Non-linear Wave Equations

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Chapter 1

Introduction

1.1 Motivation and Overview

The linear wave equation on flat space constitutes the first partial differential equation (PDE) that was ever written down (d'Alembert, 1749). The unknown is a scalar function $\phi : I \times \mathbb{R}^n \to \mathbb{R}$ (where $I \subset \mathbb{R}$ is a (time-)interval) satisfying

$$\Box\phi := -\partial_t^2\phi + c^2 \sum_{i=1}^{n} \partial_{x^i}^2 \phi = 0\,, \tag{1.1}$$

with $c > 0$ a constant corresponding to the speed of propagation of the wave. We shall set from now on $c = 1$ which can always be achieved by rescaling the space coordinates. We will also set $x^0 = t$ and use the shorthand $\partial_0 = \partial_t$ and $\partial_i = \partial_{x^i}$.[1]

Equation (1.1) is a second order, *linear* PDE. As suggested by the use of the variable t, this equation is an *evolution equation*, i.e. (as we shall see momentarily) the mathematically appropriate way to study the wave equation is in terms of the Cauchy problem:

$$\begin{cases} \Box\phi = 0 \\ \phi(0,x) = f(x) \\ \partial_t\phi(0,x) = g(x) \end{cases} \tag{1.2}$$

where f and g are prescribed functions of a certain regularity. Already this week, we shall prove existence, uniqueness and continuous dependence on the data (**Hadamard well-posedness**) for the above Cauchy problem in various regularity classes (analytic, smooth, Sobolev) using different techniques. For instance:

Theorem 1.1.1. *For f, g smooth there exists a unique smooth solution $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ to (1.2).*

This will be relatively straightforward since we can obtain *explicit* representation formulae for the solutions in terms of the data. Only basic knowledge from calculus and (later) the notion of the Fourier transform is required.

From these representation formulae we shall distill fundamental analytic and geometric properties of solutions to the wave equation: The **domain of dependence property** (which is related to the finite speed of propagation) and the global **dispersion of waves** (pointwise decay of ϕ at $t \to \infty$). We shall also see that **energy estimates** (leading to the notion of energy conservation for (1.1)) provide a way to capture these and other properties in a more robust (and less explicit) way allowing us to establish generalisations of the above properties for more general wave equations.

Now, what do we mean by "more general wave equations"? There are different ways to approach that question. One pedestrian way is to consider (say linear) "by hand" perturbations of (1.1), e.g.

$$-\partial_t^2\phi + \sum_{i,j=1}^{n} a^{ij}(t,x)\partial_{x^i}\partial_{x^j}\phi + \sum_{\mu=0}^{n} b^{\mu}(t,x)\partial_{x^\mu}\phi = 0$$

[1] Also, Latin indices typically range from $i = 1, ..., n$ while Greek indices range from $i = 0, ..., n$. More on this below.

and ask whether given appropriate assumptions on the matrix a^{ij} and the vector b^μ, the Cauchy problem is well-posed and whether the above properties hold. Similarly, one could also consider "by hand" non-linear perturbations such as

$$\Box\phi = (\partial_t\phi)^2 \tag{1.3}$$

or

$$\Box\phi = (\partial_t\phi)^2 - |\nabla\phi|^2 \tag{1.4}$$

and of course combinations thereof and ask similar questions about the local and global behaviour of solutions. However, it pays off both to think a bit more geometrically and to use physics as a guiding principle.

In this course, a general scalar wave equation for an unknown $\phi : I \times \mathbb{R}^n \to \mathbb{R}$ takes the form

$$\sum_{\mu,\nu=0}^{n} \frac{1}{\sqrt{-\det g}}\partial_\mu \left(\sqrt{-\det g}\left(g^{-1}\right)^{\mu\nu}\partial_\nu\phi\right) = F\left(\phi, \partial\phi\right) , \tag{1.5}$$

where g is a Lorentzian metric on $I \times \mathbb{R}^n$ and $F : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a function. We say that g is a Lorentzian metric if it is a symmetric $(n+1)\times(n+1)$ matrix (thus having real eigenvalues) with eigenvalues $\lambda_0 < 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. It thus makes sense to divide by $\sqrt{-\det g}$ and also define the inverse of g.

