admits two Cauchy hypersurfaces Σ_1, Σ_2, then Σ_1 is homeomorphic to Σ_2.*

Proof. Since (M, g) is time-orientable, there exists a global timelike (hence non-vanishing) vectorfield T. The integral curves of T are timelike and hence intersect each of Σ_1 and Σ_2 exactly once. The map $\pi : \Sigma_1 \to \Sigma_2$ that associates a point on Σ_1 with the corresponding point of Σ_2 lying on the same integral curve is continuous with continuous inverse. □

In turns out that the topology of the spacetime is determined by the Cauchy hypersurface:

Proposition 4.16. *If Σ is a Cauchy hypersurface for a globally hyperbolic spacetime (M, g), then M is homeomorphic to $\mathbb{R} \times \Sigma$.*

Proof. **Step 1.** Given a global timelike vectorfield T we first choose a *complete* Riemannian metric h on M and normalise T such that $\|T\|_h = 1$. This makes T a *complete* timelike vectorfield, i.e. all its integral curves are defined on $(-\infty, \infty)$. (Indeed, if the maximum time of existence for an integral curve γ through $p \in M$ was $[0, t_+)$ for some $t_+ < \infty$, then $\gamma : [0, t_+) \to M$ would remain in a compact set of M since its h-length is bounded by t_+ (recall that the manifold topology and the topology induced by a Riemannian metric agree). Hence there exists a sequence $(t_n)_n$ with $t_n \to t_+$ and $\lim_{n\to\infty} \gamma(t_n) = q \in M$. We can find an open set U around q such that the flow of T is defined on $(-\delta, \delta) \times U$ (i.e. at every point $p \in U$, the integral curves of T through p are defined on $(-\delta, \delta)$), see Theorem 2.37. We now pick N such that $t_+ - t_N < \frac{\delta}{2}$ and $\gamma(t_N)$ lies in U. Then the integral curve through $\gamma(t_+ - t_N)$ will extend the old curve past t_+.

Step 2. One now considers the continuous map $f : \mathbb{R} \times \Sigma \to M$ defined by $f(t, x) = \Phi(t, x)$ (flowing the point $x \in \Sigma$ by parameter time t along the integral curve of T). This map is injective (as integral curves cannot intersect) and surjective since through every point $p \in M$ we have an integral curve of T (Σ is a Cauchy hypersurface). By the invariance of the domain f is a homeomorphism.[19] □

We finally state without proof a theorem (quoted from Ringstroem's book) in the smooth setting that shows that the setting we studied earlier is appropriate in globally hyperbolic spacetimes

Theorem 4.17. *Let (M, g) be an oriented, time oriented, connected and globally hyperbolic Lorentzian manifold and let Σ be a smooth Cauchy hypersurface. Then there exists a smooth onto function $\tau : M \to \mathbb{R}$ such that*

- $\Sigma = \tau^{-1}(0)$
- *τ has past-directed timelike gradient*
- *each $\tau^{-1}(t)$ with $t \in \mathbb{R}$ is a Cauchy hypersurface*
- *For every inextendible causal curve $\gamma : (t_-, t_+) \to M$ we have $\tau[\gamma(t)] \to \pm\infty$ as $t \to t_\pm \mp$.*

4.6 Wellposedness for the covariant wave equation

Theorem 4.18. *Let (M, g) be a smooth, oriented, time-oriented Lorentzian-manifold. Assume that (M, g) is globally hyperbolic and that Σ is a smooth*

[19] The invariance of the domain is a deep theorem in topology (due to Brouwer) that states that given an injective continuous map f from an open subset U of $\mathbb{R}^n$ we have that $V = f(U)$ is open in $\mathbb{R}^n$ and $f : U \to V$ is a homeomorphism. This generalises to the manifold setting. In particular, if $f : M \to N$ is a continuous bijection, where M and N are manifolds of the same dimension (without boundary) then M and N are homeomorphic.

Cauchy hypersurface with future unit normal n. Let ψ_0 and ψ_1 be smooth functions on Σ and F a smooth function on M. Then the Cauchy problem

$$\begin{aligned} \Box_g \psi &= F \\ \psi|_\Sigma &= \psi_0 \\ n(\psi)|_\Sigma &= \psi_1 \end{aligned} \tag{94}$$

admits a unique smooth solution ψ on M.

Remark 4.19. *One can add first order and zeroth order terms on the left without changing the statement of the theorem. More generally, one can consider coupled systems of linear equations, e.g. $\Box_g \psi + X\phi = F_1$, $\Box_g \phi + Y\psi + \phi = F_2$ for (fixed) vectorfields X and Y.*

Remark 4.20. *Theorem 4.18 is at the basis of what we will do in the next 4-5 weeks, namely studying the* global *behaviour in time of solutions to the covariant wave equation on the Schwarzschild exterior.*

There are two main ingredients in the proof of this theorem. The first is to get existence and uniqueness in an appropriate coordinate patch (see for instance my PDE notes on non-linear wave equation). The second is to "paste together" the local solutions using the domain of dependence property and global hyperbolicity. We will not discuss any details here but I would like to sketch the main idea for the PDE part.

The first step is to observe that it suffices to solve the problem to zero initial data. Indeed, suppose we can solve the problem for zero data and we want to solve the problem with non-trivial data. Pick a function $\tilde{\psi}$ with the same initial data as ψ (but not necessarily satisfying the wave equation) and solve the problem $\Box \bar{\psi} = F - \Box \tilde{\psi}$ with zero data. Then $\psi = \bar{\psi} + \tilde{\psi}$ has the correct data and solves $\Box \psi = F$.

For the second step, we consider again the region $\mathcal{R} = \bigcup_{\tau \in [0,1]} \Sigma_\tau$ and define the space of test functions on $\mathcal{R}$:

$$C_0(\mathcal{R}) = \{\phi \in C^\infty(\mathcal{R}) \ , \ \phi|_{\Sigma_1} = \phi|_{\Sigma_0} = 0 \ , \ d\phi|_{\Sigma_0} = d\phi|_{\Sigma_1} = 0\} \, ,$$

that is smooth functions on $\mathcal{R}$ which vanish with their first derivatives at the boundary. We define a weak solution of the Cauchy problem (with $\psi_0 = \psi_1 = 0$) to be a $\psi \in L^2(\mathcal{R})$ such that

$$\int_{\mathcal{R}} \psi \Box_g \phi = \int_{\mathcal{R}} F \cdot \phi$$

holds for all $\phi \in C_0(\mathcal{R})$. (Note that if ψ were a classical (C^2) solution of the wave equation, then we have $0 = \int_{\mathcal{R}} (\Box_g \psi - F)\phi = \int_{\mathcal{R}} \psi \Box_g \phi - F\phi$ for all $\phi \in C_0(\mathcal{R})$ so a classical solution is also a weak solution.)

The idea to find a weak solution is to define a functional ψ on the space $S := \{\Box_g \phi \mid \phi \in C_0(\mathcal{R})\} \subset L^2(\mathcal{R})$ by

$$\psi \left[\Box_g \phi\right] := \int_{\mathcal{R}} F \cdot \phi \, .$$

Note that by the energy estimate we have for the equation $\Box_g \phi = \Box_g \phi$ with zero data

$$\sup_{t\in[0,1]} \|\phi\|_{L^2(\Sigma_t)} \leq \|\Box\phi\|_{L^2(\mathcal{R})} \tag{95}$$

hence ψ is well-defined. The functional ψ is also bounded on S since

$$|\psi\left[\Box_g\phi\right]| \leq \sqrt{\int_{\mathcal{R}} F^2}\sqrt{\int_{\mathcal{R}} \phi^2} \leq C\sqrt{\int_{\mathcal{R}} F^2}\sqrt{\int_0^1 dt \int_{\Sigma_t} \phi^2} \leq C_{F,\tau}\|\Box_g\phi\|_{L^2(\mathcal{R})}$$

holds, where we have again used (95). By Hahn-Banach, the functional ψ extends to a bounded linear functional on $L^2(\mathcal{R})$ which is in turn represented (Riesz representation theorem) by a function in $L^2(\mathcal{R})$. The existence of a weak solution then follows.

In a third step, one needs to improve the regularity of the solution to show that we actually obtain a classical solution. (Actually one modifies the above argument using Sobolev spaces H^{-k}. See the PDE lecture notes.)

4.7 Well-posedness for non-linear equations

When we consider *non-linear* wave equations we cannot expect a global solution on all of M as solutions can blow up in finite time. (For instance, any non-trivial solution arising from data of compact support of the equation $\Box_\eta \psi = (\partial_t \psi)^2$ blows up in finite time!) We can however expect a local in time well posedness result (and then ask about the largest domain on which the solution exists).

We first need to identify a suitable class of non-linear equation which we want to consider. A suitable class of geometric quasi-linear equations is given by equations of the form

$$\Box_{g(\psi)}\psi = \mathcal{N}(\psi, \partial\psi)\,.$$

with appropriate assumptions on the non-linearity $\mathcal{N}$ and $g(\psi)$, where again, one could in addition consider vector-valued ψ and add linear first order terms.

We will take a more pedestrian point of view, which you can view as working in a local coordinate chart (and as the "PDE part" to prove local well-posedness.) We first consider the following model equation for $\phi : I \times \mathbb{R}^n \to \mathbb{R}$:

$$\begin{cases} \partial_\alpha \left(G^{\alpha\beta}(\phi)\partial_\beta\phi\right) = F(\phi, \partial\phi) \\ \phi(0,x) = \phi_0(x) \\ \partial_t\phi(0,x) = \phi_1(x) \end{cases} \tag{96}$$

with smooth initial data $\phi_0, \phi_1 \in C^\infty(\mathbb{R}^n)$. We require $G^{\alpha\beta} = G^{\beta\alpha}$ and

$$\sum_{\alpha,\beta} |G^{\alpha\beta} - \eta^{\alpha\beta}| \leq \frac{1}{10} \quad , \quad G^{\alpha\beta}(0) = \eta^{\alpha\beta} \quad , \quad F(0,0) = 0$$

and

$G^{\alpha\beta} : \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ are smooth functions of their arguments.

We could add first order terms to (96) and also allow vector valued ϕ and if we do that the class (96) will be general enough to allow us to deduce local-wellposedness for the vacuum Einstein equations.

We have the following theorem:

Theorem 4.21. *Fix g and F satisfying the assumptions above. There exists a $T > 0$ (depending on sufficiently high Sobolev norms of ϕ_0 and ϕ_1) such that there exists a unique smooth solution of (96) on $(-T, T) \times \mathbb{R}^n$.*

Remark 4.22. *The theorem is actually proven in Sobolev spaces using a Picard iteration, energy estimates and Sobolev embedding (see the NLW notes). It also includes a statement of continuous dependence on the data and hence the entire statement of Hadamard well-posedness for (96).*

Remark 4.23. *One should think of the above as the analogue of the local existence and uniqueness part in the linear theorem (i.e. the part that one can localise to a chart of the manifold). Note however that causality now depends on the solution (the pde is* quasilinear*), so what is a spacelike hypersurface with respect to $G^{\alpha\beta}$ depends on ϕ! Note also that one cannot expect global existence in time as, for instance any solution of $\Box_\eta \psi = (\partial_t \psi)^2$ arising from data of compact support blows up in finite time. See again the NLW notes for details.*

4.8 Local existence for the Einstein equations

We recall the expression for the Riemann tensor in local coordinates:

$$R^{\rho}{}_{\alpha\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\alpha} - \partial_\nu \Gamma^{\rho}{}_{\mu\alpha} + \Gamma^{\sigma}{}_{\nu\alpha}\Gamma^{\rho}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\mu\alpha}\Gamma^{\rho}{}_{\nu\sigma}$$

Inserting the expression for the Christoffel symbols, we obtain for the Ricci-tensor

$$\begin{aligned} Ric(g)_{\mu\nu} = & -\frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu} - \frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_\mu \partial_\nu g_{\alpha\beta} \\ & + \frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \partial_\nu g_{\beta\mu} + \frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_\beta \partial_\mu g_{\alpha\nu} + F_{\mu\nu}\left(g, \partial g\right) , \end{aligned} \tag{97}$$

where $F_{\mu\nu}(g, \partial g)$ is an expression involving only the metric and first derivatives of the metric (with various contractions). The first term on the right hand side looks good (like a wave equation) but the second to fourth term prevent us from applying hyperbolic theory directly.

Remark 4.24. *More specifically, if the second, third and fourth terms were not present, then we could choose normal coordinates at a point p such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $g_{\mu\nu}|_p = \eta_{\mu\nu}$ and we it would be sufficient to solve*

$$-\frac{1}{2}\left(\eta + h)^{-1}\right)^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} + \tilde{F}_{\mu\nu}(h, \partial h) = 0\,,$$

where $\tilde{F}_{\mu\nu}$ are fixed smooth functions involving h and ∂h and being at least quadratic in derivatives of h (why?). Now the components of the inverse of $\eta+h$ can be written as a given function of the components $h_{\mu\nu}$ (using cofactors) and we could indeed apply Theorem 4.21 after specifying suitable initial data.

For a *specific* choice of coordinates, however, it turns out that we can make the hyperbolicity of the Einstein equations manifest by showing that *in these coordinates* the sum of the second, third and fourth term is actually an expression involving only *first* derivatives of g. These coordinates are called wave (or "harmonic") coordinates and defined by the condition

$$\Box_g x^\alpha = \frac{1}{\sqrt{-g}}\partial_\mu\left((g^{-1})^{\mu\nu}\sqrt{-g}\partial_\nu x^\alpha\right) = 0\,, \tag{98}$$

where we use the short hand "$-g$" to denote minus the determinant of the metric g. We compute (exercise – use the formula to differentiate an inverse matrix and the formula to differentiate the determinant!)

$$\partial_\mu\left((g^{-1})^{\mu\nu}\sqrt{-g}\partial_\nu x^\alpha\right) = \sqrt{-g}\left(-(g^{-1})^{\mu\gamma}(g^{-1})^{\alpha\delta}\partial_\mu g_{\gamma\delta} + \frac{1}{2}(g^{-1})^{\mu\alpha}(g^{-1})^{\gamma\delta}\partial_\mu g_{\gamma\delta}\right).$$

Contracting the above with $g_{\alpha\sigma}$ we see that the wave coordinates imply the relation

$$(g^{-1})^{\mu\gamma}\partial_\mu g_{\gamma\sigma} = \frac{1}{2}(g^{-1})^{\alpha\beta}\partial_\sigma g_{\alpha\beta}\,.$$

We thus define

$$\lambda_\sigma = (g^{-1})^{\mu\alpha}\partial_\mu g_{\alpha\sigma} - \frac{1}{2}(g^{-1})^{\alpha\beta}\partial_\sigma g_{\alpha\beta}$$

and we have that

$$\lambda_\mu = 0 \text{ if and only if the wave coordinate condition } \Box_g x^\mu = 0 \text{ holds.} \tag{99}$$

Note the λ_μ are *not* the components of a covector-field.

Proposition 4.25. *Define the reduced Ricci curvature to be*

$$\begin{aligned}\widetilde{Ric}\,(g)_{\mu\nu} = &-\frac{1}{2}\left(g^{-1}\right)^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}\\ &+\frac{1}{2}\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\mu g_{\sigma\rho}\partial_\beta g_{\alpha\nu} + \frac{1}{2}\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\nu g_{\sigma\rho}\partial_\alpha g_{\beta\mu}\\ &-\frac{1}{2}\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\mu g_{\sigma\rho}\partial_\nu g_{\alpha\beta} + F_{\mu\nu}\left(g,\partial g\right)\end{aligned} \tag{100}$$

where $F_{\mu\nu}(g,\partial g)$ is the expression appearing in (97).[20] *Then*

$$\widetilde{Ric}(g)_{\mu\nu} = Ric(g)_{\mu\nu} - \frac{1}{2}\partial_\mu\lambda_\nu - \frac{1}{2}\partial_\nu\lambda_\mu\,. \tag{101}$$

[20]Note that the second and third line involve only g and first derivatives of g.

Proof. We compute

$$
\begin{aligned}
-\frac{1}{2}\left(g^{-1}\right)^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} &= -\frac{1}{2}\partial_\mu\left(\left(g^{-1}\right)^{\alpha\beta}\partial_\nu g_{\alpha\beta}\right)+\frac{1}{2}\partial_\mu\left(g^{-1}\right)^{\alpha\beta}\partial_\nu g_{\alpha\beta} \\
&= \partial_\mu\lambda_\nu - \partial_\mu\left(\left(g^{-1}\right)^{\alpha\beta}\partial_\beta g_{\alpha\nu}\right)+\frac{1}{2}\partial_\mu\left(g^{-1}\right)^{\alpha\beta}\partial_\nu g_{\alpha\beta} \\
&= \partial_\mu\lambda_\nu - \left(g^{-1}\right)^{\alpha\beta}\partial_\mu\partial_\beta g_{\alpha\nu} \\
&\quad + \frac{1}{2}\partial_\mu\left(g^{-1}\right)^{\alpha\beta}\partial_\nu g_{\alpha\beta} - \partial_\mu\left(g^{-1}\right)^{\alpha\beta}\partial_\beta g_{\alpha\nu} \\
&= \partial_\mu\lambda_\nu - \left(g^{-1}\right)^{\alpha\beta}\partial_\mu\partial_\beta g_{\alpha\nu} \\
&\quad - \frac{1}{2}\partial_\mu\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\mu g_{\sigma\rho}\partial_\nu g_{\alpha\beta} \\
&\quad + \partial_\mu\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\mu g_{\sigma\rho}\partial_\beta g_{\alpha\nu}\,. \qquad (102)
\end{aligned}
$$

Since the expression is symmetric in μ and ν we obtain an analogous formula by interchanging μ and ν:

$$
\begin{aligned}
-\frac{1}{2}\left(g^{-1}\right)^{\alpha\beta}\partial_\mu\partial_\nu g_{\alpha\beta} &= \partial_\nu\lambda_\mu - \left(g^{-1}\right)^{\beta\alpha}\partial_\nu\partial_\alpha g_{\beta\mu} \\
&\quad - \frac{1}{2}\partial_\mu\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\nu g_{\sigma\rho}\partial_\mu g_{\alpha\beta} \\
&\quad + \partial_\mu\left(g^{-1}\right)^{\alpha\sigma}\left(g^{-1}\right)^{\beta\rho}\partial_\nu g_{\sigma\rho}\partial_\beta g_{\alpha\mu}\,. \qquad (103)
\end{aligned}
$$

The desired formula now follows easily from combining (97) with these two identities. □

In other words, if the wave coordinate condition holds, then the Einstein equation become the reduced Einstein equations, which are a system of non-linear wave equations that we can tackle by Theorem 4.21.

The key idea is now that we can ensure that λ satisfies a homogeneous wave equation on its own, which will allow us to ensure $\lambda = 0$ in evolution by prescribing suitable data.

Proposition 4.26. *Given a Lorentzian metric such that the reduced Einstein vacuum equations are satisfied, i.e.* $\widetilde{Ric}(g) = 0$*. Then* λ *satisfies a wave equation, namely:*

$$
\frac{1}{2}\left(g^{-1}\right)^{\sigma\mu}\partial_\sigma\partial_\mu\lambda_\nu + c_\nu^{\alpha\beta}\partial_\alpha\lambda_\beta = 0 \qquad (104)
$$

where the $c_\nu^{\alpha\beta}$ *are smooth functions of* $g_{\mu\nu}$ *and its derivatives.*

Proof. Recall first the contracted Bianchi identity $\nabla^\mu\left(\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = 0$. (To show this, use

$$
\nabla^\mu R_{\mu\nu\alpha\beta} = \left(g^{-1}\right)^{\mu\sigma}\nabla_\sigma R_{\mu\nu\alpha\beta} = -\left(g^{-1}\right)^{\mu\sigma}\nabla_\alpha R_{\mu\nu\beta\sigma} - \left(g^{-1}\right)^{\mu\sigma}\nabla_\alpha R_{\mu\nu\beta\sigma}
$$

by the Bianchi identity and contract with $(g^{-1})^{\alpha\nu}$.) Next, note that $\widetilde{\text{Ric}}(g) = 0$ implies

$$0 = Ric(g)_{\mu\nu} - \frac{1}{2}\partial_\mu \lambda_\nu - \frac{1}{2}\partial_\nu \lambda_\mu$$

hence also

$$0 = R - \left(g^{-1}\right)^{\mu\nu} \partial_\mu \lambda_\nu \, .$$

Finally, we compute using the above

$$\begin{aligned} 0 &= \nabla^\mu \left(\text{Ric}(g)_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \\ &= - \left(g^{-1}\right)^{\mu\sigma} \partial_\sigma \left(\frac{1}{2}\partial_\mu \lambda_\nu + \frac{1}{2}\partial_\nu \lambda_\mu - \frac{1}{2} g_{\mu\nu} \left(\left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \lambda_\beta \right) \right) \\ &\quad - \left(g^{-1}\right)^{\mu\sigma} \Gamma^\delta{}_{\sigma\mu} \left(\frac{1}{2}\partial_\delta \lambda \nu + \frac{1}{2}\partial_\nu \lambda_\delta - \frac{1}{2} g_{\delta\nu} \left(\left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \lambda_\beta \right) \right) \\ &\quad - \left(g^{-1}\right)^{\mu\sigma} \Gamma^\delta{}_{\sigma\nu} \left(\frac{1}{2}\partial_\mu \lambda \delta + \frac{1}{2}\partial_\delta \lambda_\mu - \frac{1}{2} g_{\mu\delta} \left(\left(g^{-1}\right)^{\alpha\beta} \partial_\alpha \lambda_\beta \right) \right) \end{aligned} \tag{105}$$

from which the result follows after observing that the terms involving second derivatives combine to produce

$$-\frac{1}{2}\left(g^{-1}\right)^{\mu\sigma} \partial_\sigma \partial_\mu \lambda_\nu - \frac{1}{2}\left(g^{-1}\right)^{\mu\sigma} \partial_\nu \partial_\sigma \lambda_\mu - \frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_\nu \partial_\alpha \lambda_\beta = -\frac{1}{2}\left(g^{-1}\right)^{\mu\sigma} \partial_\sigma \partial_\mu \lambda_\nu \, .$$

□

The strategy now is clear: We want to solve the reduced Einstein equations using Theorem 4.21 to obtain a $g_{\mu\nu}$ and then solve the wave equation with trivial data to ensure $\lambda_\nu = 0$ holds in the region where we have constructed $g_{\mu\nu}$. The $g_{\mu\nu}$ thus constructed would then be Ricci flat.

Before we can embark on that strategy, we need to discuss what the initial data for the reduced Einstein equations should actually be. To apply Theorem 4.21 we should prescribe

$$g_{\mu\nu}|_{t=0} \quad \text{and} \quad \partial_t g_{\mu\nu}|_{t=0} \, . \tag{106}$$

It turns out that, geometrically, we only need to prescribe the induced metric and the second fundamental form of the initial hypersurface (which corresponds to $t = 0$ in our coordinates). More specifically we will prescribe as data

$$g_{\mu\nu}|_{t=0} = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & g_{ij} & \\ 0 & & & \end{bmatrix} \tag{107}$$

$$\partial_t g_{\mu\nu}|_{t=0} = \begin{bmatrix} * & * & * & * \\ * & & & \\ \vdots & & 2K_{ij} & \\ * & & & \end{bmatrix} \tag{108}$$

with suitable g_{ij} (spatial metric) and K_{ij} (will turn out to be the second fundamental form of the initial hypersurface) and where $*$ denotes entries that will be determined in terms of (spatial derivatives of) g_{ij} and K_{ij} to achieve the condition $\lambda_\nu = 0$ on the initial hypersurface.

Note that indeed, with the above choice, g_{ij} is the induced spatial metric on the hypersurface $\{t = 0\}$. Moreover, we have that $g(\partial_t, \partial_t) = -1$ and ∂_t is normal to the hypersurface $\{t = 0\}$ (as $g_{0i} = 0$).[21] This choice implies that the second fundamental form of the hypersurface $\{t = 0\}$, which is geometrically defined as the symmetric 2-tensor on $\Sigma = \{t = 0\}$ (see Example Sheet 7)

$$K(X, Y) = g(\nabla_X n, Y) = -g(n, \nabla_X Y) \tag{109}$$

for X, Y vectorfields tangent to Σ, is given by[22]

$$K_{ij} = \frac{1}{2}\partial_t g_{ij}|_{t=0} \tag{110}$$

and hence the K appearing in (108) has in fact geometric significance.

We verify the claim that $\partial_t g_{00}|_{t=0}$ and $\partial_t g_{0i}|_{t=0}$ (i.e. the $*$ parts in the matrices above) are now determined by the requirement that $\lambda = 0$. Indeed, we have that

$$0 = \lambda_i|_{t=0} = \left(g^{-1}\right)^{\mu\alpha} \partial_\mu g_{\alpha i}|_{t=0} - \frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_i g_{\alpha\beta}|_{t=0} \quad \text{fixes } \partial_t g_{0i}, \tag{111}$$

$$0 = \lambda_0|_{t=0} = \left(g^{-1}\right)^{\mu\alpha} \partial_\mu g_{\alpha 0}|_{t=0} - -\frac{1}{2}\left(g^{-1}\right)^{\alpha\beta} \partial_0 g_{\alpha\beta}|_{t=0} \quad \text{fixes } \partial_t g_{00}. \tag{112}$$

(You should write out all contractions and confirm this, using the form of the metric on $t = 0$, (107).)

We need one final observation regarding constraints between the data g_{ij} and K_{ij} that need to hold if the Einstein equations hold on the hypersurface $\{t = 0\}$. One way to see their existence is to compute

$$\left(Ric - \frac{1}{2}gR\right)_{00} = ...$$

and

$$Ric_{0i} = ...$$

in terms of the metric and to observe that these expressions do not contain second time derivatives of g! Therefore (since we have specified all other second derivatives initially), the vanishing of the above expressions needs to be imposed initially on $\{t = 0\}$!

In fact, it pays off to think about this more geometrically. On Example Sheet 7 you will prove the important Gauss and Codazzi equations which govern the

[21] This should be viewed as a coordinate gauge choice relating the (geometric) normal to a coordinate vectorfield.

[22] This is an easy computation using that the normal is $n = \partial_t$.

extrinsic and intrinsic geometry of a hypersurface Σ in a spacetime M (with induced metric h on Σ being the restriction of g to TM and future timelike unit normal n). These read

$$h\,(Riem_h(X,Y)Z,W) = g\,(Riem_g(X,Y)Z,W) + K(X,Z)K(Y,W) - K(X,W)K(Y,Z) \qquad (113)$$

and

$$g\,(Riem_g(X,Y)Z,n) = (\nabla_Y K)(X,Z) - (\nabla_X K)(Y,Z)\,. \qquad (114)$$

(On Sheet 7 you will make sense of $\nabla_Y X$ (and hence the tensor $\nabla_X K$) for X, Y vectorfields on Σ. The procedure is to extend the vectorfields to spacetime vectorfields and show that the result does not depend on the extension.)

To derive from the above the constraint equations, let us work in an orthonormal frame $e_0 = n, e_1, ..., e_n$ (and recall $R_{dcab} = g(R(e_a,e_b)e_c,e_d)$). We compute from 00 component of the Einstein equations

$$Ric_{00} + \frac{1}{2}R = 0\,.$$

Since by definition (in the orthonormal frame)

$$R = -Ric_{00} + \sum_{i=1}^{n} Ric_{ii} = -2Ric_{00} + \sum_{i=1}^{n}\sum_{j=1}^{n} Riem_{ijij} \qquad (115)$$

and the condition $Ric_{00} + \frac{1}{2}R = 0$ becomes on Σ (by the Gauss equation)

$$0 = \sum_{i=1}^{n}\sum_{j=1}^{n} Riem_{ijij} = \sum_{i=1}^{n}\sum_{j=1}^{n}(Riem_h)_{ijij} - \sum_{i=1}^{n}\sum_{k=1}^{n} K_{ij}K_{ij} + \sum_{i=1}^{n} K_{ii}\sum_{j=1}^{n} K_{jj}$$

In other words, the Σ-intrinsic equation

$$R_h - |K|_h^2 + (tr_h K)^2 = 0 \qquad (116)$$

needs to hold on the $(n-1)$-dimensional hypersurface Σ in order for the Einstein equations (more specifically $\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)n^\mu n^\nu = 0$) to hold on Σ.

A similar computation for the $0i$ component of the Einstein equation, Ric_{0i}, (using now the Codazzi equation) yields

$$Ric_{0i} = -Riem_{000i} + \sum_{j=1}^{n} Riem_{j0ji} = -\sum_{j=1}^{n} Riem_{0jji} = -\nabla_i(\sum_{j=1}^{n} K_{jj}) + \sum_{j=1}^{n} \nabla_j K_{ij}\,.$$

In other words, the Σ-intrinsic equation

$$div_h\,(K - tr_h K \cdot h) = 0 \qquad (117)$$

needs to hold on the $(n-1)$-dimensional hypersurface Σ in order for the Einstein equations (more specifically $\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)n^\mu (e_i)^\nu = R_{\mu\nu}n^\mu (e_i)^\nu = 0$) to hold on Σ.

We can finally state and prove our main theorem.

Theorem 4.27. *Given initial data (h, K) on $\mathbb{R}^n$ satisfying the constraint equations (116), (117) and such that $\sum_{i,j} |h_{ij} - \delta_{ij}| \leq \frac{1}{20}$, there exists a $T > 0$ and a Lorentzian metric g on $(-T, T) \times \mathbb{R}^n$, which solves the Einstein vacuum equations and is such that the induced metric and the induced second fundamental form on $\{0\} \times \mathbb{R}^n$ coincide with h and K respectively.*

Proof. **Step 1.** We solve $\widetilde{Ric}(g)_{\mu\nu} = 0$ with data as in (107), (108) (and $g_{ij} = h_{ij}$). From Theorem 4.21 we obtain a g in $(-T, T) \times \mathbb{R}^n$ satisfying the geometric conditions (about the induced metric and second fundamental form) on the data.

Step 2. This solution would be Ricci flat ($R_{\mu\nu}(g) = 0$) if we could show $\lambda_\nu = 0$ in $(-T, T) \times \mathbb{R}^n$. Since λ satisfies a homogeneous linear wave equation this would follow if $\lambda|_{t=0} = 0$ and $\partial_t \lambda|_{t=0} = 0$. We have already set $\lambda|_{t=0} = 0$ in Step 1. To show that also $\partial_t \lambda|_{t=0} = 0$ we observe that from $\widetilde{Ric}(g)_{\mu\nu} = 0$ we have

$$0 = Ric(g)_{\mu\nu} - \frac{1}{2}\partial_\mu \lambda_\nu - \frac{1}{2}\partial_\nu \lambda_\mu \,,$$

which implies

$$R = \left(g^{-1}\right)^{\mu\nu} \partial_\mu \lambda_\nu \,.$$

Since (h, K) satisfies the constraint equations we have $Ric(g)_{i0} = 0$ and $(Ric(g)_{00} - \frac{1}{2}Rg_{00})|_{t=0} = 0$, which by the above translates to

$$\frac{1}{2}\partial_t \lambda_i|_{t=0} = -\frac{1}{2}\partial_i \lambda_0 = 0$$

(with the last equality following from having fixed $\lambda = 0$ on $\{t = 0\}$) and

$$-\partial_t \lambda_0|_{t=0} + \frac{1}{2}\partial_t \lambda_0|_{t=0} + \frac{1}{2}(h^{-1})^{ij}\partial_i \lambda_j = 0 \,,$$

which implies $\partial_t \lambda_0|_{t=0} = 0$ as desired. □

Some remarks are in order

Remark 4.28. *The assumption $\sum_{i,j=1}^n |h_{ij} - \delta_{ij}| \leq \frac{1}{20}$ may look artificial but can be removed by localising to neighbourhoods were h_{ij} is almost constant and then change coordinates where the metric has the form δ_{ij} at p. One can then use the finite speed of propagation. (Some compactness (e.g. asymptotic flatness) is needed to get a* uniform *time of existence T, otherwise one just obtains a neighbourhood of $\{t = 0\} \times \mathbb{R}^n$ where a solution exists but that neighbourhood could shrink as one moves towards infinity.)*

Remark 4.29. *Note that our black box theorem, Theorem 4.21 required $\sum_{i,j} |G^{ij} - \eta^{ij}| \leq \frac{1}{10}$ while here we only have this at $t = 0$ (with $\frac{1}{20}$ instead of $\frac{1}{10}$). However, the proof of the black box NLW theorem shows that it suffices for the condition to hold initially. (This is essentially because the proof shows that given $\epsilon > 0$ one can choose T such that $\sup_{(-T,T)\times\mathbb{R}^n} |\phi - \phi_0| < \epsilon$ holds. See again the NLW notes.)*

Remark 4.30. *The construction is essentially local using the domain of dependence property. So it should not be surprising that initial data can be imposed not only on $\mathbb{R}^n$ but on an arbitrary Riemannian manifold. A general initial data set is the a triple (Σ, h, K) consisting of a Riemannian manifold (Σ, h) and a symmetric 2-tensor K satisfying the constraint equations (116), (117).*

Remark 4.31. *There are many non-trivial (i.e. beyond those induced by known explicit solutions of the Einstein equations!) solutions to the constraint equations. It is in fact a research area on its own. A quite accessible state of the art overview can be found in* "The general relativistic constraint equations" by Alessandro Carlotto, Living Reviews in Relativity (2021) **24**:2

Remark 4.32. *One can prove the following geometric uniqueness statement: Given two solutions (M_1, g_1) and (M_2, g_2) (i.e. g_1 and g_2 Ricci flat) that induce the same Riemannian metric and fundamental form on $\{0\} \times \mathbb{R}^n$, there exist open subsets $U_i \subset M_i$ containing Σ such that (U_1, g_1) and (U_2, g_2) are isometric.*

4.9 *A sketch of the local uniqueness statement

Suppose (M_1, g_1) is the solution constructed on $(-T, T) \times \mathbb{R}^n$ by Theorem 4.27 with harmonic coordinates (t, x) on $\mathbb{R}^4$ and let M_2 be a neighbourhood of $\Sigma = \{0\} \times \mathbb{R}^n$ equipped with metric in $(\tilde{t}, \tilde{x})$ coordinates on $\mathbb{R}^4$ as

$$g_2 = (g_2)_{00}(\tilde{t}, \tilde{x}) d\tilde{t} \otimes d\tilde{t} + 2(g_2)_{0i}(\tilde{t}, \tilde{x}) d\tilde{t} \otimes d\tilde{x}^i + (g_2)_{ij}(\tilde{t}, \tilde{x}) d\tilde{x}^i \otimes d\tilde{x}^j \,, \quad (118)$$

where we can assume that $\tilde{t}|_\Sigma = 0 = t|_\Sigma$ and that the spatial coordinates on Σ are identified, $\tilde{x}|_\Sigma = x|_\Sigma$, such that $(g_2)_{ij}(\tilde{t} = 0, \tilde{x}) = (g_1)_{ij}(t = 0, x)$.[23]

Step 1. We now choose a coordinate transformation (i.e. a diffeomorphism from M_2 to itself)

$$\tau = f^0(\tilde{t}, \tilde{x}) \quad , \quad \xi = f^i(\tilde{t}, \tilde{x}) \quad (119)$$

with $f^0(0, \tilde{x}) = 0$, $f^i(0, \tilde{x}) = \tilde{x}$ and $\frac{\partial f^0}{\partial \tilde{t}}(0, \tilde{x}) = a$, $\frac{\partial f^i}{\partial \tilde{t}}(0, \tilde{x}) = b^i$. We claim (and you should verify!) that one can choose the functions $a, b^i : \Sigma \to \mathbb{R}$ such that the metric g_2 expressed in the new coordinates (τ, ξ) reads

$$g_2 = (\hat{g}_2)_{00}(\tau, \xi) d\tau d\tau + 2(\hat{g}_2)_{0i}(\tau, \xi) d\tau \otimes d\xi + (\hat{g}_2)_{ij})(\tau, \xi) d\xi^i \otimes d\xi^j \,, \quad (120)$$

with

$$\begin{aligned} &(\hat{g}_2)_{00}(0, \xi) = -1 \,, \\ &(\hat{g}_2)_{0i}(0, \xi) = 0 \,, \\ &(\hat{g}_2)_{ij})(0, \xi) = (g_2)_{ij})(0, \tilde{x} = \xi) = (g_1)_{ij}(0, x = \tilde{x} = \xi) \,. \end{aligned} \quad (121)$$

[23]Given a general metric g_2 we can change the spatial coordinates on M_2 such that the coordinate components of the induced metric on Σ agree with the components of g_1 since the induced metrics represent the same *tensor* on Σ by assumption.

Step 2. We show that (121) implies that

$$(\partial_\tau(\hat{g}_2)_{00})(0,\xi) = (\partial_t(g_1)_{00})(0, x = \xi)\,, \tag{122}$$
$$(\partial_\tau(\hat{g}_2)_{0i})(0,\xi) = (\partial_t(g_1)_{0i})(0, x = \xi)\,, \tag{123}$$
$$(\partial_\tau(\hat{g}_2)_{ij})(0,\xi) = (\partial_t(g_1)_{ij})(0, x = \xi)\,. \tag{124}$$

This is again a computation. For instance, for (124), first note that both sides represent the (respective) coordinate components of the second fundamental form (a *tensor*) of Σ in M_1 and M_2 respectively. (This is because the normal to Σ is given by $(1,0,0,0)$ and $(1,0,0,0)$ respectively, cf. (121) and (110).) So the left hand side and the right hand are coordinate components of the same *tensor* (recall that by assumption the induced second fundamental forms of the two solutions agree) and since the coordinate frames on Σ are related by $\frac{\partial}{\partial \xi^i} = \frac{\partial}{\partial x^i}$ (since $x = \tilde{x} = \xi$ on Σ) the left hand side and the right hand side have to agree. The identities (122) and (123) follow from transforming the condition $(g_1)^{\mu\nu}(\Gamma_1)^\sigma{}_{\mu\nu} = 0$ holding in (t,x) coords for g_1 to the (τ,ξ) coordinates. (Note that $\frac{\partial(t,x)}{\partial(\tau,\xi)}$ is the identity on Σ by (121).)

Step 3. We solve $\Box_{g_2}(\tilde{\tau},\tilde{\xi}) = 0$ with data $\tilde{\tau}(\tau = 0,\xi) = 0$, $\tilde{\xi}(0,\xi) = \xi$ and $\partial_\tau\tilde{\tau}(0,\xi) = 1$, $\partial_\tau\tilde{\xi}(0,\xi) = 0$ in M_2. Note that $\frac{\partial(\tilde{\tau},\tilde{\xi})}{\partial(\tau,\xi)}$ is the identity on Σ by the initial conditions chosen. In particular, (121) and hence (122)–(124) continue to hold for g_2 in $(\tilde{\tau},\tilde{\xi})$ coordinates. By standard theory for wave equations a solution $(\tilde{\tau},\tilde{\xi})$ exists in M_2 and in a subset $U_2 \subset M_2$ containing Σ we have that $\frac{\partial(\tilde{\tau},\tilde{\xi})}{\partial(\tau,\xi)}$ is invertible, so $(\tilde{\tau},\tilde{\xi})$ are indeed coordinates on U_2. More specifically, the $(\tilde{\tau},\tilde{\xi})$ coordinates are by construction *harmonic* coordinates for g_2 and since g_2 is Ricci flat is satisfies the non-linear wave equation $\widetilde{Ric}(g_2) = 0$ in $(\tilde{\tau},\tilde{\xi})$-coordinates, the same equation that g_1 satisfies in the (t,x) coordinates. Hence we identify (t,x) coordinates on an open subset U_1 of M_1 with $(\tilde{\tau},\tilde{\xi})$ coordinates on an open subset U_2 of M_2. Since g_1 and g_2 now satisfy the same non-linear wave equation and since moreover, their data (that is $(g_1)_{\mu\nu}|_{t=0}$, $\partial_t(g_1)_{\mu\nu}|_{t=0}$ in (t,x)-coordinates and $(g_2)_{\alpha\beta}|_{\tilde{\tau}=0}$, $\partial_{\tilde{\tau}}(g_2)_{\alpha\beta}|_{\tilde{\tau}=0}$ in $(\tilde{\tau},\tilde{\xi})$-coordinates respectively) agree on Σ we conclude that the map identifying (t,x) coordinates on U_1 with $(\tilde{\tau},\tilde{\xi})$ coordinates on U_2 is the desired diffeomorphism ϕ satisfying $\phi^\star g_2 = g_1$.

4.10 Local well-posedness for the Einstein equations

In this subsection we collect the well-posedness statements for the vacuum Einstein equations in purely geometric and general form which does not make reference to coordinates anymore. While this is very elegant and satisfying, let us not forget that n order to eventually arrive at these geometric statements one had to do analysis in specifically constructed coordinates! For this section, I am following the exposition in Jan Sbierski's paper "On the Existence of a Maximal Cauchy Development for the Einstein Equations: a Dezornification" *Annales Henri Poincaré* **17** *301–329 (2016).*

Definition 4.33. *A globally hyperbolic development (GHD) (M, g, i) of initial data (Σ, h, K) is a time-oriented, globally hyperbolic Lorentzian manifold (M, g) that satisfies the vacuum Einstein equations together with an embedding $i : \Sigma \to M$ such that*

1. $i^\star g = h$
2. $i^\star \overline{K} = K$
3. $i(\Sigma)$ *is a Cauchy hypersurface in* (M, g).

where $\overline{K}$ denotes the second fundamental form of $i(\Sigma)$ in M with respect to the future normal.

Definition 4.34. *Given two GHDs (M, g, i) and (M', g', i') of the same initial data (Σ, h, K), we say that (M', g', i') is an extension of (M, g, i) is there exists a time-orientation preserving isometric embedding*

$$\psi : M \to M'$$

that preserves the initial data, i.e. $\psi \circ i = i'$.

Definition 4.35. *Given two GHDs (M, g, i) and (M', g', i') of initial data (Σ, h, K) we say that a GHD (U, g_U, i_U) of the same initial data is a common globally hyperbolic development (CGHD) of (M, g, i) and (M', g', i') if both (M, g, i) and (M', g', i') are extensions of (U, g_U, i_U).*

We can then formulate our local existence and uniqueness theorem that we proved (strictly speaking we proved it for $\Sigma = \mathbb{R}^n$ with additional assumptions on h and gave a sketch of the local uniqueness) above in this entire geometric language:

Theorem 4.36. *Given initial data (Σ, h, K) for the vacuum Einstein equations, there exists a GHD and for any two GHD of the same data, there exists a CGHD.*

The original proof of the above theorem is dues to Choquet-Bruhat in 1959, hence a long time after the Einstein equations were written down! (The progress was triggered by important developments in non-linear hyperbolic equations in the 1930s associated with the names of Leray, Schauer, Friedrichs and many others.) One can globalise the above statement as follows:

Theorem 4.37. *Given initial data for the vacuum Einstein equations there exists a GHD $\tilde{M}$ that is an extension of any other GHD of the same initial data. The GHD $\tilde{M}$ is unique up to isometry and is called the maximal globally hyperbolic development of the given initial data.*

The original proof of this theorem due to Choquet-Bruhat and Geroch in 1969 uses the axiom of choice in the form of Zorn's Lemma. The idea is to introduce a partial ordering on the set of GHDs (with the partial order given by the notion of *extension* introduced above). One then concludes the existence of a maximal element from the fact that every totally ordered subset has an upper bound (namely the union of the developments in the subset). One next establishes the uniqueness of $\tilde{M}$ by showing that the assumption of a spacetime $\tilde{M}'$ that cannot be isometrically embedded into $\tilde{M}$ leads to a contradiction as one can glue $\tilde{M}$ and $\tilde{M}'$ together to produce a spacetime contradicting the maximality of $\tilde{M}$.) A constructive proof without the use of the axiom of choice was given recently by Sbierski in the aforementioned paper, which I recommend for more details on the proof of Theorems 4.36 and 4.37.

4.11 Final remarks

Theorems 4.36 and 4.37 together are sometimes called the fundamental theorem of general relativity. This is because general relativity as a mathematical subject is about studying the maximal development of given initial data. For instance one would like to know for which data the maximal development is geodesically complete and if it is not, complete whether solutions become singular at the boundary and what the structure of that boundary looks like etc. Penrose's strong cosmic censorship, one of the most difficult conjectures in the field, asserts that the maximal development of compact or asymptotically flat initial data is generically inextendible. Now Remark 4.11 makes a bit more sense, hopefully!

Note also that the Cauchy problem allow us to sensibly talk about *stability* of spacetimes.

5 The covariant wave equation on black hole spacetimes

5.1 Motivation

to be written

5.2 Warm-up: Decay for $\Box_\eta \psi = 0$ from a geometric point of view

We revisit once more the free linear wave equation on $(3+1)$-Minkowski space

$$\begin{aligned} \Box_\eta \psi &= 0 \\ \psi(t=0,x) &= \psi_0(x) \\ \partial_t \psi(t=0,x) &= \psi_1(x) \end{aligned} \tag{125}$$

where $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R}^3)$ and

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + \not{g}_{AB} d\theta^A d\theta^B \, .$$

with $\not{g}_{AB} d\theta^A d\theta^B = r^2 \gamma_{AB} d\theta^A d\theta^B = r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$ is the round metric on spheres of radius r. We have already proven the conservation of energy: For all $t \in \mathbb{R}$ we have

$$\|\psi(t,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\partial_t \psi(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} = \|\psi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\psi_1\|^2_{L^2(\mathbb{R}^3)}\,, \tag{126}$$

where we recall

$$\begin{aligned}\|\psi(t,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} &= \int_{\Sigma_t} d^3x (\partial_x\psi)^2 + (\partial_y\psi)^2 + (\partial_z\psi)^2 \\ &= \int_{\Sigma_t} r^2 dr \sin\theta d\theta d\phi \left((\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right)\,. \end{aligned} \tag{127}$$

We proved (126) by integrating the divergence identity $\nabla^\mu(T_{\mu\nu}X^\nu) = {}^{(X)}\pi^{\mu\nu}T_{\mu\nu}$ over the region enclosed between Σ_0 and Σ_T with the Killing field $X = \partial_t$. On Sheet 8 you will prove boundedness of the norm on the left of (126) by initial data for spacetimes "close" to Minkowski space in a suitable sense illustrating the robustness of the vectorfield method that we employ here. (While there is an explicit representation formula (75) for the solution of (125), this is of course not true if η is a general metric close to Minkowski space!) The next estimate captures the dispersion (decay) of the wave equation in a robust manner:

Proposition 5.1. *The solution of (125) satisfies the estimate*

$$\int_{t=0}^{\infty} dt \int_{\Sigma_t} \frac{1}{r}|\nabla\!\!\!/\psi|^2 \leq CE_0[\psi] := C\left(\|\psi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\psi_1\|^2_{L^2(\mathbb{R}^3)}\right) \tag{128}$$

for a uniform constant $C > 0$.

The estimate (128) captures that the angular derivatives of ψ decay in time in an averaged sense. On Sheet 9 you will improve (128) to the estimate

$$\int_{t=0}^{\infty} dt \int_{\Sigma_t} \left[\frac{1}{1+r^2}\left((\partial_t\psi)^2 + (\partial_r\psi)^2\right) + \frac{1}{r}|\nabla\!\!\!/\psi|^2\right] \leq CE_0[\psi]\,. \tag{129}$$

Note the importance of the r-weights – while they can be improved, the estimate would be wrong without any r-decay as it would contradict energy conservation.

Proof. We will prove (128) with $\int_{t=0}^{T}$ for any $T > 0$ with the constant C not depending on T which implies (128) as stated.

Step 1. We begin with the observation that on any Σ_t we have

$$\begin{aligned}\int_0^\infty \int_{S^2} \psi^2(t,r,\theta,\phi) dr \sin\theta d\theta d\phi &\leq C \int_0^\infty \int_{S^2} r^2(\partial_r\psi(t,r,\theta,\phi)^2 dr sin\theta d\theta d\phi \\ &\leq CE_0[\psi]\,. \end{aligned} \tag{130}$$

The second inequality is obvious from (126) and the first follows from integrating by parts and Cauchy Schwarz

$$\int_0^\infty \int_{S^2} \psi^2(t,r,\theta,\phi)\partial_r r dr \sin\theta d\theta d\phi = -2\int_0^\infty \int_{S^2} r\psi\partial_r\psi(t,r,\theta,\phi) dr \sin\theta d\theta d\phi$$

$$\leq 2\sqrt{\int_0^\infty \int_{S^2} \psi^2(t,r,\theta,\phi) dr \sin\theta d\theta d\phi}\sqrt{\int_0^\infty \int_{S^2} (\partial_r\psi(t,r,\theta,\phi))^2 r^2 dr \sin\theta d\theta d\phi}$$

from which the result easily follows.

Step 2. We integrate our fundamental divergence identity

$$\nabla^\mu (J^{(X)}_\mu[\psi]) = K^{(X)}[\psi] := {}^{(X)}\pi^{\mu\nu}T_{\mu\nu}$$

where $J^{(X)}_\mu[\psi] = T_{\mu\nu}[\psi]X^\nu$ with the vectorfield $X = \partial_r$ over the region

$$\mathcal{M}_\epsilon = \mathcal{M} \setminus \{[0,T] \times B_\epsilon\} := [0,T] \times \mathbb{R}^3 \setminus \{[0,T] \times B_\epsilon\}\,.$$

You should draw a picture. The region is the region between to constant t hypersurfaces with a small cylinder of radius ϵ excluded around the origin of the polar coordinates (where the latter are not regular). Observe that the unit *outward* normal to the cylinder is $-\partial_r$. We note $X^r = 1$ and $X_r = 1$ and compute from $2^{(X)}\pi_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu} = \partial_r g_{\mu\nu}$

$$2^{(X)}\pi_{AB} = \frac{2}{r}\not{g}_{AB} \quad , \quad 2^{(X)}\pi^{AB} = \frac{2}{r}\not{g}^{AB} \tag{131}$$

while all other components are zero, in particular ${}^{(X)}\pi^{tt} = {}^{(X)}\pi^{tr} = {}^{(X)}\pi^{rr} = {}^{(X)}\pi^{tA} = {}^{(X)}\pi^{rA} = 0$ for $A \in \{\theta, \phi\}$. It follows that

$$K^{(\partial_r)}[\psi] = \frac{1}{r}|\nabla\!\!\!/\psi|^2 - \frac{1}{2}\frac{2}{r}\left(-(\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right) = \frac{1}{r}\left((\partial_t\psi)^2 - (\partial_r\psi)^2\right).$$

Another simple computation (anticipating the boundary terms that will appear) gives

$$J^{(\partial_r)}_\mu[\psi]\,(\partial_t)^\mu = T_{tr} = \partial_t\psi\partial_r\psi \leq (\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2 \tag{132}$$

and

$$J^{(\partial_r)}_\mu[\psi]\,(\partial_r)^\mu = T_{rr} = \frac{1}{2}(\partial_t\psi)^2 + \frac{1}{2}(\partial_r\psi)^2 - \frac{1}{2}|\nabla\!\!\!/\psi|^2\,. \tag{133}$$

We conclude for the boundary term appearing on the cylinder:

$$\Big|\int_0^T dt \int_{S^2} \epsilon^2 \sin\theta d\theta d\phi J^{(\partial_r)}_\mu[\psi]\,(\partial_r)^\mu \Big| \leq \tilde{C}_{T,\psi}\epsilon^2$$

which for fixed T and ψ goes to zero as $\epsilon \to 0$. For the boundary term on constant time slices we note

$$\Big|\int_\epsilon^\infty dr \int_{S^2} r^2 \sin\theta d\theta d\phi J^{(\partial_r)}_\mu[\psi]\,(\partial_t)^\mu \Big| \leq E_0[\psi]$$

independently of ϵ. We conclude for any $\epsilon > 0$ the estimate

$$\int_{\mathcal{M}_\epsilon} dt d^3x \frac{1}{r}\left((\partial_t\psi)^2 - (\partial_r\psi)^2\right) \leq CE_0[\psi] + \tilde{C}_{T,\psi}\epsilon^2\,. \tag{134}$$

This in itself is not useful as the left hand side is not coercive.

Step 3. We add to (134) the following (easily verified) divergence identity integrated over $\mathcal{M}_\epsilon$, valid for any smooth function h on $\mathcal{M}_\epsilon$:

$$\int_{\mathcal{M}_\epsilon} dtd^3x \nabla^\mu \left(h\psi\nabla_\mu\psi - \frac{1}{2}\psi^2\nabla_\mu h \right)$$
$$= \int_{\mathcal{M}_\epsilon} dtd^3x dvol \left[h\left(-(\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2 \right) - \frac{1}{2}\psi^2\Box_\eta h \right] . \qquad (135)$$

We choose $h = \frac{1}{r}$ and obtain the estimate

$$\int_{\mathcal{M}_\epsilon} dtd^3x \frac{1}{r}|\nabla\!\!\!/\psi|^2 \leq CE_0[\psi] + \int_{\mathcal{M}_\epsilon} dtd^3x\nabla^\mu \left(h\psi\nabla_\mu\psi - \frac{1}{2}\psi^2\nabla_\mu h \right) + \tilde{C}_{T,\psi}\epsilon^2 .$$

The second term on the right is a boundary term and evaluated by Stokes' theorem. The boundary term on the hypersurfaces of constant t satisfies

$$\Big| \int_\epsilon^\infty dr \int_{S^2} r^2 \sin\theta d\theta d\phi \frac{1}{r}\psi\partial_t\psi \Big| \leq CE_0[\psi] \qquad (136)$$

independently of ϵ, where we have used Cauchy Schwarz and (130). The boundary term on the cylinder is computed to be (note the outward normal is $-\partial_r$!)

$$-\int_0^T \int_{S^2} dt\epsilon^2 \sin\theta d\theta d\phi \left(\frac{1}{r}\psi\partial_r\psi + \frac{1}{2}\frac{\psi^2}{r^2} \right) \Bigg|_{r=\epsilon}$$
$$\leq -\frac{1}{2}\int_0^T \int_{S^2} dt \sin\theta d\theta d\phi\, \psi^2(t, r=\epsilon, \theta, \phi) + \tilde{C}_{T,\psi}\epsilon \qquad (137)$$

where $\tilde{C}_{T,\psi}$ may be a different constant to the one introduced above. Note the sign of the term involving ψ^2 which is crucial because it does *not* vanish in the limit as $\epsilon \to 0$! This yields

$$\int_{\mathcal{M}_\epsilon} dtd^3x \frac{1}{r}|\nabla\!\!\!/\psi|^2 + \frac{1}{2}\int_0^T \int_{S^2} dt \sin\theta d\theta d\phi\, \psi^2(t, r=\epsilon, \theta, \phi) \leq CE_0[\psi] + \tilde{C}_{T,\psi}\epsilon .$$

We can now take the limit $\epsilon \to 0$ and obtain for any fixed T and ψ the estimate

$$\int_{[0,T]\times\mathbb{R}^3} dtd^3x \frac{1}{r}|\nabla\!\!\!/\psi|^2 + 2\pi \int_0^T \int_{S^2} dt\psi^2(t, r=0) \leq CE_0[\psi] . \qquad (138)$$

We can take the limit $T \to 0$ and obtain (128) as desired. □

5.3 The wave equation on the Schwarzschild exterior

We move on to study the covariant wave equation on the Schwarzschild exterior. We use $(t^\star, r, \theta, \phi)$ coordinates covering the regions I and II of the Penrose diagram and we will be interested in understanding the global behaviour of solutions to the problem

$$\begin{aligned} \Box_{g_S}\psi &= 0 \\ \psi(t^\star = 0, \cdot) &= \psi_0(\cdot) \\ \partial_t\psi(t^\star = 0, \cdot) &= \psi_1(\cdot) \end{aligned} \tag{139}$$

where $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R} \times S^2)$ and the metric g_S reads

$$g_S = -\left(1 - \frac{2M}{r}\right)(dt^\star)^2 + \frac{4M}{r}dt^\star dr + \left(1 + \frac{2M}{r}\right)dr^2 + \not{g}_{AB}d\theta^A d\theta^B \tag{140}$$

on the manifold $\mathcal{M} = [0,\infty)_{t^\star} \times [2M,\infty)_r \times S^2$. Note restricting to this manifold makes sense by the domain of dependence property![24]

One easily compute the non-vanishing inverse components

$$g^{t^\star t^\star} = -\left(1 - \frac{2M}{r}\right) \quad , \quad g^{t^\star r} = \frac{2M}{r} \quad , \quad g^{rr} = 1 - \frac{2M}{r} \quad , \quad g^{AB} = \not{g}^{AB} \,,$$

and the volume form $dvol = r^2 dt^\star \wedge dr \wedge d\omega$ where $d\omega = \sin\theta d\theta \wedge d\phi$. One also computes the (timelike) unit normal to constant $t^\star$ slices $\Sigma_{t^\star}$ to be

$$n = \sqrt{-g^{t^\star t^\star}}\partial_{t^\star} - \frac{g^{t^\star r}}{\sqrt{-g^{t^\star t^\star}}}\partial_r$$

and the induced volume form $dS_{\Sigma_t} = \sqrt{-g^{t^\star t^\star}} r^2 dr \wedge d\omega \sim r^2 dr \wedge d\omega$ where $\sim$ captures the fact that $1 < \sqrt{-g^{t^\star t^\star}} \leq 2M$ for $\{r \geq 2M\}$.

Proposition 5.2. *Any solution of (139) satisfies the identity*

$$\begin{aligned} &\int_{\Sigma_{t_2^\star}} \left[\left(1 + \frac{2M}{r}\right)(\partial_{t^\star}\psi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right] r^2 dr d\omega \\ &\qquad\qquad + 2\int_{t_1^\star}^{t_2^\star}\int_{S^2} dt^\star d\omega (\partial_{t^\star}\psi)^2\,(t^\star, r = 2M, \theta, \phi) \\ &= \int_{\Sigma_{t_1^\star}} \left[\left(1 + \frac{2M}{r}\right)(\partial_{t^\star}\psi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right] r^2 dr d\omega \end{aligned}$$

Proof. This is a simple computational exercise. Apply the fundamental divergence identity with the Killing vectorfield $T = \partial_{t^\star}$. Use the version of Stokes' theorem on Example sheet 5 which allows for null hypersurfaces. □

We make the following important observations:

- Away from the horizon we control all derivatives, however, the control for the transversal derivatives degenerates near the horizon.

[24] The behaviour of solutions in the interior of the black hole, i.e. region II, is another interesting problem but not currently our concern.

- The boundary term on the horizon only controls the $\partial_{t^\star}$-derivative of ψ.
- The estimate states that the energy on spacelike slices is non-increasing. The boundary term on the horizon captures the fact that energy can disperse through the black hole event horizon. However, note that the above estimate is still consistent with $\partial_r \psi$ growing in time along the event horizon (because of the degenerating factor of $(1 - \frac{2M}{r})$)!

The question is: Can we improve on the above?

5.4 The boundedness statement

This section follows closely the "Lecture Notes on Black Holes and Linear Waves" by Dafermos–Rodnianski.

Theorem 5.3. *Any solution of (139) satisfies the non-degenerate boundedness estimate*

$$\int_{\Sigma_{t^\star}} r^2 dr d\omega \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right]$$
$$\leq C \int_{\Sigma_0} r^2 dr d\omega \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] \tag{141}$$

for any $t^\star \geq 0$ and a constant C depending only on M.

In a nut-shell, the proof of Theorem 5.3 consist in constructing a suitable vectorfield N on $\mathcal{M}$ whose associated divergence identity (when combined with Theorem 5.2 in a clever way) produces the estimate of Theorem 5.3.

The vectorfield N (which is also called the *red-shift* vectorfield) in constructed in Proposition 5.4 immediately below. First, however, we give some heuristics on why one should expect to be able to overcome the degeneration at $r = 2M$ in the statement of Proposition 5.2. For this consider two observers A and B which follow the orbits of the timelike Killing field $\partial_{t^\star}$. In other words

$$\gamma_A(t^\star) = (t^\star, r = r_A, \theta, \phi) \quad , \quad \gamma_B(t^\star) = (t^\star, r = r_B, \theta, \phi) \, , \tag{142}$$

where θ, ϕ is some fixed point on S^2. Now suppose A sends two consecutive signals with coordinate distance $t^\star_2 - t^\star_1$ towards B. Observer B will receive them at points on her orbit which are also coordinate distance $t^\star_2 - t^\star_1$ away (draw a picture!). However, the physically relevant *proper time* for A between the two signals is $\int_{t^\star_1}^{t^\star_2} dt^\star \sqrt{-g(\dot{\gamma}_A, \dot{\gamma}_A)} = \sqrt{1 - \frac{2M}{r_A}} \, (t^\star_2 - t^\star_1)$ while for B the time passed between receiving the two signals (measured, of course, with respect to her proper time) is $\int_{t^\star_1}^{t^\star_2} dt^\star \sqrt{-g(\dot{\gamma}_B, \dot{\gamma}_B)} = \sqrt{1 - \frac{2M}{r_B}} \, (t^\star_2 - t^\star_1)$. We conclude that if A moves very close to the event horizon $r = 2M$, then the time B has to wait between the two signals is $\frac{\sqrt{1-\frac{2M}{r_B}}}{\sqrt{1-\frac{2M}{r_A}}}$-times longer than the time elapsed for A when sending them! If we consider the two signals as the distance between two maxima of a wave, we see that the frequency of waves gets shifted towards longer wave lengths (i.e. towards the *less energetic* (infra)-red part of the spectrum).[25] In yet other words, waves lose energy as they propagate near the event horizon. (Of course, this is associated with energy falling behind the event horizon and no longer contributing to the energy.) The strength of the above effect will be measured by the so-called surface gravity κ. To define it, recall from Exercise Sheet 7 that the horizon $\{r = 2M\}$ is a null hypersurface and that T is a generator. The vectorfield T therefore satisfies on $\mathcal{H}^+$ the equation

$$\nabla_T T = \kappa T \tag{143}$$

for some function κ on $\mathcal{H}^+$. One easily computes $\kappa = \frac{1}{4M}$ on Schwarzschild (and also sees that the computation involves the radial derivative of $g_{t^\star t^\star}$, which is what was crucial in the above heuristics.

Proposition 5.4. *There exists a vectorfield N such that*

1. $(\phi_{t^\star})_\star N = N$*, where $\phi_{t^\star}$ is the one-parameter group of diffeomorphisms associated with the Killing field $T = \partial_{t^\star}$.*[26]
2. *N is future-directed timelike.*
3. *$N = T$ in $\Sigma_{t^\star} \cap \{r \geq r_1\}$ for some $r_1 > 2M$.*
4. *There exists $r_0 > 2M$ such that on $\Sigma_{t^\star} \cap \{r \leq r_0\}$ we have*

$$K^N[\psi] \geq c J^{(N)}_\mu N^\mu \sim c\left[(\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right]$$

for a constant c depending only on r_0 and M.

Remark 5.5. *We can already anticipate the potential usefulness of the vectorfield in the divergence identity. Property 2 implies that the boundary terms on $\Sigma_{t^\star}$ will be coercive (why?) and Property 4 states that the bulk term has a good sign at least in a neighbourhood of the event horizon!*

Proof. We remark that one pedestrian way of proving the proposition would be to make an ansatz $N = \alpha(r)\partial_{t^\star} + \beta(r)\partial_r$ and to try and construct the smooth functions α and β such that the above conditions hold. We choose a more conceptual approach.

Observation 1. It suffices to construct a future-directed timelike N_0 along Σ_0 which satisfies Condition 4 on $\Sigma_0 \cap \{r \geq r_1\}$ for some $r_1 > 2M$. This is because with this done, given N_0 we define

$$\tilde{N}_0 = \chi(r)N_0 + (1 - \chi(r))T$$

[25] You need a little bit of physics here: Shorter wavelengths are associated with higher energy according to Planck's formula $E = \frac{hc}{\lambda}$ for the energy of a photon.

[26] In other words, the vectorfield N is time independent, $\mathcal{L}_T N = 0$, or $\partial_{t^\star} N^\mu = 0$ in the $(t^\star, r, \theta, \phi)$ coordinates.

where χ is a radial cut-off function being equal to 1 in $r \leq r_0 = \frac{r_1 - 2M}{2}$ and vanishing identically for $r \geq r_1$. This is future-directed timelike (why?) and pushing it forward along the integral curves of $\partial_{t^\star}$ produces the desired N.

Observation 2. It suffices for property 4 to hold on the sphere $S_0 = \Sigma_0 \cap \{r = 2M\}$ because of continuity. To see this, observe that $K^N[\psi] = T_{\mu\nu}{}^{(N)}\pi^{\mu\nu}$ is a quadratic form in derivatives of ψ with coefficients depending smoothly on derivatives of the metric and derivatives of the components of N. Hence if the quadratic form is positive definite at $r = 2M$ then it is so also in a neighbourhood of that sphere.

The actual construction. We first note that the vectorfield

$$\overline{Y} = -2\partial_r + 2\partial_{t^\star}$$

is null on the horizon (why?) and $g(\overline{Y}, \partial_{t^\star})|_{r=2M} = -2$. (Draw this vectorfield in the Penrose-diagram!) We extend $\overline{Y}$ off the horizon to a vectorfield Y by imposing the condition

$$\nabla_Y Y = -\sigma(Y + T) \quad \text{on } S_0 \tag{144}$$

for some large positive constant σ that we will choose below.[27] We now claim that $N_0 = Y + T$ satisfies condition 4 at S_0 provided we choose σ sufficiently large depending only on M. To see this we compute at $p \in S_0$ in a null frame (T, Y, E_1, E_2) the following derivatives (which are in turn essential to compute the components of the deformation tensor of N).

$$\nabla_T Y = -\kappa Y + a^1 E_1 + a^2 E_2 \quad \text{note there is no } T\text{-term (why?)} \tag{145}$$

$$\nabla_Y Y = -\sigma T - \sigma Y \quad \text{by definition} \tag{146}$$

$$\nabla_{E_1} Y = h_1^1 E_1 + h_1^2 E_2 - \tilde{a}^1 Y \quad \text{note there is no } T\text{-term (why?)} \tag{147}$$

$$\nabla_{E_2} Y = h_2^1 E_1 + h_2^2 E_2 - \tilde{a}^2 Y \quad \text{note there is no } T\text{-term (why?)} \tag{148}$$

Note that it is indeed the surface gravity $\kappa = \frac{1}{4M}$ appearing in the first identity since $g(T, \nabla_T Y) = -g(\nabla_T T, Y) = 2\kappa$.

We now compute the components of the deformation tensor (check this!)

$$2^{(Y+T)}\pi(V,W) = 2^{(Y)}\pi(V,W) = (\mathcal{L}_Y g)(V,W) = g(\nabla_W Y, V) + g(\nabla_V Y, W)\,.$$

We now compute the desired contraction of the energy momentum tensor and the deformation tensor in the null frame (T, Y, E_1, E_2) at p (writing $\mathbf{T}$ for the energy momentum tensor to avoid confusion with the vectorfield T)

$$\begin{aligned} K^N[\psi] = \mathbf{T}_{\mu\nu}{}^{(Y)}\pi^{\mu\nu} &= g^{\alpha\mu} g^{\beta\nu} \mathbf{T}_{\mu\nu}{}^{(Y)}\pi_{\alpha\beta} \\ &= \frac{1}{2}\frac{1}{2}\mathbf{T}(Y,Y)^{(Y)}\pi(T,T) + 2\frac{1}{2}\frac{1}{2}\mathbf{T}(Y,T)^{(Y)}\pi(T,Y) \\ &+ \frac{1}{2}\frac{1}{2}\mathbf{T}(T,T)^{(Y)}\pi(Y,Y) + \dots\ , \end{aligned} \tag{149}$$

[27]Note we can for instance solve an ODE in a neighbourhood of S_0 to construct a Y in a neighbourhood and achieve this condition at S_0.

where you should complete the computation, resulting in

$$\begin{aligned}K^N[\psi] &= \frac{1}{2}\kappa|Y\psi|^2 + \frac{1}{2}\sigma\left(|E_1(\psi)|^2 + |E_2(\psi)|^2\right) + \frac{1}{2}\sigma|T\psi|^2 \\ &- \frac{1}{2}\mathbf{T}(E_1,Y)a^1 - \frac{1}{2}\mathbf{T}(E_2,Y)a^2 + \frac{1}{2}\mathbf{T}(E_1,T)\tilde{a}^1 + \frac{1}{2}\mathbf{T}(E_2,T)\tilde{a}^2 \\ &+ \mathbf{T}(E_1,E_1)h_1^1 + \mathbf{T}(E_2,E_2)h_2^2 + \mathbf{T}(E_1,E_2)\left(h_2^1 + h_1^2\right) \end{aligned} \tag{150}$$

It is now clear[28] that by choosing σ sufficiently large (depending only on the value of the constants $a^1, a^2, \tilde{a}^1, \tilde{a}^2, h_1^1, h_2^1, h_1^2, h_2^2$) we can achieve (applying Cauchy's inequality with ϵ for the terms in the second and third line) that

$$K^N[\psi] \geq c\left(|Y(\psi)|^2 + |T\psi|^2 + |E_1(\psi)|^2 + |E_2(\psi)|^2\right) \tag{151}$$

for some c depending only on M, thereby proving that property 4 holds on S_0 which by the previous considerations implies the proposition. Note that it is the positivity of the surface gravity which is essential for the proof to work! □

We can finally prove Theorem 5.3:

Proof. We write out the fundamental divergence identity for the vectorfield N (we now think of r_0 and r_1 as having been *fixed* in the construction of N) integrated over the region $\mathcal{M}(t_1^\star, t_2^\star) := \mathcal{M} \cap \{t_1^\star \leq t^\star \leq t_2^\star\}$. This produces

$$\begin{aligned}\int_{\Sigma_{t_2^\star}} J_\mu^{(N)} n_\Sigma^\mu + \int_{\mathcal{H}(t_1^\star,t_2^\star)} J_\mu^{(N)} n_{\mathcal{H}^+}^\mu + \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} K^N[\psi] = \\ -\int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r_0\leq r\leq r_1\}} K^N[\psi] + \int_{\Sigma_{t_1^\star}} J_\mu^{(N)} n_\Sigma^\mu\,.\end{aligned} \tag{152}$$

Dropping the second term and using Proposition 5.4 we infer

$$\begin{aligned}\int_{\Sigma_{t_2^\star}} J_\mu^{(N)} n_\Sigma^\mu + c\int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} J_\mu^{(N)} n_\Sigma^\mu \\ \leq C\int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r_0\leq r\leq r_1\}} J_\mu^{(N)} n_\Sigma^\mu + \int_{\Sigma_{t_1^\star}} J_\mu^{(N)} n_\Sigma^\mu\end{aligned} \tag{153}$$

for constants C and c depending only on M. To estimate the "bad" term on the right, we recall the T-estimate

$$\int_{\Sigma_{t^\star}} J_\mu^{(T)} n_\Sigma^\mu \leq \int_{\Sigma_0} J_\mu^{(T)} n_\Sigma^\mu =: D\,, \tag{154}$$

valid for $t^\star \geq 0$ from which we deduce

$$c\int_{\Sigma_{t^\star}\cap\{r\geq r_0\}} J_\mu^{(N)} n_\Sigma^\mu \leq \int_{\Sigma_{t^\star}\cap\{r\geq r_0\}} J_\mu^{(T)} n_\Sigma^\mu \leq \int_{\Sigma_{t^\star}} J_\mu^{(T)} n_\Sigma^\mu \leq D\,. \tag{155}$$

[28] Note in this context that the only way a term of the form $|Y\psi|^2$ can arise is through $\mathbf{T}(Y,Y)$ itself. This follows from the form of the energy momentum tensor and $g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi = -T\psi Y\psi + |E_1(\psi)|^2 + |E_2(\psi)|^2$.

Integrating this estimate in $t^\star$ from $t_1^\star$ to $t_2^\star$ we conclude (using $dvol_{\mathcal{M}} = r^2 dt^\star dr d\omega$ and $d\Sigma_{t^\star} = \sqrt{1+\frac{2M}{r}} r^2 dr d\omega$)

$$c \int_{\mathcal{M}(t_1^\star, t_2^\star) \cap \{r \geq r_0\}} J_\mu^{(N)} n_\Sigma^\mu \leq D(t_2^\star - t_1^\star) \, . \tag{156}$$

Combining this with (153) we arrive at the estimate[29]

$$\int_{\Sigma_{t_2^\star}} J_\mu^{(N)} n_\Sigma^\mu + c \int_{\mathcal{M}(t_1^\star, t_2^\star)} J_\mu^{(N)} n_\Sigma^\mu \leq CD(t_2^\star - t_1^\star) + \int_{\Sigma_{t_1^\star}} J_\mu^{(N)} n_\Sigma^\mu \, , \tag{157}$$

valid for $t_2^\star \geq t_1^\star \geq 0$. Defining $f(t^\star) := \int_{\Sigma_{t^\star}} J_\mu^{(N)} n_\Sigma^\mu$ we can write this as

$$f(t_2^\star) + c \int_{t_1^\star}^{t_2^\star} dt^\star f(t^\star) \leq f(t_1^\star) + CD(t_2^\star - t_1^\star) \, . \tag{158}$$

A straightforward ODE computation (exercise) now yields the desired statement

$$f(t^\star) \leq C f(0)$$

and the Theorem is proven. □

5.5 Higher derivatives and pointwise bounds

Theorem 5.3 proves boundedness of the $\dot{H}^1$-energy. However, often one is interested in pointwise bounds. We now discuss how one can prove higher order versions of the above energy estimates, which when combined with (simple versions of) Sobolev embedding theorems, produce pointwise bounds.

5.5.1 Sobolev embedding

We start with two propositions which contain the Sobolev embedding statements tailored to our setting. These will be proven on Sheet 10.

Proposition 5.6. *Let (S^2, γ) denote the round unit sphere. For $u : S^2 \to \mathbb{R}$ a smooth function we have the estimate*

$$\sup_{S^2} |u| \leq C \|u\|_{H^2(S^2)} \tag{159}$$

for a uniform constant C.

Since the spheres $S^2_{t^\star, r}$ in $\mathcal{M}$ are round spheres of radius r (with metric $\not{g} = r^2 \gamma$), a simple rescaling yields an estimate applicable to our setting:

[29] Note that constants C and c are allowed to change their value from line to line. What is important is that there exists constants depending only on M such that the stated estimate holds.

Corollary 5.7. *For a smooth function $u : S^2_{t^\star,r} \to \mathbb{R}$ we have the estimate*

$$\sup_{S^2_{t^\star,r}} |u|^2 \leq C \int_{S^2_{t^\star,r}} \sin\theta d\theta d\phi \left[|r^2 \nabla\!\!\!/ \nabla\!\!\!/ u|^2 + |r \nabla\!\!\!/ u|^2 + |u|^2 \right] . \tag{160}$$

The second proposition concerns Sobolev embedding on the 3-dimensional spacelike slices $\Sigma_{t^\star}$:

Proposition 5.8. *Consider the slices $\Sigma_{t^\star}$. For $u : \Sigma_{t^\star} \to \mathbb{R}$ smooth and of compact support, we have*

$$\sup_{\Sigma_{t^\star}} |u| \leq C \left(\|u\|_{\dot{H}^2(\Sigma_{t^\star})} + \|u\|_{\dot{H}^1(\Sigma_{t^\star})} \right) \tag{161}$$

for a uniform C (in particular not depending on the size of the support of u).

5.5.2 The commutation lemma

The third ingredient to derive higher order estimates and pointwise bounds is the following commutation Lemma, which will also be proven on Sheet 10.

Lemma 5.9. *Let ψ be a smooth solution of the equation $\Box_g \psi = 0$ and X a vectorfield. Then*

$$\Box_g (X\psi) = 2^{(X)}\pi^{\alpha\beta} \nabla_\alpha \nabla_\beta \psi + \left(2\nabla^{\alpha (X)}\pi_{\alpha\mu} - \nabla_\mu (tr^{(X)}\pi) \right) \nabla^\mu \psi \tag{162}$$

Corollary 5.10. *If X is Killing, then it commutes with the covariant wave operator.*

5.5.3 Warm-up: Minkowski space

We illustrate the basic idea to obtain pointwise estimates from higher order L^2 estimates for Minkowski space. Recall that in Minkowski space we proved the basic estimate (for all $t \in \mathbb{R}$)

$$\|\psi(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t \psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\psi(0,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t \psi(0,\cdot)\|_{L^2(\mathbb{R}^n)}$$

Since ∂_t is Killing it commutes with the wave operator and we obtain

$$\|\partial_t \psi(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t \partial_t \psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\partial_t \psi(0,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\Delta \psi(0,\cdot)\|_{L^2(\mathbb{R}^n)}$$

Writing now the wave equation as $\Delta\psi(t,\cdot) = \partial_t^2 \psi(t,\cdot)$ and noting that the right hand side is in L^2 on each constant t-slice, we deduce from standard elliptic estimates

$$\|\psi(t,\cdot)\|_{\dot{H}^2(\mathbb{R}^n)} \leq C \|\partial_t \partial_t \psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|\partial_t \psi(0,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\Delta \psi(0,\cdot)\|_{L^2(\mathbb{R}^n)}$$

Using the Sobolev embedding you proved on $\mathbb{R}^n$ we deduce, again for any $t \in \mathbb{R}$

$$\begin{aligned} \sup_x |\psi(t,x)| &\leq \|\psi(t,\cdot)\|_{\dot{H}^2(\mathbb{R}^n)} + \|\psi(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} \\ &\leq \|\partial_t \psi(0,\cdot)\|_{H^1(\mathbb{R}^n)} + \|\psi(0,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\Delta\psi(0,\cdot)\|_{L^2(\mathbb{R}^n)} . \end{aligned} \tag{163}$$

The last estimate is clearly a uniform bound on ψ in all of $\mathbb{R}^{1+3}$ from an initial data norm. (Of course you could derive a similar estimate from commuting with spatial isometries (how?).)

5.5.4 Pointwise estimate for ψ using angular commutation

In Schwarzschild we have a 4-dimensional Lie-algebra of Killing vectors generated by the rotations (spherical symmetry!), which in the standard coordinates take the form

$$\Omega_1 = \partial_\phi \quad , \quad \Omega_2 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi \quad , \quad \Omega_3 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$$

and the timelike Killing field $T = \partial_{t^\star}$. We first show how to obtain pointwise boundedness for ψ using only the rotations.

For this, first observe that on any $\Sigma_{t^\star}$ we have for any $R \geq 2M$ the estimate[30]

$$\int_R^\infty \int_{S^2_{t^\star,r}} \psi^2 dr d\omega + \int_{S^2_{t^\star,R}} \psi^2 r d\omega \leq C \int_R^\infty \int_{S^2_{t^\star,r}} r^2 \left(\partial_r \psi\right)^2 dr d\omega \, , \tag{164}$$

hence in particular on any sphere $S^2_{t^\star,R}$ with $t^\star \geq 0$ and $R \geq 2M$ we have

$$\int_{S^2_{t^\star,R}} \psi^2 d\omega \leq C E_0[\psi] := C \int_{\Sigma_0} J^{(N)}_\mu[\psi] n^\mu_\Sigma \, . \tag{165}$$

From the fact that the angular momentum operators commute with the wave operator, it follows that

$$\sum_{i=1}^3 \int_{S^2_{t^\star,R}} |\Omega_i \psi|^2 d\omega \leq C \sum_{i=1}^3 E_0[\Omega_i \psi] \tag{166}$$

$$\sum_{i,j=1}^3 \int_{S^2_{t^\star,R}} |\Omega_i \Omega_j \psi|^2 d\omega \leq C \sum_{i=1}^3 E_0[\Omega_i \Omega_j \psi] \tag{167}$$

Summing the last three estimates and using the results of Sheet 10 we deduce

$$\sup_{S^2_{t^\star,R}} |\psi|^2 \leq C \left(E_0[\psi] + \sum_{i=1}^3 E_0[\Omega_i \psi] + \sum_{i=1}^3 E_0[\Omega_i \Omega_j \psi] \right) \tag{168}$$

Since this holds independently of $t^\star \geq 0$ and $R \geq 2M$ we infer a uniform bound on ψ on all of $\mathcal{M}$ in terms of initial data, which has been our goal.

[30] This is proven exactly as Step 1 of Proposition 5.1 above carrying through the additional boundary term at $r = R$. Exercise!

5.5.5 Pointwise estimate for ψ using elliptic estimates

Let us write the wave equation as[31]

$$\frac{1}{r^2}\partial_r\left(r^2\left(1-\frac{2M}{r}\right)\partial_r\psi\right)+\not\Delta\psi$$
$$=\left(1+\frac{2M}{r}\right)\partial_{t^\star}\partial_{t^\star}\psi-\frac{2M}{r}\partial_{t^\star}\partial_r\psi-\frac{2M}{r^2}\partial_{t^\star}\psi \tag{169}$$

Strictly away from the horizon the right hand side is a uniformly elliptic operator on $\Sigma_{t^\star}$ and the right hand side is in $L^2(\Sigma_{t^\star})$ for all $t^\star \geq 0$ by previous estimates. This leads to the estimate

$$\|\psi\|^2_{\dot{H}^2\left(\Sigma_{t^\star}\cap\{r\geq r_0'\}\right)}+\|\psi\|^2_{\dot{H}^1\left(\Sigma_{t^\star}\cap\{r\geq r_0'\}\right)}\leq C_{r_0'}\left(E_0[\psi]+E_0[T\psi]\right), \tag{170}$$

for any $r_0' > 2M$ with the constant $C_{r_0'}$ blowing up as $r_0' \to 2M$. If you do not have much PDE experience (and "uniformly elliptic" doesn't mean anything to you), you can easily prove (170) by hand (in fact, a stronger statement!). Note the only difficulty is to establish the estimate for $\|\psi\|^2_{\dot{H}^2(\Sigma_{t^\star}\cap\{r\geq r_0\})}$ as we have already controlled the $\|\psi\|^2_{\dot{H}^1(\Sigma_{t^\star}\cap\{r\geq r_0\})}$-norm in Theorem 5.3. You multiply (169) by $r^2\not\Delta\psi$ and integrate over $\Sigma_{t^\star}$. Integrating by parts will produce control over all derivatives involving at least one *angular* derivative. The other terms can be put to the right hand side and controlled by Cauchy's inequality with ϵ and previous estimates. (If you haven't seen elliptic estimates before this is a really good exercise!) The missing $\partial_r\partial_r$ derivative can then be obtained directly from the equation (169). Note however, that this derivative will only be controlled *degenerately* at $r = 2M$, i.e. one will only control $\left(1-\frac{2M}{r}\right)\partial_r\partial_r$ in L^2.

5.5.6 Pointwise estimate for ψ using redshift commutation

The elliptic estimates of the previous section did not (and cannot) produce non-degenerate control over the second transversal derivatives at the horizon. To remedy this problem we commute the wave equation with the vectorfield $\hat{Y} = -\partial_r$ and deduce:

Lemma 5.11. *The solution of (139) satisfies the commuted equation*

$$\Box_g\hat{Y}\psi=\left(\frac{2}{r}-\frac{2M}{r^2}\right)\hat{Y}\hat{Y}\psi-\frac{4}{r}\hat{Y}T\psi+\frac{2}{r^2}\left(T\psi-\hat{Y}\psi\right) \tag{171}$$

Proof. Direct computation using Lemma 5.9. □

The important point here is that the factor $\left(\frac{2}{r}-\frac{2M}{r^2}\right)$ is positive in a neighbourhood of the horizon. In particular, we can choose the r_0 in the construction of the vectorfield N such that $\frac{2}{r}-\frac{2M}{r^2}\geq\frac{1}{4M}$ holds in $r \leq r_0$! Note that the

[31] This is easily seen from the form $\Box_g\psi=\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right)$ using $g = g_S$ from (140).

factor $\left(\frac{2}{r} - \frac{2M}{r^2}\right)$ (which arises from the $\hat{Y}\hat{Y}$ contraction of $2^{-\partial_r}\pi^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi$) *on the horizon itself* reduces precisely to (twice) the surface gravity κ so **there is again geometric significance to the positive sign that occurs here**!

We now prove the following commuted analogue of Theorem 5.3:

Theorem 5.12. *Any solution of (139) satisfies the non-degenerate boundedness estimate*

$$\int_{\Sigma_{t^\star}} r^2 dr d\omega \left[(\partial_t \hat{Y}\psi)^2 + (\partial_r \hat{Y}\psi)^2 + |\nabla\!\!\!/ \hat{Y}\psi|^2\right] \leq C \int_{\Sigma_0} r^2 dr d\omega \left[(\partial_t \hat{Y}\psi)^2 + (\partial_r \hat{Y}\psi)^2 + |\nabla\!\!\!/ \hat{Y}\psi|^2\right] \tag{172}$$

for any $t^\star \geq 0$ and a constant C depending only on M.

Remark 5.13. *Note that an equivalent way of stating the above estimate is $\int_{\Sigma_{t^\star}} J^{(N)}_\mu[\hat{Y}\psi] n^\mu_{\Sigma_{t^\star}} \leq C \int_{\Sigma_0} J^{(N)}_\mu[\hat{Y}\psi] n^\mu_{\Sigma_{t^\star}}$, which is also what we will prove.*

Proof. We write the divergence identity[32] associated with the vectorfield N for the commuted equation (171). This yields

$$\begin{aligned}\int_{\Sigma_{t_2^\star}} J^{(N)}_\mu[\hat{Y}\psi] n^\mu_\Sigma + \int_{\mathcal{H}(t_1^\star,t_2^\star)} J^{(N)}_\mu[\hat{Y}\psi] n^\mu_{\mathcal{H}^+} + \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} K^N[\hat{Y}\psi] \\ = - \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r_0\leq r\leq r_1\}} K^N[\hat{Y}\psi] + \int_{\Sigma_{t_1^\star}} J^{(N)}_\mu[\hat{Y}\psi] n^\mu_\Sigma \\ + \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} \mathcal{E}^N[\hat{Y}\psi] + \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\geq r_0\}} \mathcal{E}^N[\hat{Y}\psi]\end{aligned}$$

where

$$\mathcal{E}^N[\hat{Y}\psi] = -N\hat{Y}\psi\left[\left(\frac{2}{r} - \frac{2M}{r^2}\right)\hat{Y}\hat{Y}\psi - \frac{4}{r}\hat{Y}T\psi + \frac{2}{r^2}\left(T\psi - \hat{Y}\psi\right)\right]. \tag{173}$$

Observe carefully the signs for the term involving $\mathcal{E}^N$! Now we can drop the good term on the horizon and we can observe that away from the horizon we have from (170) integrated in time the estimates

$$\left|\int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\geq r_0\}} K^N[\hat{Y}\psi]\right| + \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\geq r_0\}} \mathcal{E}^N[\hat{Y}\psi] \leq C\left(t_2^\star - t_1^\star\right)\left(E_0[\psi] + E_0[T\psi]\right) \tag{174}$$

For the critical term $+\int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} \mathcal{E}^N[\hat{Y}\psi]$ we make the observation that the vectorfield N can be related to the vectorfield $\hat{Y}$ by the relation

$$N = Y + T = \left(2 + k_1\left(1 - \frac{2M}{r}\right)\right)\hat{Y} + k_2\left(1 - \frac{2M}{r}\right)T \tag{175}$$

[32]Observe that the divergence identity associated with the inhomogeneous equation $\Box_g\phi = F$ becomes $\nabla^\mu(T_{\mu\nu}[\phi]N^\nu) = {}^{(N)}\pi^{\mu\nu}T_{\mu\nu}[\phi] + FN(\phi)$, which we apply here with $\phi = \hat{Y}\psi$ and F the right hand side of (171).

where k_1 and k_2 are constants chosen such that condition (144) holds on $\mathcal{H}^+$. We can moreover choose r_0 small enough such that $2+k_1\left(1-\frac{2M}{r}\right) \geq 1$ holds in $r \geq r_0$.[33] We can hence estimate **in the region** $r \leq r_0$ after applying Cauchy's inequality with weights (borrowing from the good $\hat{Y}\hat{Y}\psi$-term)

$$-\mathcal{E}^N[\hat{Y}\psi] \geq \frac{1}{8M}|\hat{Y}\hat{Y}\psi|^2 - c_1|T\hat{Y}\psi|^2 - c_2|T\psi|^2 - c_3|\hat{Y}\psi|^2 \tag{176}$$

Now by the uniform boundedness statement for ψ and (the trivially commuted $T\psi$) we infer

$$-\int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} \mathcal{E}^N[\hat{Y}\psi] \geq \int_{\mathcal{M}(t_1^\star,t_2^\star)\cap\{r\leq r_0\}} \frac{1}{8M}|\hat{Y}\hat{Y}\psi|^2 - C\left(t_2^\star - t_1^\star\right)\left(E_0[\psi] + E_0[T\psi]\right) . \tag{177}$$

Combining the above leads us to the conclusion

$$\int_{\Sigma_{t_2^\star}} J_\mu^{(N)}[\hat{Y}\psi]n^\mu_\Sigma + \int_{\mathcal{M}(t_1^\star,t_2^\star)} K^N[\hat{Y}\psi] \leq \int_{\Sigma_{t_1^\star}} J_\mu^{(N)}[\hat{Y}\psi]n^\mu_\Sigma + C\left(t_2^\star - t_1^\star\right)\left(E_0[\psi] + E_0[T\psi]\right) ,$$

from which the boundedness statement follows as at the end of the proof of Theorem 5.3. □

With $\int_{\Sigma_{t^\star}} J_\mu^{(N)}[\hat{Y}\psi]n^\mu_{\Sigma_{t^\star}}$ (and after trivial commutation $\int_{\Sigma_{t^\star}} J_\mu^{(N)}[T\psi]n^\mu_{\Sigma_{t^\star}}$) both controlled by initial data, we only need to control second angular derivatives in order to apply Proposition 5.8 on the slices $\Sigma_{t^\star}$ to infer a pointwise bound for ψ.

To achieve this, we can read (169) as an equation on the spheres $S^2_{t^\star,r}$ and obtain the second angular derivatives via estimates on spheres (see Sheet 10) or use the direct elliptic argument below equation (170) which actually produces control on the second angular derivatives *non-degenerately* all the way to $r = 2M$. Note these elliptic estimates, unlike the commutation with angular momentum operators, do *not* depend on the spherical symmetry and are hence much more robust. Similarly the commutation with the redshift vectorfield outlined above is very robust and applies to any Killing horizon with positive surface gravity. In particular this argument can be adapted to the Kerr family of black holes. More on this later.

Finally, you should appreciate that using the above procedure we can clearly estimate derivatives of arbitrary high order by commuting sufficiently many times with the vectorfield $\hat{Y}$.

5.6 Integrated decay estimates

We now know how to prove uniform boundedness of solutions to the covariant wave equation on the black hole exterior. In this section, I would like to discuss

[33] Recall that if the proof of Proposition 5.4 works for some $r_0' > 2M$, then it works for all $2M < r_0 \leq r_0'$ with the same σ in (144).

two estimates regarding the *decay* of solutions. It turns out that proving decay is intimately tied to understanding the (null) geodesics flow on spacetime.

Theorem 5.14. *Solutions to (139) satisfy the degenerate estimate*[34]

$$\int_{\mathcal{M}(t_1^\star,t_2^\star)} \frac{1}{r^2}\left(1-\frac{3M}{r^2}\right)\left((\partial_{t^\star}\psi)^2+(\partial_r\psi)^2+|\nabla\!\!\!/\psi|^2\right) \leq C\cdot E^N[\psi](t_1^\star) \quad (178)$$

for any $t_2^\star \geq t_1^\star \geq 0$*. We also have the non-degenerate estimate*

$$\int_{\mathcal{M}(t_1^\star,t_2^\star)} \frac{1}{r^2}\left((\partial_{t^\star}\psi)^2+(\partial_r\psi)^2+|\nabla\!\!\!/\psi|^2\right) \leq C\left(E^N[\psi](t_1^\star)+E^N[T\psi](t_1^\star)\right).$$

Remark 5.15. *In the first estimate once can actually control the normal derivative to constant r (which is* $\left(1-\frac{2M}{r}\right)\partial_r - \frac{2M}{r}\partial_{t^\star}$ *in the* $(t^\star, r, \theta, \phi)$ *coordinates) at* $r = 3M$ without *any degeneration. Also the weights in* $\frac{1}{r}$ *are non optimal but (just as in the Minkowskian case) some degeneration near infinity is necessary for the estimate to hold.*

Of course, our main task when discussing Theorem 5.14 will be to explain the degeneration at $r = 3M$ and why it can be overcome by controlling an additional derivative on the data. The fact that *some* degeneration[35] at $r = 3M$ is necessary will be a corollary of the following Theorem which is due to Jan Sbierski (2013).

Theorem 5.16. *There exists a sequence of initial data for (139) with corresponding solutions* (ψ_n) *such that*

- $E_0^N[\psi_n] := E^N[\psi_n](t^\star = 0) = 1$ *for all* $n \in \mathbb{N}$.
- *There exists a constant* $b > 0$ *(independent of* n*) such that given any time* $T^\star > 0$ *there exists an* n *such that*

$$\int_{\Sigma_{t^\star}\cap\{|r-3M|<\frac{1}{10}M\}} \left((\partial_{t^\star}\psi)^2+(\partial_r\psi)^2+|\nabla\!\!\!/\psi|^2\right) \geq bE_0^N[\psi_n] = b\,.$$

holds for any $t^\star \in [0, T^\star]$.

In other words, if you give me a time $T^\star$ (large) then I can give you a solution with initial energy equal to 1 such that a uniform fraction of the initial energy remains in a uniform neighbourhood of $r = 3M$ for up to time $T^\star$. In yet other words, the energy can concentrate near $r = 3M$ for arbitrarily long times!

It is not hard to see that Theorem 5.16 implies that the second estimate of Theorem 5.14 cannot hold without the control on the second derivatives on the right. Indeed, suppose it was true that the estimate

$$\int_{\mathcal{M}(t_1^\star,t_2^\star)} \frac{1}{r^2}\left((\partial_{t^\star}\psi)^2+(\partial_r\psi)^2+|\nabla\!\!\!/\psi|^2\right) \leq CE^N[\psi](t_1^\star)\,.$$

[34]Previously, we have been using the word "degenerate" for degeneration of estimate at the horizon, i.e. at $r = 2M$. Here the degeneration happens at a different location, $r = 3M$.

[35]Using microlocal techniques the degeneration can be reduced to a logarithmic loss.

held for a uniform constant C and all solution to (139). Then, using a dyadic decomposition, i.e. defining $t^\star_0 = 1$ and $t^\star_{i+1} = 2t^\star_i = 2^i$ (note in particular $t^\star_{i+1} - t^\star_i = t_i$, i.e. the size of each slab is of the size of the $t^\star$ coordinate the slab is at), we see that from

$$\int_{t^\star_i}^{t^\star_{i+1}} dt^\star \int_{\Sigma_{t^\star}} \frac{1}{r^2}\left((\partial_{t^\star}\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right) \leq CE^N[\psi](t^\star_1) \leq CE^N_0[\psi]\,,$$

which is valid for all $i \in \mathbb{N}$ and where the second inequality follows from the boundedness statement, we infer (how?) the existence of a sequence of times $t^\star_i \leq \tilde{t}^\star_i \leq t^\star_{i+1}$ such that

$$\int_{\Sigma_{\tilde{t}^\star}} \frac{1}{r^2}\left((\partial_{t^\star}\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right) \leq \frac{C}{t^\star_i}E^N_0[\psi]$$

Applying this to the (ψ_n) of Sbierski's theorem we see that given any $\epsilon > 0$ there is a time $T^\star$ such that for any $n \in \mathbb{N}$ there is a slice $\tilde{T}^\star_n \in [\frac{T^\star}{2}, T^\star]$ (the actual slice may depending on n) such that

$$\int_{\Sigma_{\tilde{T}^\star_n}} \frac{1}{r^2}\left((\partial_{t^\star}\psi_n)^2 + (\partial_r\psi_n)^2 + |\nabla\!\!\!/\psi_n|^2\right) \leq \epsilon$$

Since this holds for all $n \in \mathbb{N}$ this violates Sbierski's theorem which promises that there is an $n \in \mathbb{N}$ such that

$$\int_{\Sigma_{\tilde{T}^\star} \cap |r-3M| \leq \frac{1}{10}M\}} \frac{1}{r^2}\left((\partial_{t^\star}\psi_n)^2 + (\partial_r\psi_n)^2 + |\nabla\!\!\!/\psi_n|^2\right) \geq b$$

holds for all $\tilde{T}^\star \in [0, T^\star]$.

I will now try to sketch for you the proofs of Theorems 5.14 and 5.16.

5.6.1 Sketch of the proof of Theorem 5.14

The basic idea to prove Theorem 5.14 is to exploit the divergence identity generated by a suitably chosen vectorfield X, which in standard Schwarzschild (t, r, θ, ϕ) or so-called Regge-Wheeler coordinates[36] $(t, r^\star, \theta, \phi)$ takes the form:

$$X = f(r)\partial_r = f(r)\partial_{r^\star}$$

for an appropriate smooth bounded function f. Denoting by a prime a derivative with respect to $r^\star$ and writing all terms in Regge-Wheeler coordinates we find for the bulk term (exercise)

$$K^X[\psi] = \frac{f'}{1-\frac{2M}{r}}\left(\partial_{r^\star}\psi\right)^2 + \frac{f}{r}\left(1 - \frac{3M}{r}\right)|\nabla\!\!\!/\psi|^2 - \frac{1}{4}\left(2f' + 4\frac{1-\frac{2M}{r}}{r}f\right)\nabla^\alpha\psi\nabla_\alpha\psi\,. \tag{179}$$

[36] These are related to the standard Schwarzschild coordinates by a rescaling for r, i.e. $r^\star = r + 2M\log(r-2M) + c$. They are valid on the exterior region I excluding the horizon and behave like $r^\star \to -\infty$ as the horizon is approached. The metric takes the form $g = -\left(1 - \frac{2M}{r}\right)(dt^2 - (dr^\star)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ which is computationally very convenient.

We combine this with the following divergence identity for a smooth function h (compare with (135))

$$\nabla^\mu J^{aux,h}_\mu := \nabla^\mu \left(\psi \nabla_\mu \psi h - \frac{1}{2}\psi^2 \nabla_\mu h\right) = h \nabla^\alpha \psi \nabla_\alpha \psi - \frac{1}{2}\psi^2 \Box_g h =: K^{aux,h}[\psi]$$

We now choose $h = f' + \frac{2}{r}\left(1-\frac{2M}{r}\right) f + \frac{\delta}{r^4}\left(1-\frac{2M}{r}\right)\left(1-\frac{3M}{r}\right) f$ and consider the divergence identity

$$K^{X,h}[\psi] := K^X[\psi] + K^{aux,h}[\psi] = \nabla^\mu \left(T_{\mu\nu}X^\nu + J^{aux,h}_\mu\right) . \tag{180}$$

We see that (try to understand this first with $\delta = 0$!)

$$\begin{aligned} K^{X,h}[\psi] :=& \left(\frac{f'}{1-\frac{2M}{r}} - \frac{\delta f}{2r^4}\left(1-\frac{3M}{r}\right)\right)(\partial_{r^\star}\psi)^2 \\ &+ \frac{f}{r}\left(1-\frac{3M}{r}\right)\left(\left(1-\frac{\delta\left(1-\frac{2M}{r}\right)}{2r^4}\right)|\nabla\!\!\!\!/\,\psi|^2 + \frac{\delta}{2r^3}(\partial_t\psi)^2\right) \\ &- \left(\frac{1}{2}\Box_g\left(f' + 2\frac{1-\frac{2M}{r}}{r} f + \dots\right)\right)\psi^2 \end{aligned} \tag{181}$$

We now see that if we choose f to be monotonically increasing and to change sign at $3M$ then all derivative terms will be non-negative, provided we also choose δ sufficiently small (depending only on M and the exact choice of f). Moreover, the function f should also be uniformly bounded (with appropriate decay in $\frac{1}{r}$ for derivatives) in order for the boundary terms in the divergence identity to be controlled from the N-energy (here even the T-energy is sufficient), which is uniformly bounded by data. Clearly, lots of such f exists! The point now is that one can choose such an f such that also the term in the last line becomes non-negative! See the lecture notes of Dafermos-Rodnianski for an explicit f that works. This (modulo non-optimal weights near infinity and the horizon) gives the first estimate of Theorem 5.14. Improving the estimates near the horizon (using the redshift) and near infinity is relatively straightforward.

Remark 5.17. *It is clear that the biggest worry here is the zeroth order term in ψ, i.e. the low frequencies. If we use the spherical symmetry of the background, we can convince ourselves that at least for sufficiently high angular modes we can, by borrowing from the $|\nabla\!\!\!\!/\,\psi|^2$-term, produce a non-negative zeroth order term (after integration over the spheres) as follows.*

We can decompose the solution into spherical harmonics

$$\psi = \sum_{\ell \geq 0, |m| \leq \ell} \psi_{\ell,m}(t,r) Y^\ell_m(\theta,\phi)$$

where $\psi_{\ell,m}(t,r) = \int_{S^2} \sin\theta d\theta d\phi Y^\ell_m(\theta,\phi)\psi(t,r,\theta,\phi)$. Each $\psi_{\ell,m}$ satisfies the wave equation (why?) and the convergence of the sum is in L^2 on the spheres. For fixed L we can then decompose the solution into

$$\psi = \psi_{\ell \leq L} + \psi_{\ell \geq L}$$

i.e. into parts with angular momentum larger and smaller than some fixed L. For $\psi_{\ell\geq L}$ we have

$$\int_{S^2} |\nabla\!\!\!/ \psi_{\geq L}|^2 \geq \frac{L(L+1)}{r^2} \int_{S^2} |\psi_{\geq L}|^2 \,.$$

Therefore we can use the second line of (181) to produce a (large if L is large) $|\psi|^2$ term which will make establishing positivity of the $|\psi|^2$-term relatively simply (it essentially reduces the problem to choosing f such that the expression in the last line of (181) is non-negative at $r = 3M$. Of course, for finitely many modes one still needs a (separate) argument!

We turn briefly to the second estimate of Theorem 5.14. For this we remark (without proof) that the previous argument gives the following quantitative estimate

$$\int_0^{t^\star} d\bar{t}^\star \int_{\Sigma_{\bar{t}^\star}} \frac{|\psi|^2}{r^4} \leq E_0^N[\psi] \,. \tag{182}$$

We first remark that if we are willing to add also $\sum_{i=1}^N E_0^N[\Omega_i\psi]$ on the right hand side of the second estimate (that is in addition to $E_0^N[\psi] + E_0^N[T\psi]$), then the statement follows immediately from (182) and the fact that T, Ω_i commute trivially for $i = 1, 2, 3$ and the fact that (as promised in Remark 5.15 and seen in the proof) the $\partial_{r^\star}$-derivative is controlled non-degenerately at $r = 3M$ from the first estimate.

The claim that commuting with the T-derivative is actually sufficient is a consequence of the Lagrangian multiplier identity (180). Indeed, observe that we only need to control the T and the angular derivatives non-degenerately in a neighbourhood of $r = 3M$. For the T derivative this follows from the commuted (182) and for the other derivatives from applying (180) with h a bump-function near $r = 3M$ (why?).

5.6.2 Sketch of the proof of Theorem 5.16

We already proved on Sheet 5 the existence of affine complete null geodesics on the Schwarzschild manifold which remain tangent to the timelike hypersurface $r = 3M$ for all times. Moreover, we also showed that this phenomenon is unstable in a suitable sense. If you haven't done the exercise back then you can check now directly that

$$\gamma(s) = \left(t = s, r = 3M, \theta = \frac{\pi}{2}, \phi = \frac{1}{3\sqrt{3}M} s \right)$$

is such a "trapped" null geodesic.

The idea to prove Theorem 5.16 is to consider a high frequency (i.e. highly oscillatory) ansatz

$$\psi_\lambda = e^{i\lambda\phi(x)} a(x) \tag{183}$$

where $\phi : \mathcal{M} \to \mathbb{R}$ and $a : \mathcal{M} \to \mathbb{R}$ are smooth functions and where a will eventually be compactly supported in a neighbourhood of the trapped geodesic γ. We expect that in the high frequency approximation, solutions to the wave equation propagate along null geodesics (cf. the approximation of light rays in geometric optics). Now, for (183) to be a solution to $\Box_g \psi_\lambda = 0$ we need

$$0 = e^{i\lambda\phi(x)}\Big(\lambda^2 \cdot a(x) \cdot g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi + \lambda\left(2g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu a + a\Box_g\phi\right) + \Box_g a\Big).$$

If we can construct ϕ and a such that both

- $g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi = 0$ (eikonal equation) and
- $2g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu a + a\Box_g\phi = 0$ (transport equation for a)

hold, then we would have that ψ_λ satisfies the wave equation up to $O(1)$ in λ and in particular the estimate

$$\|\Box_g\psi_\lambda\|_{L^2(\mathcal{R}[0,T])} \leq C_T \tag{184}$$

for C_T a constant independent on λ (but dependent on T) would hold. Here $\mathcal{R}[0,T] = [0,T] \times \mathcal{A}$ where $\mathcal{A}$ denotes a small annular region around $r = 3M$ where the function a is (going to be) supported.

We now normalise the approximate solution ψ_λ to have initial energy 1 independently of λ by considering

$$\tilde{\psi}_\lambda = \frac{\psi_\lambda}{\sqrt{E_0^T[\psi_\lambda]}} \tag{185}$$

where $E_0^T[\psi_\lambda]$ denote the T-energy at time $t^\star = 0$ of the solution ψ_λ,[37] which is easily seen to satisfy $E_0^T[\psi_\lambda] \sim \lambda^2$. Therefore (184) becomes

$$\|\Box_g\tilde{\psi}_\lambda\|_{L^2(\mathcal{R}[0,T])} \leq \frac{C_T}{\lambda} \to 0 \tag{186}$$

as $\lambda \to \infty$ for fixed T. We comment briefly on solving the eikonal equation and the transport equation.

Step 1. To solve $g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi = 0$, we recognise it as a first order PDE

$$F\left(x^\mu, \not{\phi}, \partial_{x^\mu}\phi = p_\mu\right) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu = 0\,.$$

The characteristic ODEs are

$$\frac{dx^\mu}{ds} = g^{\mu\nu}p_\mu \tag{187}$$

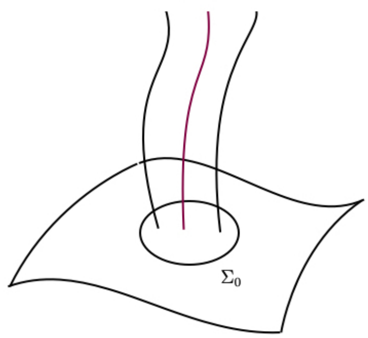
$$\frac{d\phi}{ds} = 0 \tag{188}$$

$$\frac{dp_\tau}{ds} = -\frac{1}{2}\partial_\tau(g^{\mu\nu})p_\mu p_\nu \tag{189}$$

[37]Note that the solution is supported near $3M$ and hence using the T energy is appropriate here. In particular, the T-energy is bounded above and below by the N-energy with constants depending only on M.

In other words, ϕ is constant along the characteristic curves (this is (188)), which are in turn nothing but the geodesic flow on the tangent bundle TM (this is (187) and (189)).[38]

In other words, we can specify ϕ along Σ_0 in a neighbourhood of $r = 3M$ and then ϕ will be constant along the null geodesics emanating from Σ_0 (which are the characteristic curves with tangent proportional to the null vector $\nabla\phi$).



Locally in time a solution indeed exists but globally there could be caustics, i.e. null geodesics with different ϕ initial values for ϕ could intersect and the solution would cease to exist globally. Unfortunately, this is what happens here and the construction above needs more care (either with so-called Gaussian beams (see Sheet 11) or one needs to bridge the caustics in an appropriate manner. We will not provide the details here and pretend that we can construct a ϕ globally.

Step 2. With ϕ constructed we can solve the linear transport equation for a along the integral curves of $\nabla\phi$ (which are the null geodesics). Indeed we need to solve

$$2\frac{d}{ds}a + a\Box\phi = 0 \tag{190}$$

which we can view as ODEs along each null geodesics. In particular, if $a = 0$ initially, then $a = 0$ along the entire integral curves. Therefore, if $a = 0$ initially, then $a = 0$ along the integral curves. It follows that (again pretending no caustics for the spray of null geodesics) we can specify a of compact support on Σ_0 such that a remains compactly supported in a small tubular neighbourhood of the geodesic γ up to time T. (Draw again a picture!).

Note finally that the approximate solution $\tilde{\psi}_\lambda$ thus constructed has the property that given $T^\star$ and $\epsilon > 0$ we can find a λ such that for all $\lambda > \Lambda$ and all

[38] Indeed, you should be able to derive the geodesic equation from (187) and (189) establishing that the projections of the characteristic curves to the manifold are indeed (null) geodesics. This is the connection with geometric optics alluded to above.

$t^\star \in [0,T]$ we have

$$E^T[\tilde{\psi}_\lambda](t^\star) \geq E^T[\tilde{\psi}_\lambda](t^\star = 0) - \epsilon\,. \tag{191}$$

This is easily seen by applying the T-energy estimate for $\Box\tilde{\psi}_\lambda = \Box\tilde{\psi}_\lambda$ (sic!) in the region $\mathcal{R}[0,T^\star]$ and using (186) for the right hand side. In other words, the approximate solution keeps the energy concentrated near $3M$ up to time $T^\star$ if we choose λ sufficiently large.

Step 3. We now construct from the approximate solution $\tilde{\psi}_\lambda$ which concentrates the energy as desired an actual solution $\overline{\psi}_\lambda$ to the wave equation which still concentrates the energy (with a small error). This is achieved by solving the Cauchy problem

$$\begin{aligned} \Box\overline{\psi}_\lambda &= 0\,, \\ \overline{\psi}_\lambda|_{\Sigma_0} &= \tilde{\psi}_\lambda|_{\Sigma_0}\,, \\ (n_{\Sigma_0}\overline{\psi}_\lambda)|_{\Sigma_0} &= (n_{\Sigma_0}\tilde{\psi}_\lambda)|_{\Sigma_0}\,. \end{aligned} \tag{192}$$

Note that $(\overline{\psi}_\lambda - \tilde{\psi}_\lambda$ then satisfies $\Box(\overline{\psi}_\lambda - \tilde{\psi}_\lambda) = -\Box\tilde{\psi}_\lambda$ with trivial data on Σ_0. We can now apply the T-energy estimate in a region enclosed by Σ_0, the slice $\Sigma_{T^\star} \cap \{|r-3M| \leq \frac{1}{10}M\}$ and the past light cones of the spheres $r = 3M \pm \frac{1}{10}M$ (draw a picture in the Penrose diagram!). This yields (with the obvious notation) for any $\overline{T}^\star \in [0,T^\star]$

$$E^T_{|r-3M|\leq\frac{1}{10}M}[(\overline{\psi}_\lambda - \tilde{\psi}_\lambda](\overline{T}^\star) \leq 0 + \int_0^{\overline{T}^\star} dt^\star \int_{\Sigma_{t^\star}} |\Box\tilde{\psi}_\lambda|\; |T\left((\overline{\psi}_\lambda - \tilde{\psi}_\lambda\right)|r^2 dr \sin\theta d\theta d\phi$$

Noting that the integrand is supported near $r = 3M$ (i.e. contained in $|r-3M| \leq \frac{1}{10}M$) we deduce from Cauchy-Schwarz in spacetime and (186) the estimate

$$E^T_{|r-3M|\leq\frac{1}{10}M}[(\overline{\psi}_\lambda - \tilde{\psi}_\lambda](\overline{T}^\star) \leq \frac{C_{\overline{T}^\star}}{\lambda} \sup_{t^\star\in[0,\overline{T}^\star]} \sqrt{E^T_{|r-3M|\leq\frac{1}{10}M}[(\overline{\psi}_\lambda - \tilde{\psi}_\lambda](t^\star)}\,,$$

again valid for any $\overline{T}^\star \in [0,T^\star]$. We conclude

$$\sup_{t^\star\in[0,T^\star]} E^T_{|r-3M|\leq\frac{1}{10}M}[(\overline{\psi}_\lambda - \tilde{\psi}_\lambda](t^\star) \leq \frac{C_{T^\star}}{\lambda^2}\,.$$

Therefore given a time $T^\star$ (large) we can choose λ such that the localised (near $r = 3M$) energy of $(\overline{\psi}_\lambda - \tilde{\psi}_\lambda)$ is as small as we like for all times up to $T^\star$. Combining this statement with (191) and the reverse triangle inequality,

$$\sqrt{E^T_{|r-3M|\leq\frac{1}{10}M}[\overline{\psi}_\lambda](t^\star)} \geq \sqrt{E^T_{|r-3M|\leq\frac{1}{10}M}[\tilde{\psi}_\lambda](t^\star)} - \sqrt{E^T_{|r-3M|\leq\frac{1}{10}M}[\overline{\psi}_\lambda - \tilde{\psi}_\lambda](t^\star)}\,,$$

we infer that given $T^\star$ we can find a Λ such that for all $\lambda > \Lambda$ we have

$$\sqrt{E^T_{|r-3M|\leq\frac{1}{10}M}[\overline{\psi}_\lambda](t^\star)} \geq \frac{1}{2}$$

for all $t^\star \in [0,T^\star]$ which proves the claim since all the $\overline{\psi}_\lambda$ have initial energy equal to 1 independently of λ (since the $\tilde{\psi}_\lambda$ have by construction).

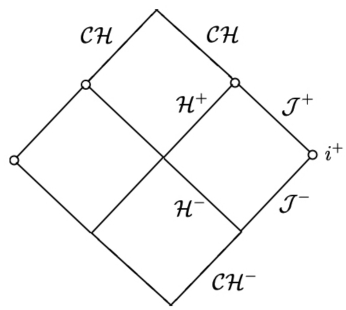


Figure 1: Penrose diagram of subextremal Reissner-Nordstroem

5.7 Extremal black holes and the Aretakis instability

We consider the so-called Reissner-Nordstroem metric, whose line element in "Schwarzschild coordinates" reads

$$g_{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{193}$$

where $|Q| \leq M$ is the charge. The metric g_{RN} is not Ricci flat but instead solves the Einstein-Maxwell equations, which are the Einstein equations with energy momentum tensor given by Maxwell's theory of electromagnetism. Defining

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

we see that for $|Q| < M$ (the sub-extremal case) the expression $D(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}$ has two distinct roots, while for $|Q| = M$, (the so-called extremal case), we have $r_+ = r_- = M$. We can draw the Penrose diagram of the above spacetime which can be "derived" by similar argument that we employed in the Schwarzschild case.

In particular, the analogue of the ingoing Eddington-Finkelstein coordinates is

$$g_{RN} = -D(r)dv^2 + 2dvdr + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

defined on the manifold $(0,\infty)_v \times (0,\infty)_r \times S^2$ which in particular covers the event horizon $\mathcal{H}^+$ at $r = r_+$.

Note that the vectorfield $V = \partial_v$ is Killing and a null generator of the event horizon $\mathcal{H}^+$. Therefore, the surface gravity of the horizon is computed to be

$$\kappa = (\nabla_V V)^v \Big|_{r=r_+} = \Gamma^v{}_{vv}\Big|_{r=r_+} = -\frac{1}{2} g^{vr} \partial_r g_{vv}\Big|_{r=r_+} = \frac{1}{2}\frac{r_+ - r_-}{r_+^2} \tag{194}$$

and hence vanishes exactly in the extremal case. In the sub-extremal case the surface gravity is positive and we could prove the same results for solutions to the covariant wave equation on the black hole exterior than we did for $Q = 0$!

In the extremal case, however, there is an elementary argument showing that certain derivatives of ψ satisfying $\Box_{g_{RN}} \psi = 0$ cannot decay along the event horizon. The instability mechanism is due to Stefanos Aretakis and has been generalised to many other "extremal" black holes. The idea is to write the covariant wave equation $\Box_{g_{RN}} \psi = 0$ as

$$D\partial_r\partial_r\psi + 2\partial_v\partial_r\psi + \frac{2}{r}\partial_v\psi + \left(\partial_r D + \frac{2}{r}D\right)\partial_r\psi + \not{\Delta}\psi = 0 \tag{195}$$

and restrict it to the event horizon where it becomes (since in the extremal case $D = 0$ *and* $\partial_r D = 0$ at $r = r_+$):

$$2\partial_v\partial_r\psi + \frac{2}{r}\partial_v\psi = -\not{\Delta}\psi \quad \text{on } \mathcal{H}^+. \tag{196}$$

It follows that the quantity

$$\int_{S^2} \sin\theta d\theta d\phi \left(\partial_r\psi + \frac{1}{M}\psi\right)$$

is constant along $\mathcal{H}^+$ and hence does not decay along $\mathcal{H}^+$ (unless it is initially zero, which is a non-generic assumption on initial data.). It turns out that in fact ψ itself does decay along $\mathcal{H}^+$ so the quantity which does not decay is (the spherical average of) $\partial_r\psi$. A similar argument (exericse) shows blow-up for second transversal derivatives if suitable assumptions on the horizon are being made.

5.8 From integrated decay to inverse polynomial decay

We now discuss briefly how to go from the integrated local energy decay estimates of Theorem 5.14 to inverse polynomial decay rates for the energy (and for the field ψ itself). Such decay rates are not only a perhaps more intuitive measure of decay but also crucial for applications to non-linear problems. This section follows very closely the (very readable and short) paper of Dafermos–Rodnianski ("A new physical space approach to decay for the wave equation", arXiv:0910.4957).

5.8.1 Slices ending at null infinity

To state the results we first need to re-consider the choice of slices that we should use to measure the decay. Indeed, starting with Minkowski space (where an integrated decay estimate for solutions to the wave equation was proven on Sheet 9) , it is clear that the energy through slices of constant time t cannot decay because it is conserved! To capture decay we need to choose slices "ending at null infinity" as seen in the picture. A simple choice in Minkowski space is

$$\Sigma_\tau := (\{t = \tau\} \cap \{r \leq R\}) \cap (\{u := t - r = \tau - R\} \cap \{r \geq R\}) \ ,$$

which is a constant $t=\tau$ slice up to $r=R$ with an outgoing null hypersurface emanating from the sphere $S^2_{\tau,R}$ attached (see the picture).

On Schwarzschild, recall the Regge-Wheeler coordinates $(t, r^\star, \theta, \phi)$ and the double null coordinates $(u = t-r, v = t+r, \theta, \phi)$ (in which the metric takes the form $g_S = -4\left(1-\frac{2M}{r}\right) dudv + r^2 d\omega^2$). We may define a coordinate (compare Question 2 on Sheet 5)

$$t^\star = t + \chi(r) \cdot 2M \log(r-2M)$$

where $\chi(r)$ is a smooth cut-off function equal to 1 for $r \leq 7M$ and equal to zero for $r \geq 8M$. Then we define the slices

$$\Sigma_\tau := \left(\{t^\star = \tau\} \cap \{r \leq R\}\right) \cap \left(\{u := t - r^\star = \tau - R^\star\} \cap \{r \geq R\}\right) .$$

These slices intersect the horizon and are spacelike in the region $r \leq R$ with a piece of outgoing null hypersurface emanating from the sphere $S^2_{t,R}$ attached.

We define the following energy for solutions of $\Box_g \psi = 0$:

$$E[\psi](\tau) := \int_{\Sigma_\tau} J^N_\mu n^\mu_{\Sigma_\tau} = \int_{\Sigma_\tau \cap \{r \leq R\}} J^N_\mu n^\mu_{\Sigma_\tau} + \int_{\mathcal{N}_\tau} J^N_\mu n^\mu_{\Sigma_\tau} , \tag{197}$$

where in Minkowski $N = T = \partial_t$ while in Schwarzschild N is the familiar timelike vectorfield generating the non-degenerate energy. Note that on the null piece $\mathcal{N}_\tau := \{u = \tau - R^{(\star)}\} \cap \{r \geq R\}$, the expression $J^N_\mu n^\mu_{\Sigma_\tau} \sim (\partial_v \psi)^2 + |\nabla\!\!\!/ \psi|^2$ involves only derivatives *tangential* to the cone (as the normal is tangential to the null cone). The choice of slices is by no means unique. For instance, you can find smooth and purely spacelike slices approximating the slices above:

Exercise 5.18. *Find an explicit parametrisation of smooth spacelike slices $\tilde{\Sigma}_\tau$ ending at null infinity (as indicated in the picture) in both the Minkowski and the Schwarzschild case.*

5.8.2 Boundedness and integrated decay for the new slices

One now easily (re)proves that the energy boundedness and integrated local energy decay estimates we proved for slices of constant t (or $t^\star$ in the Schwarzschild case) generalise to the slices Σ_τ:

Proposition 5.19. *For ψ a solution of $\Box_\eta \psi = 0$ arising from data of compact support one has the estimates*

$$E[\psi](\tau_2) \leq E[\psi](\tau_1) \tag{198}$$

and

$$\int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \frac{1}{(1+r^2)^2} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \right] \leq C E[\psi](\tau_1) \tag{199}$$

for $\tau_2 \geq \tau_1 \geq 0$. Similarly, for ψ a solution of $\Box_{g_S}\psi = 0$ arising from data of compact support one has the estimates

$$E[\psi](\tau_2) \leq C \cdot E[\psi](\tau_1) \tag{200}$$

and

$$\int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \frac{1}{(1+r^2)^2}\left(1-\frac{3M}{r}\right)\left[(\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right] \leq CE[\psi](\tau_1)$$

$$\int_{\tau_1}^{\tau_2} d\tau \int_{\Sigma_\tau} \frac{1}{(1+r^2)^2}\left[(\partial_t\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2\right] \leq C\sum_{i=0}^{1} E[T^i\psi](\tau_1)$$

Proof. See Sheet 11. □

5.8.3 Proving polynomial decay in the Minkowski case

We start with the case of Minkowski space and prove:

Theorem 5.20. *Let ψ be a solution of $\Box_\eta\psi = 0$ arising from data of compact support. Then there exists a constant C depending only on R and M such that the following holds (where $N = \partial_t$)*

$$\int_{\Sigma_\tau} J^N_\mu[\psi] n^\mu_{\Sigma_\tau} \leq \frac{C}{\tau^2}\int_{\Sigma_{\tau=0}} (1+r^2) J^N_\mu[\psi] n^\mu_{\Sigma} \tag{201}$$

Proof. We write the wave equation in null coordinates $u = t-r$, $v = t+r$ (which imply $\partial_u = \frac{1}{2}\partial_t - \frac{1}{2}\partial_r$, $\partial_v = \frac{1}{2}\partial_t + \frac{1}{2}\partial_r$ and hence $\partial_u r = -\frac{1}{2}$, $\partial_v r = \frac{1}{2}$) and for $\phi = \psi r$ as

$$-4\partial_u\partial_v\phi + \frac{1}{r^2}\mathring{\Delta\!\!\!/}\phi = 0 \tag{202}$$

where $\mathring{\Delta\!\!\!/}$ denotes the covariant Laplacian on the *unit* sphere (with metric $\mathring{g\!\!\!/} = d\theta^2 + \sin^2\theta d\varphi^2$).

Step 1. Deriving a p-weighted identity. Multiplying (202) by $-\frac{1}{2}r^p\partial_v\phi$ yields the identity

$$\begin{aligned}&\partial_u\left(r^p(\partial_v\phi)^2\right) + \frac{1}{2}pr^{p-1}(\partial_v\phi)^2 + \mathring{\nabla\!\!\!/}^A\left(-\frac{1}{2}r^{p-2}\partial_v\phi\mathring{\nabla\!\!\!/}_A\phi\right)\\ &+\frac{1}{4}\partial_v\left(r^{p-2}\mathring{\nabla\!\!\!/}^A\phi\mathring{\nabla\!\!\!/}_A\phi\right) + \frac{1}{8}(2-p)r^{p-3}\mathring{\nabla\!\!\!/}^A\phi\mathring{\nabla\!\!\!/}_A\phi = 0\,.\end{aligned} \tag{203}$$

Integrating over the region $\mathcal{D}^{\tau_2}_{\tau_1} = \bigcup_{\tau_1\leq\tau\leq\tau_2}(\Sigma_\tau \cap \{r \geq R\}) = \bigcup_{\tau_1\leq\tau\leq\tau_2}\mathcal{N}_\tau$ with respect to the measure $dudv\sin\theta d\theta d\varphi$ we see that the term involving the angular divergence on the unit sphere will vanish and we will produce for any $p \leq 2$ the identity

$$\int_{u=\tau_2-R, v\geq \tau_2+R} r^p (\partial_v \phi)^2 \sin\theta d\theta d\varphi dv + \frac{1}{4}\int_{\mathcal{I}^{\tau_2-R}_{\tau_1-R}} r^p |\nabla\!\!\!/ \psi|^2 \sin\theta d\theta d\varphi du$$
$$+ \int_{\mathcal{D}^{\tau_2}_{\tau_1}} r^{p-1}\left(\frac{1}{2}p(\partial_v\phi)^2 + \frac{1}{8}(2-p)|\nabla\!\!\!/ \phi|^2\right)\sin\theta d\theta d\varphi du dv$$
$$\leq \int_{u=\tau_1-R, v\geq \tau_1+R} r^p(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv$$
$$+ \int_{\tau_1}^{\tau_2} r^p\left(\frac{1}{4}|\nabla\!\!\!/ \phi|^2 - (\partial_v\phi)^2\right)\sin\theta d\theta d\varphi d\tau\Big|_{r=R} \tag{204}$$

Note that the left hand side is non-negative for $p \leq 2$!

Remark 5.21. *We derived this identity renormalising the wave equation and integrating by parts "by hand". However, you can of course also obtain the identity (204) using the familiar vectorfield currents J^X_μ and $K^X[\psi]$ combined with the Lagrangian identity for an appropriate vectorfield X and a Lagrangian function. This is a computational exercise.*

Step 2. Stepping down the hierarchy and the dyadic argument. We first claim that for any $p \leq 2$

$$\int_{\tau_1}^{\tau_2} r^p\left(\frac{1}{4}|\nabla\!\!\!/ \phi|^2 - (\partial_v\phi)^2\right)\sin\theta d\theta d\varphi d\tau\Big|_{r=R} \leq C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi]n^\mu_\Sigma \tag{205}$$

with C depending on R only. This will be proven on Sheet 11.

Therefore, applying (204) with $p = 2$ produces the estimate

$$\int_{u=\tau_2-R, v\geq\tau_2+R} r^2(\partial_v\phi)^2\sin\theta d\theta d\varphi dv + \int_{\mathcal{D}^{\tau_2}_{\tau_1}} r(\partial_v\phi)^2 \sin\theta d\theta d\varphi du dv$$
$$\leq \int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2\sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi]n^\mu_\Sigma\,. \tag{206}$$

Consider now a dyadic sequence of times $\overline{\tau}_n \to \infty$ (for instance $\overline{\tau}_n = 2^n\tau_0$ with $\tau_0 \geq 1$). Applying (206) in each region between $\overline{\tau}_{n-1}$ and $\overline{\tau}_n$ we can extract (how?) a dyadic sequence[39] of times $\tau_n \to \infty$ such that

$$\int_{u=\tau_n-R, v\geq\tau_n+R} r(\partial_v\phi)^2\sin\theta d\theta d\varphi dv$$
$$\leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2\sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi]n^\mu_\Sigma\right]. \tag{207}$$

[39]for instance, this can be arranged to satisfy $\frac{1}{4}\tau_n \leq \tau_n - \tau_{n-1} \leq \frac{7}{8}\tau_n$ by choosing every other good slice.

The idea, now is to apply the identity (204) now with $p = 1$ and in between two good slices, i.e. the region $\mathcal{D}^{\tau_n}_{\tau_{n-1}}$. Note that the boundary terms on the outgoing null cones can now be estimated by (216). This yields using again also (205) the estimate

$$\int_{\mathcal{D}^{\tau_n}_{\tau_{n-1}}} \left(\frac{1}{2}(\partial_v \phi)^2 + \frac{1}{8}|\nabla\!\!\!/ \phi|^2\right) \sin\theta d\theta d\varphi du dv \tag{208}$$
$$\leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\right] + C\int_{\Sigma_{\tau_{n-1}}} J^T_\mu[\psi] n^\mu_\Sigma.$$

We claim that we can convert ϕ into ψ, i.e. insert $\phi = \psi r$, integrate by parts and deduce

$$\int_{\mathcal{D}^{\tau_n}_{\tau_{n-1}}} \left(\frac{1}{2}(\partial_v \psi)^2 + \frac{1}{8}|\nabla\!\!\!/ \psi|^2\right) r^2 \sin\theta d\theta d\varphi du dv \tag{209}$$
$$\leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\right] + C\int_{\Sigma_{\tau_{n-1}}} J^T_\mu[\psi] n^\mu_\Sigma.$$

This will be proven in detail on Sheet 11. Note that now it is indeed the characteristic energy of ψ on the truncated outgoing null cones integrated in u which appear as the first term! Adding to this (199) we deduce

$$\int_{\tau_{n-1}}^{\tau_n} d\tau \int_{\Sigma_\tau} J^T_\mu[\psi] n^\mu_\Sigma \tag{210}$$
$$\leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\right] + C\int_{\Sigma_{\tau_{n-1}}} J^T_\mu[\psi] n^\mu_\Sigma.$$

Note that we could replace τ_{n-1} by τ_0 in the last term by energy conservation. We can now conclude the argument. In every interval $[\tau_{n-1}, \tau_n]$ we find a $\tilde{\tau}_n$ such that

$$\int_{\Sigma_{\tilde{\tau}_n}} J^T_\mu[\psi] n^\mu_\Sigma \leq \frac{C}{\tilde{\tau}_n}\left[\int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\right]$$

obtaining thus a dyadic sequence $\tilde{\tau}_n \to \infty$ along which the energy decays like $\frac{1}{\tau}$. Using (198) we immediately deduce the estimate for any slice:

$$\int_{\Sigma_\tau} J^T_\mu[\psi] n^\mu_\Sigma \leq \frac{C}{\tau}\left[\int_{u=\tau_1-R, v\geq\tau_1+R} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\right]. \tag{211}$$

Indeed, given any slice Σ_τ we take the slice $\Sigma_{\tilde{\tau}_n}$ closest to Σ_τ in the past and apply (198) between the two slices. This yields (211) with τ replaced by $\tilde{\tau}_n$ on the right hand side. However, $\tilde{\tau}_n \geq \lambda\tau$ holds for some $\lambda > 0$ (for instance $\lambda = \frac{1}{16}$ suffices) by the dyadic property.

With (211) established, we can repeat the argument one more time: From (210) we find yet another dyadic sequence along which

$$\int_{\Sigma_{\hat{\tau}_n}} J^T_\mu[\psi] n^\mu_\Sigma \leq \frac{C}{(\hat{\tau}_n)^2} \left[\int_{u=\tau_1 - R, v \geq \tau_1 + R} r^2 (\partial_v \phi)^2 \sin\theta d\theta d\varphi dv + C \int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma \right]$$

holds. Repeating the energy conservation argument (and converting ϕ into ψ for the term on the outgoing null hypersurface, we finally establish (201). Note that one cannot improve the decay further (why?) and that the rate of τ^{-2} is obviously related to the fact that we could only choose $p = 2$ at the top of the hierarchy. □

5.8.4 Proving polynomial decay in the Schwarzschild case

Theorem 5.22. *Let ψ be a solution of $\Box_{g_S} \psi = 0$ arising from data of compact support. Then there exists a constant C depending only on R and M such that the following holds (where N is the timelike vectorfield constructed earlier)*

$$\int_{\Sigma_\tau} J^N_\mu[\psi] n^\mu_{\Sigma_\tau} \leq \frac{C}{\tau^2} \int_{\Sigma_{\tau=0}} r^2 J^N_\mu[\psi] n^\mu_\Sigma . \tag{212}$$

Proof. We write the wave equations for the rescaled $\phi = r\psi$ as

$$-4 \frac{1}{1 - \frac{2M}{r}} \partial_u \partial_v \phi + \frac{1}{r^2} \mathring{\Delta} \phi - \frac{2M}{r^3} \phi = 0 . \tag{213}$$

The fundamental identity obtained by multiplying with $-\frac{1}{2} r^p \partial_v \phi$ now becomes

$$\begin{aligned}
&\partial_u \left(\frac{r^p}{1 - \frac{2M}{r}} (\partial_v \phi)^2 \right) + \left[\partial_u \frac{r^p}{1 - \frac{2M}{r}} \right] (\partial_v \phi)^2 + \mathring{\nabla}^A \left(-\frac{1}{2} \frac{r^{p-2}}{1 - \frac{2M}{r}} \partial_v \phi \mathring{\nabla}_A \phi \right) \\
&+ \frac{1}{4} \partial_v \left(\frac{r^{p-2}}{1 - \frac{2M}{r}} \mathring{\nabla}^A \phi \mathring{\nabla}_A \phi \right) - \frac{1}{4} \left[\partial_v \frac{r^{p-2}}{1 - \frac{2M}{r}} \right] \mathring{\nabla}^A \phi \mathring{\nabla}_A \phi \\
&+ \frac{1}{4} \partial_v \left(\frac{2M}{r^3} r^p \phi^2 \right) - \frac{M}{2} \left[\partial_v r^{-3+p} \right] \phi^2 = 0 .
\end{aligned} \tag{214}$$

Recall in Schwarzschild we have $\partial_v r = \frac{1}{2}\left(1 - \frac{2M}{r}\right)$ and $\partial_u r = -\frac{1}{2}\left(1 - \frac{2M}{r}\right)$. Note that since $M > 0$, we can choose R sufficiently large such that all the square brackets are positive in the region $r \geq R$. The terms on $r = R$, including the (wrong signed) additional lower oder ϕ^2 term from the last line) can be controlled as on Exercise Sheet 11 (note that the degenerate (at $r = 3M$) integrated decay estimate is sufficient here since we are in a region of large R). Integrating over $\mathcal{D}^{\tau_2}_{\tau_1}$ therefore yields for $p = 2$

$$\begin{aligned}
&\int_{u=\tau_2 - R^\star, v \geq \tau_2 + R^\star} r^2 (\partial_v \phi)^2 \sin\theta d\theta d\varphi dv + \int_{\mathcal{D}^{\tau_2}_{\tau_1}} r (\partial_v \phi)^2 \sin\theta d\theta d\varphi du dv \\
&\leq \int_{u=\tau_1 - R^\star, v \geq \tau_1 + R^\star} r^2 (\partial_v \phi)^2 \sin\theta d\theta d\varphi dv + C \int_{\Sigma_{\tau_1}} J^N_\mu[\psi] n^\mu_\Sigma .
\end{aligned} \tag{215}$$

As in the Minkowski case, we find a dyadic sequence $\tau_n \to \infty$

$$\int_{u=\tau_n - R^\star, v\geq \tau_n+R^\star} r(\partial_v \phi)^2 \sin\theta d\theta d\varphi dv$$
$$\leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R^\star, v\geq\tau_1+R^\star} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\right]. \tag{216}$$

Applying the fundamental identity now with $p = 1$ between two slices (and converting from ϕ to ψ as before) we infer

$$\int_{\mathcal{D}^{\tau_n}_{\tau_{n-1}}} \left(\frac{1}{2}(\partial_v\psi)^2 + \frac{1}{8}|\nabla\!\!\!/ \psi|^2\right) r^2 \sin\theta d\theta d\varphi du dv \tag{217}$$
$$\leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R^\star, v\geq\tau_1+R^\star} r^2(\partial_v\phi)^2 \sin\theta d\theta d\varphi dv + C\int_{\Sigma_{\tau_1}} J^N_\mu[\psi]n^\mu_\Sigma\right] + C\int_{\Sigma_{\tau_{n-1}}} J^N_\mu[\psi]n^\mu_\Sigma.$$

Now adding the *non-degenerate* integrated decay estimate (and converting ϕ to ψ in the square bracket using a Hardy inequality and the fact that $r \geq 2M$ is bounded below – Exercise!) we deduce

$$\int_{\tau_n}^{\tau_{n-1}} d\tau \int_{\Sigma_\tau} J^N_\mu n^\mu_{\Sigma_\tau} \leq \frac{C}{\tau_n}\left[\int_{u=\tau_1-R^\star, v\geq\tau_1+R^\star} r^2 J^N_\mu[\psi] n^\mu_{\Sigma_\tau}\right]$$
$$+ C\int_{\Sigma_{\tau_{n-1}}} J^N_\mu[\psi]n^\mu_\Sigma + \int_{\Sigma_{\tau_{n-1}}} J^N_\mu[T\psi]n^\mu_\Sigma.$$

We now run the argument as in the Minkowski case choosing a sequence of good slices and exporting the decay to all slices using (200) and the dyadic property. This eventually proves (212). □

5.9 Outlook: The Kerr metric

In this final (non-examinable) subsection, we will introduce the Kerr metric and discuss, rather informally, some of its properties. (Indeed, you shouldn't have taken a course on black holes without having heard about the Kerr metric but discussing what we did in Chapter 4 for that metric is more complicated, which is why we have postponed its introduction until now.)

We let $M > 0$ and $a \in \mathbb{R}$ with $M \geq |a|$ and define

$$\Delta = r^2 - 2Mr + a^2 \quad , \quad \Sigma = r^2 + a^2\cos^2\theta\,. \tag{218}$$

We define the manifold $\mathcal{M} = (-\infty,\infty)_t \times (r_+,\infty)_r \times S^2_{\theta,\phi}$ where $r_+ := M + \sqrt{M^2-a^2}$ and equip it with a 2-parameter (M,a) family of metrics of the form

$$g = g_{tt}dt^2 + g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2\,, \tag{219}$$

with

$$
\begin{aligned}
g_{tt} &= -\left(1-\frac{2Mr}{\Sigma}\right) \quad , \quad g_{rr} = \frac{\Sigma}{\Delta} \quad , \quad g_{\theta\theta} = \Sigma \, , \\
g_{t\phi} &= -\frac{2Mr}{\Sigma} a \sin^2\theta \quad , \qquad g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta \, . \qquad (220)
\end{aligned}
$$

The coordinates (t, r, θ, ϕ) are called Boyer-Lindquist coordinates. One may check that the above metric is Ricci-flat. We collect also the inverse (noting that $g_{tt}g_{\phi\phi} - (g_{t\phi})^2 = -\Delta \sin^2\theta$)

$$
\begin{aligned}
g^{tt} &= -\frac{g_{\phi\phi}}{\Delta \sin^2\theta} \quad , \quad g^{rr} = \frac{\Delta}{\Sigma} \quad , \quad g^{\theta\theta} = \Sigma^{-1} \, , \\
g^{t\phi} &= \frac{g_{t\phi}}{\Delta \sin^2\theta} \quad , \qquad g^{\phi\phi} = -\frac{g_{tt}}{\Delta \sin^2\theta} \, . \qquad (221)
\end{aligned}
$$

We collect some facts about the Kerr metric:

1. **History and basic facts.** The Kerr metric was discovered in 1963, many years after Schwarzschild. It is, obviously, algebraically much more complicated! Note that for $a = 0$ the metric reduces to the Schwarzschild metric in the original Schwarzschild coordinates, so the Kerr family contains the Schwarzschild family. For $|a| = M$, the function Δ has a double root at $r = M$. This is reminiscent of the extremal Reissner-Nordstroem black hole of Section 5.7 and indeed the case $|a| = M$ is called *extremal Kerr*. For $M = 0$ (and a arbitrary) the metric (219) is actually the flat metric in so-called spheroidal coordinates.

2. **Asymptotic flatness.** The metric is "asymptotically flat" (I won't give a formal definition here). You can change coordinates to asymptotically Euclidean coordinates by setting $\tilde{r} = \frac{1}{2}\sqrt{r - M + r\left(1 - \frac{2M}{r}\right)}$ and $x = \tilde{r}\sin\theta\cos\varphi$, $y = \tilde{r}\sin\theta\sin\varphi$, $z = \tilde{r}\cos\theta$. The metric becomes

$$
\begin{aligned}
g = &-\left(1 - \frac{2M}{\tilde{r}} + \mathcal{O}(\tilde{r}^{-2})\right)dt^2 + \left(1 + \frac{2M}{\tilde{r}} + \mathcal{O}(\tilde{r}^{-2})\right)\left(dx^2 + dy^2 + dz^2\right) \\
&+ \mathcal{O}_{\mu\nu}(\tilde{r}^{-2})\,(dx^\mu \otimes dx^\nu) \qquad (222)
\end{aligned}
$$

3. **Killing symmetries.** The Kerr metric has two Killing vectors ∂_t and ∂_ϕ, which moreover commute. The vectorfield ∂_ϕ is spacelike and generates rotations around the z-axis, so Kerr is *axisymmetric*. Remarkably, the Kerr metric admits also a *Killing-tensor* whose discovery is due to Carter (in the 1960s). A Killing 2-tensor is a symmetric 2-tensor K which satisfies $\nabla_{(\alpha}K_{\beta\gamma)} = 0$. A Killing tensor is called irreducible if it cannot be constructed from the metric (which is always a Killing tensor) and Killing vectors. Note that a Killing tensor K produces a conserved quantity along geodesics since $K_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta$ is conserved along γ (why?).

The existence of the Killing tensor (sometimes called a "hidden symmetry" in the physics literature) has several consequences. In a Ricci flat spacetime (such as Kerr) it implies the separability of the wave equation. It (the existence of an additional conserved quantity) also implies the integrability of the geodesic flow. Both statements should be understood that the PDE problems can be reduced to ODEs. This plays a fundamental role in the analysis of the covariant wave equation on Kerr. In addition, a Killing tensor gives rise to a (second-order) symmetry operator $K = \nabla_\alpha \left(K^{\alpha\beta} \nabla_\beta \right)$ which commutes with the wave equation: $[\Box_g, K] = 0$. In Kerr we have with $L = \frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi + \partial_r$ and $\underline{L} = \frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi - \partial_r$ denoting the principal null directions:

$$K_{\mu\nu} = \Delta \cdot L_\mu \underline{L}_\nu + r^2 g_{\mu\nu} \, ,$$
$$K\psi = \mathring{\Delta}_{S^2}\psi - \Phi^2\psi + (a^2 \sin^2\theta) T^2 \psi \, . \tag{223}$$

I invite you to check the above properties! Note already, however, that commuting twice with T and Φ and with K (the commutation of all of which is trivial), we will be able to estimate $\mathring{\Delta}_{S^2}\psi$ and hence angular derivatives of ψ by elliptic estimates.

4. **Superradiance.** The vectorfield ∂_t is timelike for large r (the metric is stationary) and in general satisfies

$$g(\partial_t, \partial_t) < 0 \quad \text{for } 2Mr < \Sigma, \tag{224}$$
$$g(\partial_t, \partial_t) = 0 \quad \text{for } 2Mr = \Sigma, \tag{225}$$
$$g(\partial_t, \partial_t) > 0 \quad \text{for } 2Mr > \Sigma. \tag{226}$$

The region where ∂_t is spacelike is called the ergoregion.[40] It is the range

$$M + \sqrt{M^2 - a^2} < r < M + \sqrt{M^2 - a^2 \cos^2\theta} \, .$$

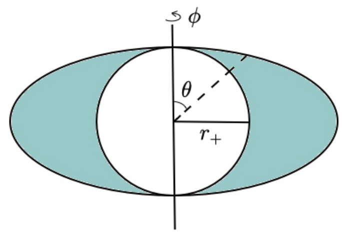
Note that ∂_t is also spacelike on $\mathcal{H}^+$ (except at the poles of the sphere)

[40]Inside the ergosphere an observer cannot follow the stationary Killing field ∂_t, he or she would have to go faster than light. In fact consider any timelike curve $\gamma(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ inside the ergosphere. Then, since the tangent vector is timelike,

$$\sum_{\mu\nu} g_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu < 0 \, . \tag{227}$$

In the sum on the left, all terms are positive except $2g_{t\phi}\frac{dt}{d\tau}\frac{d\phi}{d\tau}$, so we clearly need $\frac{d\phi}{d\tau} \neq 0$ along the curve. Hence an observer in the ergoregion necessarily needs to rotate!

since $2Mr_+ > r_+^2 + a^2 - a^2 \sin^2\theta$ on the horizon becomes $0 > -a^2 \sin^2\theta$.



For the study of the wave equation $\Box_g \psi = 0$ this is somehow bad news because the boundary term on the horizon in the estimate associated with ∂_t may be negative, so we can have

$$\int_{\mathcal{I}} J^T_\mu n^\mu_\Sigma > \int_{\Sigma_{t^\star=0}} J^T_\mu n^\mu_\Sigma \tag{228}$$

This phenomenon is called *superradiance*. There is also a particle model for this (the so-called *Penrose process*, see the book of Wald) which exhibits energy extraction from black holes.

5. **Trapping.** As mentioned, geodesic flow is integrable for Kerr so we can compute the geodesics by solving ODEs. You derived the resulting system for Schwarzschild on Sheet ... Doing this for Kerr results for $\gamma(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ in the ODE system (see for instance the PhD thesis of Sbierski)

$$\Sigma \dot{t} = a\mathbb{D} + (r^2 + a^2)\frac{\mathbb{P}}{\Delta} \tag{229}$$

$$\Sigma^2 \dot{r}^2 = R(r) := -K\Delta + \mathbb{P}^2 \tag{230}$$

$$\Sigma^2 \dot{\theta}^2 = \Theta(\theta) := K - \frac{\mathbb{D}}{\sin^2\theta} \tag{231}$$

$$\Sigma \dot{\phi} = \frac{\mathbb{D}}{\sin^2\theta} + \frac{a\mathbb{P}}{\Delta} \tag{232}$$

where K is the Carter constant of γ, $\mathbb{P}(r) = (r^2 + a^2)E - La$ and $\mathbb{D}(\theta) = L - Ea\sin^2\theta$ and finally $E = -g(\dot{\gamma}, \partial_t)$, $L = g(\dot{\gamma}, \partial_\phi)$.

Below we sketch the ODE analysis necessary to infer the existence of trapped geodesics for r-values in a full interval $[r_1, r_2]$ where r_1 and r_2 are the roots of the polynomial $p(r) = r(r - 3M)^2 - 4a^2M$ which lie in the interval $[r_+, \infty)$.[41] In summary trapped geodesics exist for r values in an

[41] The first observation is that simple zeros of $R(r)$ correspond to turning points while double zeros correspond to trapped geodesics. Since by (231) we have $K > 0$ we have that $R(r)$ is

entire closed interval around $r = 3M$, i.e. they fill out a set of non-trivial measure in the manifold! Our construction of physical space multipliers degenerating at $r = 3M$ only will therefore fail!

The good news is that the phenomenon is still unstable when viewed in the tangent bundle: Perturbing the tangent vector of a fixed trapped geodesic (which you can view as changing the conserved quantities slightly) will result in the geodesic leaving through the event horizon or null infinity.

6. **Global structure of the Kerr metric.** We can extend the metric through r_+ in a similar fashion to what we did in Schwarzschild. One defines

$$t^\star = t + T(r) \qquad \text{with } \frac{dT}{dr} = \frac{r^2 + a^2}{\Delta} \tag{233}$$

$$\phi^\star = \phi + A(r) \mod 2\pi \qquad \text{with } \frac{dA}{dr} = \frac{a}{\Delta}. \tag{234}$$

The metric in $(t^\star, r, \theta, \phi^\star)$ coordinates can be defined on a larger manifold into which the original $\mathcal{M}$ embeds isometrically. The global structure of the Penrose diagram (the part which is globally hyperbolic) looks as the one for subextremal Reissner-Nordstroem in Figure 1. In particular, we see that the global structure (especially that of the black hole *interior*) is quite different from that of the Schwarzschild metric!

7. **Surface Gravity of the horizon** One computes that $r = r_+$ is a Killing horizon, i.e. a null hypersurface whose generators (see Sheet 5) are

$$K = \partial_t + \frac{a}{r_+^2 + a^2}\partial_\phi \, .$$

The surface gravity, defined by $\nabla_K K = \kappa K$ is computed to be

$$\kappa = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2} \, . \tag{235}$$

The computations establishing the above, should of course be done in the $(t^\star, r, \theta, \phi^\star)$ coordinates, which are regular on the horizon. (The point is that the vectorfield K extends regularly to r_+ which can be seen by going to the regular coordinates.)

We state informally some theorems and conjectures about the Kerr metric.

positive for $r \in (r_-, r_+)$. It is also positive near infinity and viewed as a 4^{th} order polynomial in r does not have an r^3 term. It follows that R has an even number of zeros in (r_+, ∞). It can't have four because by the absence of the r^3 term the sum of the roots has to equal zero. If it has zero roots clearly the geodesic is not trapped. If it has two single roots r_1, r_2 in (r_+, ∞) then R is negative in (r_1, r_2) and again the geodesic is not trapped (but turns around at these points). One now has to check that one can choose the parameters K,E and L such that $R(r)$ has a double root. To do this, set wlog $E = 1$ and solve $R(\bar{r}) = 0$ and $\frac{d}{dr}R(\bar{r}) = 0$ for $L(r)$ and $K(r)$. One needs to check consistency with the other equations and this works for r in the aforementioned interval.

Conjecture 5.23. *The two parameter family of Kerr black holes comprises* <u>*all*</u> *stationary, asymptotically flat vacuum black holes.*

The conjecture has been proven rigorously under the additional assumption of axisymmetry (by Carter-Robinson around 1970) and under the assumption of analyticity of the spacetime (which allows one to infer the axisymmetry by an argument of Hawking). More recently, it has also been shown in the class of spacetimes sufficiently close to the Kerr family (by Alexakis–Ionescu–Klainerman). Note that it is very remarkable and surprising that all equilibrium (="stationary") should be described by just two parameters, the mass and the angular momentum of the black hole!

In the recent (2016) gravitational wave experiments that you will have undoubtedly heard about, the signal comes from the merger of two black holes which settle down in a Kerr black hole. From the way the signal settles down to zero one can infer the parameters a and M of the black hole! This leads into the subject of quasinormal modes for black holes which is far from the scope of this introductory course.

In any case, Conjecture 5.23 clearly illustrates the central role of the Kerr family for gravitational dynamics! We saw that to investigate the *stability* of the Kerr metric, a natural toy-problem is the study of the covariant wave equation (since the Einstein equations themselves are, as we have also seen, (non-linear) wave equations. For the toy problem we have the following result:

Theorem 5.24 (Dafermos–Rodnianski-Shlapentokh-Rothman 2014)**.** *Solutions to the wave equation* $\Box_{g_{Kerr}}\psi = 0$ *on a* subextremal *Kerr exterior arising from suitable decaying initial data decay inverse polynomially in time.*

There has been a lot of progress on moving from the toy problem to the problem of proving stability of the Kerr family for the Einstein vacuum equations. This is a topic of ongoing research and could be the topic of another course.

6 The Penrose Incompleteness Theorem

6.1 Closed trapped surfaces

Let (M, g) be a spacetime and consider $S \subset M$ as spacelike 2-surface, which for simplicity we assume to be contained in a 3-dimensional spacelike hypersurface $\Sigma \subset M$ and that S bounds a compact domain K in Σ. If R is the (spacelike) outward normal of S in Σ and N is the (future directed timelike) unit normal of Σ in M then $L = N + R$ and $\underline{L} = N - R$ are future directed null vectors defined along S satisfying $g(L, \underline{L}) = -2$. For obvious reasons we call L the outgoing normal null direction to S and $\underline{L}$ the ingoing normal null direction of S.[42] Given L and $\underline{L}$ along S we can look at the ingoing and outgoing null geodesics from S

[42] We could start more abstractly with S (without Σ) and pick the two null directions in $(T_pS)^\perp \subset T_pM$.

which (at least locally) generate an ingoing and an outgoing null hypersurface $\underline{C}$ and C respectively.

To L and $\underline{L}$ we can associate the second fundamental forms χ and $\underline{\chi}$ defined at each $p \in S$ as follows

$$\chi(X,Y) = g(\nabla_X L, Y) \quad , \quad \underline{\chi}(X,Y) = g(\nabla_X \underline{L}, Y) \tag{236}$$

for all $X, Y \in T_p S$. Note that this indeed defines a tensorfield on S.

Definition 6.1. *A closed spacelike surface S as above is called* <u>*trapped*</u> *if the traces of the two associated null second fundamental forms are negative, i.e.*

$$tr\chi < 0 \quad , \quad tr\underline{\chi} < 0 \quad \text{at each } p \in S.$$

As we shall see below, this condition implies that the area of the surface decreases when deformed in each of the two null directions. Note that a standard sphere in Minkowski space has $tr\chi > 0$ and $tr\chi < 0$.

Exercise 6.2. *Show that any sphere of symmetry in region II of Schwarzschild is trapped. Show that any sphere in the black hole interior of subextremal Kerr is trapped.*

The notion of a trapped surface is intimately tied to the notion of a black hole. However, why the latter is a global notion (depending on the existence of an appropriate asymptotic boundary $\mathcal{I}^+$ of spacetime), the notion of a trapped surface is *local*.

6.2 Statement of the theorem

We can now directly state the Penrose incompleteness theorem:

Theorem 6.3 (Penrose, 1965)**.** *Let (M,g) be a $3{+}1$ dimensional globally hyperbolic Lorentzian manifold with a non-compact Cauchy hypersurface whose Ricci curvature satisfies*

$$Ric(V,V) \geq 0 \quad \text{for all null vectors } V.$$

Suppose moreover that (M,g) contains a closed trapped surface. Then (M,g) is future geodesically incomplete.

Remark 6.4. *The curvature assumption (trivially) holds for any vacuum spacetime. It also holds for spacetimes satisfying the so-called null energy condition, which is a very reasonable assumption on the matter content in spacetime.*

Remark 6.5. *The theorem is sometimes called a singularity theorem. Note, however, that it is geodesic incompleteness that is asserted and that the geometry of (M,g) does not necessarily blow-up (in the sense of exhibiting infinite curvature) towards the future. Compare the conclusion of the theorem for the trapped surfaces in Schwarzschild and Kerr respectively!*

Remark 6.6. *The statement and the proof that we're about to see should be compared with the theorem of Bonnet-Myers in Riemannian geometry. That heorem states that if (M, g) is a complete and connected Riemannian manifold and $Ric(X, X) \geq \frac{n-1}{\rho^2} g(X, X)$ holds for all $X \in \mathcal{X}(M)$, then M is compact and $diam(M) \leq \pi\rho$.*[43] *Idea of the proof: Assume that the diameter bound does not hold. Then there exists (by a Corollary of Hopf-Rinow) a length minimizing geodesic γ connecting p and q with $L(\gamma) > \rho\pi$. However, the curvature assumption implies that the geodesic must develop a conjugate point at or before $\rho\pi$ after the geodesic is no longer minimising. Contradiction. The diameter bound implies the compactness (by Hopf Rinow, the closed and bounded subsets of M are compact).*

By the above, one may view Bonnet-Myers as: "completeness and a lower bound of the Ricci curvature implying compactness" and the Penrose theorem as "completeness and a lower bound on Ricci curvature in null directions + existence of a trapped surface implying completeness of every Cauchy hypersurface."

Remark 6.7. *The assumption of a trapped surface is of course a very strong one. A celebrated (much more recent!) result of Christodoulou proves that trapped surfaces can indeed form in the evolution of the Einstein equations from "arbitrary dispersed" initial data. The proof requires a deep understanding of the structure of non-linearities appearing in the Einstein equations as an evolutionary PDE.*

6.3 Analysis of Null Geodesics emanating from S

The main analytic ingredient in the proof of Penrose's theorem consists in a detailed analysis of the null geodesics emanating orthogonally from the trapped surface S. In this subsection we provide this analysis.

At each $p \in S$ there exists a unique maximal geodesic $\gamma_p : (T_-(p), T_+(p)) \to M$ with $\gamma_p(0) = p$, $\dot{\gamma}_p(0) = L_p$. Similarly, there is a unique maximal geodesic $\underline{\gamma}_p : (\underline{T}_-(p), \underline{T}_+(p)) \to M$ such that $\underline{\gamma}_p(0) = p$, $\dot{\underline{\gamma}}_p(0) = \underline{L}_p$. We consider the sets

$$C = \bigcup_{p \in S} \gamma_p \quad , \quad \underline{C} = \bigcup_{p \in S} \underline{\gamma}_p \, .$$

The geodesics γ_p, $\underline{\gamma}_p$ are called the null generators of C and $\underline{C}$ respectively. The sets C and $\underline{C}$ are smooth hypersurfaces locally near S but not necessarily globally as the example of a standard sphere in Minkowski space shows.

Let us look, for the moment, at an appropriate restriction of C near S such that C is actually smooth (similar considerations hold for $\underline{C}$). We will continue to use the notation C for the restriction. Clearly, we can define L on all of C by setting $L_{\gamma_p(\tau)} = \dot{\gamma}_{\gamma(\tau)}$.

We can normalise the affine parameter τ of γ_p such that $\tau = 0$ on S and $L\tau = 1$.[44] We thus obtain an affine foliation of the cone C by spheres: We define

[43] $diam(M) := \sup\{d_g(p, q) \mid p, q \in M\}$ is the diameter of the manifold.

[44] Recall that the affine parameter is unique up to linear transformations $\tau' = a\tau + b$. In

$S_\tau = \mathcal{F}_\tau(S = S_0)$ with $\mathcal{F}_\tau$ the flow along the affinely parametrised geodesics. Now, given any $y \in C$ we have $y = \gamma_p(\tau)$ for some τ so $y \in S_\tau$ for some τ. If E_1, E_2 is a basis of T_pS, then (L, E_1, E_2) is a basis for T_pC. We may propagate this basis by Lie propagation, i.e. according to[45]

$$\mathcal{L}_L E_i = 0$$

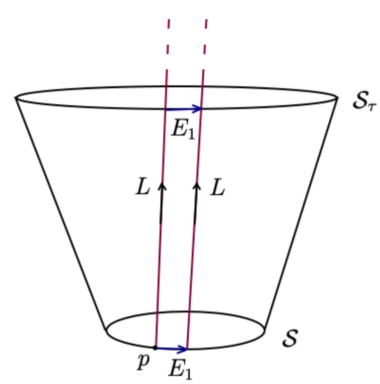
The E_1, E_2 are tangent to the section S_τ (note $LE_A(\tau) = E_A L(\tau) = E_A(1) = 0$ so $E_A(\tau) = 0$ everywhere since $E_i(\tau) = 0$ holds at $\tau = 0$; alternatively, compute $\nabla_L(g(L, E_A)) = 0 + g(L, \nabla_L E_A) = g(L, \nabla_{E_A} L) = \frac{1}{2}\nabla_{E_A} g(L, L) = 0$.) and we have

$$(E_i)_{\gamma_p(\tau)} = d\mathcal{F}_\tau(E_i)_p \, .$$

The E_1 and E_2 are called *normal Jacobi fields* along the geodesics. The point is that they measure the (infinitesimal) displacement between nearby null geodesics. Indeed, suppose $\lambda : (-\epsilon, \epsilon) \to$ is a curve on S such that $\lambda(0) = p$ and $\dot{\lambda}(0) = E_1$, then we have a mapping

$$\Phi : (-\epsilon, \epsilon) \times [0, T) \to C \quad , \quad \Phi(s, \tau) = \gamma_{\lambda(s)}(\tau) \, .$$

If we fix τ, then $\alpha(s) = \gamma_{\lambda(s)}(\tau)$ corresponds to the displacement of the null generators at affine time τ. The infinitesimal displacement for a fixed geodesic (say γ_p itself) is $\frac{d\alpha}{ds}(s = 0) = E_i$ (evaluated at a fixed but arbitrary τ). All this is illustrated in the picture below:



Note finally that C is indeed a null hypersurface. To see this, we show that given $y \in C$ arbitrary, T_yC is a null hyperplane in M. We have $y = \gamma_p(\tau)$ for some p and τ. Since clearly $L \in T_yC$, it suffices to show that $g(L, X) = 0$ for all vectors X in T_yC (as this shows that the induced metric is indeed degenerate).

the timelike/ spacelike case one easily see that one can choose a so that the tangent vector has length ± 1 along the geodesic (recall the length is conserved along γ).

[45] Note this amounts to solving a first order ODE for E_i along γ_p.

We compute for $X \in \{L, E_1, E_2\}$

$$\nabla_L g(L, X) = 0 + g(L, \nabla_L X) = g(L, \nabla_X L) = \frac{1}{2}\nabla_X (g(L, L)) = 0 \tag{237}$$

along γ_p. Since at p we have $g(L, X) = 0$ for all $X \in \{L, E_1, E_2\}$, we conclude that $g(L, X) = 0$ along γ_p, hence at $T_y C$.

Remark 6.8. *Note that it is very important that L is <u>normal</u> to S. The hypersurface $r = 3M$ in Schwarzschild for instance is spanned by the trapped null geodesics but is timelike. The null vectors of the trapped geodesics are not normal to the spheres of symmetry!*

Remark 6.9. *The frame (L, E_1, E_2) at $T_y C$ can (at any point $y \in C$) be completed (uniquely) to a spacetime null frame $(L, \underline{L}, E_1, E_2)$ with $\underline{L}$ a null vector orthogonal to the S_τ (hence to E_1, E_2) and $g(L, \underline{L}) = -2$. Note that L and $\underline{L}$ are determined independently of how one choses the frame on S_τ, so one can speak about L and $\underline{L}$ without having chosen a frame E_1, E_2.*

Remark 6.10. *Having established the geometric picture above we can also clarify the trapped surface condition: If $\not{g}$ denotes the metric induced on the hypersurfaces S_τ, i.e. $\not{g}(X, Y) = g(X, Y)$ for $X, Y \in T_{\gamma_\tau(p)} S_\tau$ then differentiating the metric along L yields:*

$$\begin{aligned}\frac{d}{d\tau}(\not{g}(E_A, E_B)) &= \nabla_L (g(E_A, E_B)) = g(\nabla_L E_A, E_B) + g(E_A, \nabla_L E_B) \\ &= g(\nabla_{E_A} L, E_B) + g(E_A, \nabla_{E_B} L) = 2\chi(E_A, E_B),\end{aligned} \tag{238}$$

hence by the formula for differentiating a determinant

$$\frac{d}{d\tau}\sqrt{\not{g}} = \frac{1}{2}\sqrt{\not{g}}\not{g}^{AB}\frac{d}{d\tau}\not{g}_{AB} = \sqrt{\not{g}} tr\chi \,.$$

This makes manifest that for a trapped surface any deformation of (infinitesimal) area in the direction of the null generators results in a decrease of that area.

Proposition 6.11. *We have $\partial J^+(S) \subset C \cup \underline{C}$.*

Proof. We sketch the argument. By global hyperbolicity we have that $J^+(S)$ is closed (Sheet 12), therefore $\partial J^+(S) \subset J^+(S)$. Given $y \in \partial J^+(S) \subset J^+(S)$ there exists a causal curve γ from some $p \in S$ to y. If γ was not null we could deform it into an overall timelike curve contradicting $y \in \partial J^+(S)$ so γ has to be a null curve. If it wasn't a null geodesic, we could also deform it into a timelike curve. Finally, if it wasn't orthogonal to S we could also deform it into a timelike curve. □

Remark 6.12. *Note that $C \cup \underline{C} \subset \partial J^+(S)$ does not necessarily hold as the example of a standard sphere in Minkowski shows. The ingoing cone $\underline{C}$ forms a caustic and the points in the causal future of that singularity lie in $I^+(S)$.*

6.3.1 Jacobi fields

The proof of Penrose's theorem is based on understanding the behaviour of certain Jacobi fields along the null generators γ_p and $\underline{\gamma}_p$.

Definition 6.13. *Given $p \in S$ a point $q = \gamma_p(\tau_\star)$ for some $\tau_\star \in (0, T_+(p))$ is called a <u>focal point</u> to p if there exists a non-trivial (i.e. not identically vanishing) normal <u>Jacobi field</u> J along γ_p such that*

$$[L(\tau), J(\tau)] = 0 \quad \textit{for all } \tau \in [0, T_+(p)) \textit{ and } J(\tau_\star) = 0.$$

Similarly, a point $q = \underline{\gamma}_p(\underline{\tau}_\star)$ for some $\underline{\tau}_\star \in (0, \underline{T}_+(p))$ is called a <u>focal point</u> to p if there exists a non-trivial (i.e. not identically vanishing) normal Jacobi field $\underline{J}$ along $\underline{\gamma}_p$ such that

$$[\underline{L}(\underline{\tau}), \underline{J}(\underline{\tau})] = 0 \quad \textit{for all } \underline{\tau} \in [0, \underline{T}_+(p)) \textit{ and } \underline{J}(\underline{\tau}_\star) = 0.$$

Note that J satisfies the Jacobi equation (use $[L, J] = 0$ and $\nabla_L L = 0$):

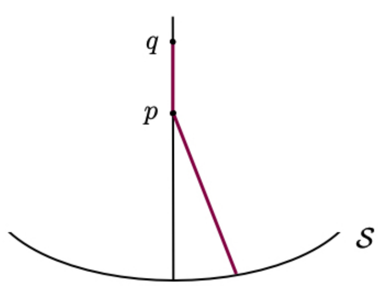
$$\nabla_L \nabla_L J = \nabla_L \nabla_J L - \nabla_J \nabla_L L = Riem(L, J)L$$

and of course similarly $\nabla_{\underline{L}} \nabla_{\underline{L}} \underline{J} = Riem(\underline{L}, \underline{J})\underline{L}$.

Proposition 6.14. *Consider $p \in S$ and let $\tau_\star > 0$ be such that $\gamma_p(\tau_\star)$ is a focal point to p along γ_p. Then for any $\epsilon > 0$ we have $\gamma_p(\tau_\star + \epsilon) \in I^+(S)$. Similarly, if $\underline{\tau}_\star > 0$ is such that $\underline{\gamma}_p(\underline{\tau}_\star)$ is a focal point to p along $\underline{\gamma}_p$, then for any $\epsilon > 0$ we have $\underline{\gamma}_p(\underline{\tau}_\star + \epsilon) \in I^+(S)$.*

Proof. postponed. □

The theorem is easily explained intuitively and should be compared with the theorem in Riemannian geometry that geodesics stop minimizing length after a conjugate point.



The above picture shows (in Euclidean geometry) how to "cut the corner" to turn the broken geodesic into a shorter unbroken one. In the Lorentzian case it is the reverse triangle inequality that enters the cutting the corner argument turning a broken null geodesic into a "longer" timelike one.

Remark 6.15. *We note the related notions of conjugate points and that of cut-locus in Riemannian geometry. Recall that the cut-locus of p is the collection of points after which geodesics emanating from p stop minimising length. While conjugate points are in the cut-locus, points in the cut-locus do not have to be conjugate/ focal points as the example of the torus shows.*

Proposition 6.14 allows to strengthen the conclusion of Proposition 6.11:

Corollary 6.16. *If $\tau_\star > 0$ is such that γ_p and $\underline{\gamma}_p$ each contain a focal point before time $\tau_\star$ for all $p \in S$, then*

$$\partial J^+(S) \subset C(\tau_\star) \cup \underline{C}(\tau_\star)$$

where $C(\tau_\star) = \bigcup_{\tau \in [0,\tau_\star]} \bigcup_{p \in S} \gamma_p(\tau)$ and similarly for $\underline{C}(\tau_\star)$.

6.3.2 The Raychaudhuri equations

Recall now that we denoted by $\not{g}$ the Riemannian metric induced on the spheres S_τ (and $\underline{S}_{\underline{\tau}}$). We define

$$\hat{\chi} = \chi - \frac{1}{2} tr\chi \not{g} \quad \text{and} \quad \hat{\underline{\chi}} = \underline{\chi} - \frac{1}{2} tr\underline{\chi} \not{g}$$

which are the tracefree parts of the second fundamental forms χ and $\underline{\chi}$ respectively. The next Lemma establishes an important monotonicity property of the traces:

Lemma 6.17. *The quantities $tr\chi$ and $tr\underline{\chi}$ satisfy the Raychaudhuri equations*

$$\begin{aligned} L(tr\chi) &= -\frac{1}{2}(tr\chi)^2 - |\hat{\chi}|^2_{\not{g}} - Ric(L,L)\,, \\ \underline{L}(tr\chi) &= -\frac{1}{2}(tr\underline{\chi})^2 - |\hat{\underline{\chi}}|^2_{\not{g}} - Ric(\underline{L},\underline{L})\,. \end{aligned} \tag{239}$$

Remark 6.18. *Note the right hand side is indeed negative if the curvature assumption of Penrose's theorem holds. It follows that $tr\chi$ will remain negative (and will in fact go to $-\infty$ in finite time) along the cone.*

Proof. We prove the first identity, the second being entirely analogous. We have (use again $[L, E_A] = 0$ and $\nabla_L L = 0$)

$$\begin{aligned} L(\chi(E_A, E_B)) &= g(\nabla_L \nabla_{E_A} L, E_B) + g(\nabla_{E_A} L, \nabla_L E_B) \\ &= Riem(L, E_A, L, E_B) + g(\nabla_{E_A} L, \nabla_{E_B} L). \end{aligned} \tag{240}$$

On the other hand, from the Leibniz rule

$$\nabla_L \chi(E_A, E_B) = L(\chi(E_A, E_B)) - \chi(\nabla_L E_A, E_B) - \chi(E_A, \nabla_L E_B) \tag{241}$$

and hence after inserting the previous identity for the first term

$$\begin{aligned} \nabla_L \chi(E_A, E_B) =& Riem(L, E_A, L, E_B) + g(\nabla_{E_A} L, \nabla_{E_B} L) \\ &- g(\nabla_{\nabla_L E_A} L, E_B) - g(\nabla_{E_A} L, \nabla_L E_B). \end{aligned} \tag{242}$$

After the cancellation and using that $\nabla_{E_A} L = {\chi_A}^C E_C$ we deduce

$$\nabla_L \chi(E_A, E_B) = -{\chi_A}^C \chi_{CB} - Riem(E_A, L, L, E_B)\,. \tag{243}$$

It follows that

$$L(tr\chi) = \nabla_L (g^{AB} \chi_{AB}) = g^{AB} (\nabla_L \chi(e_A, e_B)) = -\chi^{AB} \chi_{AB} - Ric(L, L)$$

□

6.3.3 Proving the existence of focal points

Assume the assumptions of Penrose's theorem. Since S is compact, there exists $k > 0$ such that both

$$\sup_{p \in S} tr\chi(p) \leq -k \quad \text{and} \quad \sup_{p \in S} tr\underline{\chi}(p) \leq k$$

hold.

Proposition 6.19. *Assume the assumptions of Penrose's theorem. If for all $p \in S$, $T_+(p) > \frac{2}{k}$ and $\underline{T}_+(p) > \frac{2}{k}$, then for all $p \in S$ the null generators γ_p and $\underline{\gamma}_p$ both contain a focal point to p.*

In other words, if the null geodesic γ_p exists for sufficiently long affine time, then it must contain a conjugate point.

Proof. We prove the conclusion for the null generators γ_p, the conclusion for $\underline{\gamma}_p$ is of course proven entirely analogously.

Step 1. A quantitative upper bound on $tr\chi$. We have

$$L(-(tr\chi)^{-1}) \leq -\frac{1}{2}$$

along γ_p by Lemma 6.17 and the curvature assumption in Penrose's theorem. Integrating yields

$$-\left[(tr\chi)(\gamma_p(\tau))\right]^{-1} \leq -\frac{\tau}{2} + \frac{1}{k}$$

and hence

$$tr\chi(\gamma_p(\tau)) \leq \left(\frac{\tau}{2} - \frac{1}{k}\right)^{-1} \quad \text{for all } \tau > 0. \tag{244}$$

In particular $tr\chi(\gamma_p(\tau))$ goes to $-\infty$ on or before affine time $\frac{2}{k}$.

Step 2. Construction of the Jacobi field. Instead of the Lie propagated frame for the S_τ, it will be convenient to work with a different frame E_1, E_2, which is *orthonormal* (Fermi propagated). Specifically, given L and $\underline{L}$ above, we choose an orthonormal frame at T_pS and extend the frame to $T_{\gamma_p(\tau)}S_\tau$ by (the ODE along γ_p)

$$\nabla_L E_A = -\frac{1}{2} g(\nabla_{E_A} L, \underline{L}) L \tag{245}$$

for $A = 1, 2$. Note that with this choice we have $\nabla_L(g(E_A, K))$ for $K \in \{E_1, E_2, L, \underline{L}\}$ so that E_1, E_2 is indeed an orthonormal frame for each $T_{\gamma_p(\tau)}S_\tau$.[46] Consider now an arbitrary non-trivial vector $v \in T_pS$. Define a vectorfield J along γ_p by solving $[L, J] = 0$ with $J(0) = v$. One checks that $L(g(J, L)) = L(g(J, \underline{L})) = 0$ (exercise, use the footnote) hence that $J(\tau) \in T_{\gamma_p(\tau)}S_\tau$ for all τ. Now since $[L, J] = 0$ is a first order linear (vector-valued) ODE for J we can write for the components of the solution in the basis E_1, E_2:

$$J^A(\tau) = M^A{}_B(\tau)v^B$$

for some matrix $M^A(\tau)$ independent of v. Clearly $M(0) = id$, $\det M(0) = 1$, $detM(\tau) \neq 0$ for τ sufficiently small. If we can find a $\tau_\star > 0$ with $\det M(\tau_\star) = 0$, then there exists a $v \in T_pS$ with $v \neq 0$ such that $J(\tau^\star) = 0$, hence p has a focal point along γ_p. We now show that this is the case.

$$L(J^A) = L(g(J, E_A)) - g(\nabla_L J, E_A) + g(J, \nabla_L E_A) \tag{246}$$

Inserting (245) and using $[L, J] = 0$ we infer

$$L(J^A) = g(\nabla_J L, E_A) - \chi(J, E_A)\,.$$

It follows that

$$L(M^A{}_B(\tau))v^B = \chi(J(\tau), E_A) = \chi(E_A, E_C)M^C{}_B v^B\,.$$

Since this holds for all v^B we deduce

$$L(M^A{}_B(\tau)) = \chi_{AC}M^C{}_B\,.$$

Using the well known formula $\frac{d}{ds}detM = detM \cdot tr\left(M^{-1}\frac{dM}{ds}\right)$ we infer

$$\frac{d}{d\tau}\log\det M = tr\chi\,. \tag{247}$$

By the initial condition $\log\det M(0) = 0$ and the estimate (244) we finally obtain

$$\log detM(\tau) \leq 2\log\left(\frac{\tau}{2} - \frac{1}{k}\right) \implies detM(\tau) \leq \left(\frac{\tau}{2} - \frac{1}{k}\right)^2\,. \tag{248}$$

Therefore, there exists a focal point along γ_p on or before affine time $\tau = \frac{2}{k}$. $\square$

[46] Indeed, using (245) one easily computes that $\nabla_L(g(E_A, L)) = 0$ and $\nabla_L(g(E_A, E_B)) = 0$. Moreover, in view of the easily established identities $g(\nabla_L\underline{L}, L) = g(\nabla_L\underline{L}, \underline{L}) = 0$ we infer that $\nabla_L\underline{L}$ is tangent to the sections S_τ. This implies for $X \in T_{\gamma_p(\tau)}S_\tau$ that $\not{g}(\nabla_L\underline{L}, X) = g(\nabla_L\underline{L}, X) = \nabla_L(g(\underline{L}, X)) - g(\underline{L}, \nabla_L X) = -g(\underline{L}, \nabla_L X)$ and hence that

$$\nabla_L(g(E_A, \underline{L})) = -\frac{1}{2}g(\nabla_{E_A}L, \underline{L})g(L, \underline{L}) + g(E_A, \nabla_L\underline{L}) = g(\nabla_{E_A}L, \underline{L}) - g(\nabla_L E_A, \underline{L}) = 0$$

with the last step following from inserting (245).

6.4 Summary of the main ingredients of the proof

The proof of the theorem has four ingredients:

1. The analytical ingredient manifest in Proposition 6.19 (establishing the existence of focal points within affine time $\frac{2}{k}$ along all null generators emanating orthogonally from S).

2. The variational ingredient manifest in Proposition 6.11 and Corollary 6.14 (establishing that $p \in \partial J^+(S)$ iff p lies on a null geodesic starting orthogonally from S and not containing focal points).

3. A Lorentzian geometric ingredient, manifest in the following proposition that we will prove later.

   Definition 6.20. *A set $K \subset M$ is a* <u>*future set*</u> *if $p \in K$ implies $I^+(p) \subset K$. A set $A \subset M$ is called* <u>*achronal*</u> *if there is no pair $p, q \in A$ with $q \in I^+(p)$.*

   Our main example of a future set will be $J^+(S)$. Note also that if $A \subset M$ is achronal, then no timelike curve can intersect A twice.

   Proposition 6.21. *If $K \subset M$ is a future set, then its topological boundary $\partial K \subset M$ (defined as $\overline{K} \cap \overline{M \setminus K}$) is a closed, achronal, 3-dimensional, locally Lipschitz submanifold without boundary (in the sense of (topological) manifolds).*

   We postpone the proof and further comments to Section 6.6 but merely illustrate the situation for K being a standard sphere in Minkowski space from which the above properties are immediately read off.

4. A (simple) topological ingredient manifest in the following standard result in topology (proven on Sheet 13 if you haven't seen this in a topology course).

   Proposition 6.22. *Let X be a compact topological space and Y a Hausdorff topological space. If $h : X \to Y$ is a continuous bijection, then h has a continuous inverse, i.e. it is a homeomorphism.*

We have seen that the assumptions of a trapped surface as well as the curvature condition were crucial to establish the first ingredient. The assumption of a non-compact Cauchy hypersurface will come in when combining the above ingredients in the proof.

6.5 Proof of the Theorem

We can now prove Theorem 6.3. Suppose for contradiction that (M, g) is future geodesically complete. Then γ_p and $\underline{\gamma}_p$ are future complete, hence $T_+(p) = \infty$,

$\underline{T}_+(p) = \infty$ for all $p \in S$. By Proposition 6.19 (Ingredient 1) we have focal points to p before $\frac{2}{k}$ along any generator and hence by Corollary 6.14 (Ingredient 2)

$$\partial J^+(S) \subset C\left(\frac{2}{k}\right) \cup \underline{C}\left(\frac{2}{k}\right) .$$

The right hand side is the union of two compact sets.[47] Since $\partial J^+(S)$ is closed (by Proposition 6.21) it is compact (being a closed subset of a compact space).

Since (M, g) is time-oriented, there exists a globally timelike vectorfield T. By assumption M admits a non-compact Cauchy hypersurface Σ. It follows that for each $q \in \partial J^+(S)$, the integral curve of T through q intersect Σ exactly once and it cannot intersect $\partial J^+(S)$ again since $\partial J^+(S)$ is achronal by Proposition 6.21. Therefore, we obtain a continuous injection

$$\psi : \partial J^+(S) \to \Sigma$$

which by Proposition 6.22 (Ingredient 4) is a homeomorphism on its image $N = \psi(\partial J^+(S))$. It follows from Proposition 6.21 that N is a compact, embedded, 3-dimensional topological submanifold (without boundary) in Σ. It follows that N is open in Σ. Being compact it is also closed in Σ. Since Σ is connected we infer $N = \Sigma$ which is contradiction since Σ is non-compact.

6.6 Proof of Proposition 6.21

The topological boundary of a set is by definition closed. To show the achronality, let $x, y \in \partial K$ such that $y \in I^+(x)$. Then by Corollary 2.73 there exists an open neighbourhood U_x of x in M such that $U_x \subset I^-(y)$. Since $x \in \partial K$ there exists a $p \in U_x$ such that $p \in K$ and $y \in I^+(p)$. There also exists an open set U_y of y such that $U_y \subset I^+(p)$. Since $y \in \partial K$ we have[48] $U_y \cap (M \setminus K) \neq \emptyset$. But this contradicts the futureness of K which demands $U_y \subset K$.

The remaining properties of ∂K are *local* notions so we restrict to a normal neighbourhood U_x of a point $x \in \partial K$. More specifically, let $V_x \subset T_xM$ be such that $\exp_x : V_x \to U_x \subset M$ is a diffeomorphism and consider normal coordinates on V_x. Consider the lines $\gamma(t) = (t, c_1, c_2, c_3)$ in V_x, i.e. the integral curves of ∂_t. These are all timelike. Consider also their image $\exp_x(\gamma(t))$. These curves are generally not timelike but since $(\exp_x)_v$ is the identity at $v = 0$ we can choose the neighbourhood V_x sufficiently small such that the curves $(\exp_x)_v$ are timelike.[49] By achronality, these curves intersect ∂K at most once once. We will now show that the curves intersect ∂K at least (hence exactly) once.

To see this note first that for $x \in \partial K$ we have $I^+(x) \subset K$ and $I^-(x) \subset M \setminus K$. To show the former, indeed, if $q \in I^+(x)$, then there is an $x' \in K$ close to x such that $q \in I^+(x')$ and hence $q \in K$ by the fact that K is a future set. A similar

[47] the first being the image of the continuous map $[0, \frac{2}{k}] \times S \to C$ defined by $(\tau, p) \mapsto \gamma_p(\tau)$, the second being the image of the continuous map $[0, \frac{2}{k}] \times S \to \underline{C}$ defined by $(\tau, p) \mapsto \underline{\gamma}_p(\tau)$.

[48] Recall any neighbourhood of a point on the boundary ∂K contains at least one point in K and one point in K^c.

[49] Recall that the tangent to the curve $\exp_x(\gamma(t))$ at t is $(d\exp_x)_{(t,c_1,c_2,c_3)}\partial_t$.

argument can be made for the latter inclusion (exercise). With this established we have the following picture

Obviously, every γ intersects both $I_x^+ \cap V_x$ and $I_x^- \cap V_x$. By Proposition 2.70 every $\exp_x(\gamma)$ intersects $I^+(x) \subset K$ and $I^-(x) \subset M \setminus K$ hence ∂K.

The integral curves can thus feature to construct a local chart for ∂K near x (establishing that it is indeed a 3-dimensional manifold). More precisely we will show that $\exp_x^{-1}(\partial K \cap U_x)$ is a locally Lipschitz manifold without boundary, which in view of $\exp_x$ being a local diffeomorphism will establish this property for $\partial K \cap U_x$ itself.

Since each integral curve γ of ∂_t intersects $\exp_x^{-1}(\partial K \cap U_x)$ exactly once, we have that $\exp_x^{-1}(\partial K \cap U_x)$ is the graph of the function

$$\exp_x^{-1}(\partial K \cap U_x) = \{(t(x^1, x^2, x^3), x^1, x^2, x^3) \mid (x^1, x^2, x^3) \in (\{t = 0\} \cap V_x)\}.$$

We claim that for any $(x^1, x^2, x^3), (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \{t = 0\} \cap V_x$ we have

$$\frac{|t(x^1, x^2, x^3) - t(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)|}{\|(x^1, x^2, x^3) - (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\|} \leq \frac{3}{2}\,, \tag{249}$$

from which it follows that $\exp_x^{-1}(\partial K \cap U_x)$ is the graph of a Lipschitz function. Assuming (249), in summary we have shown that any point x of ∂K has a neighbourhood $\partial K \cap U_x$ homemorphic to $\mathbb{R}^3$ which establishes that ∂K is a 3-dimensional topological manifold without boundary. It remains to prove (249). We sketch the argument: Assume there exist $(x^1, x^2, x^3), (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \{t = 0\} \cap V_x$ with

$$\frac{|t(x^1, x^2, x^3) - t(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)|}{\|(x^1, x^2, x^3) - (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\|} > \frac{3}{2}$$

Then there exists a timelike curve (a timelike line of slope bounded away from 1 in $V_x \subset T_xM$) connecting $(t(x^1, x^2, x^3), x^1, x^2, x^3)$ with $((\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$. Now $(\exp_p)$ maps the timelike line into a timelike curve in M (use that $(\exp_p)_0$ is the identity and choose the neighbourhoods sufficiently small). This contradicts the futureness of ∂K as two points in ∂K would be connected by a timelike curve.

6.7 Proof of Proposition 6.14

to be added (see notes of Christodoulou)

A The Gauss Lemma

Lemma A.1. *Let $p \in M$, $v \in T_pM$, $w \in T_pM \approx T_v(T_pM)$. We have*

$$g_p(v,w) = g_{\exp_p(v)}\left((d\exp_p)_v v, (d\exp_p)_v w\right) \tag{250}$$

Remark A.2. *The equality can be viewed as the exponential map being some kind of a partial isometry. Note for a proper isometry (see Definition 2.58) we would need $g_p(v_1,v_2) = g_{\exp_p(v)}\left((d\exp_p)_v v_1, (d\exp_p)_v v_2\right)$ for all $v_1, v_2 \in T_pM$. Note also that $g_p(v,v) = g_{\exp_p(v)}\left((d\exp_p)_v v, (d\exp_p)_v v\right)$ holds trivially by the fact that the length of the v is preserved along a geodesic*[50]*, which implies that is suffices to prove the lemma for w being orthogonal to v (and that the exponential map preserves orthogonality in the radial directions).*

The following picture is instructive: picture here

Proof. See for instance do Carmo's book. □

B More on global hyperbolicity

Recall that $C(p,q)$ is the space of continuous causal curves from p to q where we identify curves which are continuous reparametrisation of one another. To define a continuous causal curve we make the following definition.[51]

Definition B.1. *The curve $\gamma : I \to M$ is a continuous causal curve if at every point $t \in I$ we can find an interval $\tilde{I}$ around t and a normal neighbourhood U around $\gamma(t)$ such that $\gamma(\tilde{t}) \in J^-(\gamma(t)) \setminus \{\gamma(t)\}$ if $\tilde{t} < t$ and $\gamma(\tilde{t}) \in J^+(\gamma(t)) \setminus \{\gamma(t)\}$ if $\tilde{t} > t$ hold for $\tilde{t} \in \tilde{I}$.*

In other words, *locally* there should be a causal curve in the old sense connecting two points on the curve. An analogous definition could be made for timelike curves replacing $J^\pm$ by $I^\pm$ in the above.

[50] Indeed, $(d\exp_p)_v v = \frac{d}{dt}\exp_p(tv)\big|_{t=1}$ is the tangent vector at the point $\exp_p(1\cdot v)$ to the radial geodesic emanating from p in the direction v.

[51] Recall that so far we have defined the notion of causal curve via the tangent vector, so the standard definition makes sense only for (piecewise (at the corners, the limit of the two tangent vectors should point in the same light cone)) differentiable curves.

We would like to install a topology on $C(p,q)$. Perhaps the cleanest is to declare, for any $U \in M$, the sets

$$O(\mathcal{U}) = \{\lambda \in C(p,q) \mid \lambda \subset U\}$$

to be open in $C(p,q)$ and to form a basis of the desired topology. (Note $O(\mathcal{U})$ consists of all causal curves from p to q entirely contained in $U \subset M$.) Note that $O(\mathcal{U}_1 \cap \mathcal{U}_2) = O(\mathcal{U}_1) \cap O(\mathcal{U}_2)$ so the basis indeed generates a topology.

Alternatively (and equivalently) one can install an auxiliary Riemannian metric on M and define the distance between (the image of) two curves with respect to that Riemannian metric to induce a topology.

Remark B.2. *Note that two curves which are close in this topology can have tangent vectors which are far away, as illustrated by a zig zag curve approximating a timelike curve. One could have entertained the idea of looking at the space $C'(p,q)$ of piecewise differentiable curves and installed a topology where also the tangent vectors are required to be close but the $C'(p,q)$ is not compact!*

We now claim

Theorem B.3. *If (M,g) is globally hyperbolic, then $C(p,q)$ is compact.*

For the following, note that with our definition of globally hyperbolic, any inextendible curve must enter $I^+(\Sigma)$ and $I^-(\Sigma)$. This follows from local considerations at Σ (normal neighbourhoods), which the curve must intersect.

Sketch of proof. We can assume $q \in J^+(p)$ as otherwise $C(p,q)$ is empty.

Step 1. We claim that given $p, q \in M$ and a sequence (λ_n) in $C(p,q)$, there exists a limiting causal curve λ which either has q as an endpoint or is inextendible and does not reach q.

We take a normal ball $B_p(a)$ around p not containing q. We consider the sequence of points $\lambda_n|_{S_p(a)}$, where $S_p(a)$ denotes the (compact) boundary of $B_p(a)$. This sequence has a convergent subsequence and we denote the corresponding limit point by x_1. The corresponding subsequence of (λ_n^1) therefore has x_1 as a convergence point and $x_1 \in J^+(p)$ since $J^+(p)$ is closed for a normal neighbourhood. We now look at all rational balls $B_{\lambda a}(p)$ for $\lambda \in \mathbb{Q} \cap (0,1)$ and repeat the construction. The diagonal sequence λ_n^n has convergence points on all rational balls and the closure of these sets of points produces a continuous causal curve from p to x_1. We then repeat the construction from x_1. This way we either reach q or obtain an inextendible curve which does not reach q.[52]

Step 2. We now consider three cases (with corresponding $(\lambda)_n$ given)

(1) $p \in I^-(\Sigma)$ and $q \in I^-(\Sigma) \cup \Sigma$,

(2) $p \in I^-(\Sigma)$ and $q \in I^+(\Sigma)$,

[52]Indeed, any extendible curve which has not eached q can be extended further according to the above procedure (albeit by potentially smaller and smaller amounts as the normal balls may get smaller), hence we obtain an inextendible limit curve by taking the union of all extensions.

(3) $p \in I^+(\Sigma) \cup \Sigma$ and $q \in I^+(\Sigma)$,

and we show that in each case we can obtain a limiting causal curve from p to q. Since the third case can be treated exactly as the first (exchanging p and q and I^+ and I^-), we will only consider the cases (1) and (2) below.

In case (1), the limit curve cannot enter $I^+(\Sigma)$ (which we recall is open) since none of the $(\lambda)_n$ does. Moreover, if it did not extend all the way to q, then we would have constructed an inextendible causal curve which does not enter $I^+(\Sigma)$ which is impossible.[53]

In case (2), the limit curve λ has to enter $I^+(\Sigma)$, as otherwise we would again have constructed an inextendible causal curve not entering $I^+(\Sigma)$. Pick a point $r \in \{\lambda\} \cap I^+(\Sigma)$ and a subsequence (λ'_n) such that each point on the segment from p to r is a convergence point for the (λ'_n) . Consider now the sequence (λ'_n) as a sequence of past directed causal curve from q to p. The limiting curve λ' must enter $I^-(\Sigma)$ as otherwise we would again produce an intextendible curve not entering $I^-(\Sigma)$ in contradiction with global hyperbolicity. By construction, r is a convergence point for the (λ'_n) and the limiting curve must actually extend to r: If it did not, λ' would be inextendible and every point on λ' would be in $\overline{J^+(r)} \subset \overline{J^+(\Sigma)}$,[54] which contradicts that the curve enters $I^-(\Sigma)$.

We concatenate the restriction of the curve λ from p to r with the restriction of the curve λ' from r to q to obtain a causal limiting curve Λ in $C(p,q)$ as desired. □

Theorem B.4. *Let (M,g) be globally hyperbolic. Then $J^+(p) \cap J^-(q)$ is compact in the manifold topology.*

Proof. Take a sequence (p_n) in $J^+(p) \cap J^-(q)$. To prove the claim, we need to extract a subsequence converging to a $p \in J^+(p) \cap J^-(q)$.

Take a sequence (λ_n) in $C(p,q)$ with p_n on λ_n. By Theorem B.3 we obtain a subsequence (λ'_n) converging to a limiting curve $\lambda \in C(p,q)$. We consider λ as a compact set in M and cover it by finitely many open sets with compact closure to obtain $\lambda \subset \mathcal{U} \subset \overline{\mathcal{U}}$ with $\mathcal{U}$ open with compact closure. Since the (λ'_n) converge to λ, there is an $N \in \mathbb{N}$ such that $\lambda'_n \subset \mathcal{U}$ for all $n \geq N$. Hence (p'_n) for $n \geq N$ is a sequence in the compact set $\overline{\mathcal{U}}$ and we extract a convergent subsequence converging to a $p \in \mathcal{U}$. Clearly, p must lie on λ (if it didn't, p would have a neighbourhood disjoint form λ, which contains infinitely many (p'_n) in contradiction with the corresponding curves (λ'_n) to converge to λ) and hence $p \in J^+(p) \cap J^-(q)$ is the desired limit point. □

Theorem B.5. *Let (M,g) by globally hyperbolic. Then, if $p, q \in M$ are such that $p \in I^-(q)$, there exists a timelike geodesic from p to q that maximises the proper time.*

[53] We are using here that since any inextendible curve intersects Σ exactly once, we must have that any such curve enters bot $I^+(\Sigma)$ and $I^-(\Sigma)$. This can for instance be shown by normal neighbourhood considerations at Σ.

[54] Since $r \in I^+(\Sigma)$ is open....

Sketch of proof. Step 1. Let $\tilde{C}(p,q)$ be the space of C^1 timelike curves from p to q equipped with the $C(p,q)$ topology. We claim that $\tilde{C}(p,q)$ is dense in $C(p,q)$: Any continuous causal curve can be arbitrarily well approximated in the $C(p,q)$ topology by a C^1 timelike curve. (Exercise: Can you show this?)

Step 2. The proper time $\tau[\lambda]$ is a function on $\tilde{C}(p,q)$ but it is not continuous (recall the zig-zag curves). However, we can show that $\tau[\lambda]$ is upper semicontinuous on $\tilde{C}(p,q)$.[55] We can then extend the function $\tau[\lambda]$ as an upper semicontinuous function to $C(p,q)$ as follows. Given a $\lambda \in C(p,q)$ we define

$$\tau[\lambda] = \inf_{\mathcal{U} \text{ containing } \lambda} \sup_{\mu \in \tilde{C}(p,q) \cap \mathcal{U}} \tau[\mu] \,. \tag{251}$$

Step 3. Now $\tau[\lambda]$ is an upper semicontinuous function of the compact set $C(p,q)$ and hence assumes a maximum $\tau[\lambda]$. This maximum is positive since $p \in I^-(q)$, so there is a curve of positive proper time from p to q.

Step 4. We have shown the existence of a maximum length curve λ and it remains to show that λ is actually a geodesic.

If the limiting λ was C^1, this would follow by assuming (for contradiction) that λ was not a geodesic segment near $p \in \lambda$ and to then connect, in a small convex normal neighbourhood around p, two points on λ by the geodesic segment between them to produce a curve strictly longer than λ (since any curve which is not a reparametrisation of the geodesic segment has strictly longer length, cf. Proposition 2.74). Here we need to show in addition that competing curves that are merely continuous also have strictly smaller length. It is clear that is suffices to establish this in a convex normal neighbourhood. So let γ be the geodesic segment from r to s (with $s \in I^+(r)$) in a convex normal neighbourhood and μ be a competing continuous causal curve of maximal length from r to s. We have $\tau[\gamma] \geq \tau[\mu]$ because all C^1-curves approximating μ must have shorter or equal length that γ (by the maximising property in convex normal neighbourhoods) hence the inequality must hold for μ itself. We need to exclude the equality case. So let us assume that $\gamma \neq \mu$ but $\tau[\gamma] = \tau[\mu]$ and pick a point w on μ not on γ. Consider the piecewise C^1-curve from r to s consisting of the geodesic segments from r to w and from w to s. By the previous argument we have $\tau[\gamma_1]+\tau[\gamma_2] \geq \tau[\mu]$ which using $\tau[\gamma] = \tau[\mu]$ combines to give $\tau[\gamma_1]+\tau[\gamma_2] \geq \tau[\gamma]$. However, that is a contradiction with the fact that γ is strictly longer than any piecewise smooth curve from r to s that is not a reparametrisation of γ (cf. Exercise 2.75). □

[55] See Sheet 7: Given $\lambda \in \tilde{C}(p,q)$, for every $\epsilon > 0$ there exists a neighbourhood $\mathcal{U}$ with $\lambda \subset \mathcal{U}$ such that $\tau(\tilde{\lambda}) \leq \tau(\lambda) + \epsilon$. In other words nearby curves in the $C(p,q)$ topology can have arbitrarily small length but they cannot be much longer.

C The Hopf-Rinow Theorem

D A heuristic derivation of the Einstein equations

E Further topics

We mention some further topics that could be added to the notes in the future

1. A more detailed discussion of variational principles
2. A more thorough discussion of the Lorentzian geometry bits entering the global part of Theorem 4.18.
3. Proof of the equivalence of the different notions of globally hyperbolicity
4. Proof of the existence of a maximal globally hyperbolic development
5. Proper discussion of the theory of Gaussian beams
6. The Einstein Equations in spherical symmetry (including a proof of Birkhoff's theorem)