We say that the equation (1.5) is linear if g is independent of ϕ and F is a linear function of both of its arguments; the equation is semilinear if g is independent of ϕ and F is a nonlinear function; and the equation is quasilinear if g is a function of ϕ and or $\partial\phi$.

If you have never seen (1.5) before the expression might look very cumbersome and artificial.[2] However, (for those who know) the left hand side is precisely the covariant wave operator associated with the Lorentzian metric g, which can also be written $\left(g^{-1}\right)^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0$, which is an operator that appears in many geometric applications. One also easily sees that it includes all the "by hand" generalisations above.

You might rightfully expect that there are no explicit representation formulae for (1.3) and (1.4), let alone for the general scalar wave equation (1.5). It is here were functional analytic methods enter and where we need the theory of L^2-based Sobolev spaces, the Hahn-Banach theorem and Banach's fixed point theorem to prove (local in time) existence of solutions to general wave equations. Our strategy will be roughly as follows

1. Develop a robust theory for general linear equations.
2. Understand the general local in time theory for non-linear equations.
3. Understand the global in time theory of non-linear equations.

The global behaviour of non-linear wave equations can be very different from their linear counterparts. In particular, solutions do not necessarily exist globally in time but can form singularities. To give you a taste, we shall prove the following two complementary theorems:

Theorem 1.1.2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth and compactly supported. Then, for $\epsilon > 0$ sufficiently small, the Cauchy problem*

$$\Box\phi = (\partial_t\phi)^2 - |\nabla\phi|^2 \quad , \quad \phi(0,x) = \epsilon f(x) \quad , \quad \partial_t\phi(0,x) = \epsilon g(x)$$

has smooth global in time solutions.

Theorem 1.1.3. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth, compactly supported and not identically zero. Then the Cauchy problem*

$$\Box\phi = (\partial_t\phi)^2 \quad , \quad \phi(0,x) = f(x) \quad , \quad \partial_t\phi(0,x) = g(x)$$

does not have global in time (C^3)-solutions.

[2] It is also not immediately clear how to formulate the Cauchy problem in this generality!

Clearly, there is a structural property on the non-linearity that distinguishes the two results and our goal will be to understand this condition (it is the so-called **null-condition**). Let me also emphasise that the point here is not the *particular* equations appearing in the theorems but to understand large classes of non-linearities which lead to similar global behaviour. Because of its very special algebraic structure, one can actually prove Theorem 1.1.3 in a few lines and you'll be able to do so in about a week's time using the transformation $\psi = e^{\phi} - 1$, which transforms the non-linear equation into a linear one. This is a happy coincidence, however!

The Cauchy problems in the above theorems are toy-problems for the more fundamental Euler equations (describing fluid mechanics) and the Einstein equations of general relativity. At the end of the course we shall make this connection and address some "real world" problems.

1.2 Prerequisites

The course will require a good knowledge in basic calculus and ODE theory. Furthermore, I will assume that you have seen L^p spaces and also the L^2-based Sobolev spaces, as well as the notion of the Fourier transform. If you haven't, please let me know. Some elementary properties as well as the basic Sobolev embedding theorems will also be recalled in the examples classes. Riemannian geometry will be helpful for the later parts of the course but it is certainly not required.

1.3 Books and References

I will follow (sometimes closely, sometimes in spirit) the Stanford lecture notes of Jonathan Luk. We will move at a slightly slower pace and spend a bit more time on the linear theory. I am planing to post lecture notes accompanying the course.

For the first 2-3 weeks I can also recommend the book "Partial Differential Equations" by Fritz John (Springer) covering the standard theory of the linear wave equation.

Chapter 2

The Linear Wave Equation

2.1 Dimension $1+1$: d'Alembert's formula

We consider the Cauchy problem

$$\phi_{tt} - \phi_{xx} = 0 \quad \text{with data} \quad \phi\left(0,x\right) = f\left(x\right) \quad \phi_t\left(0,x\right) = g\left(x\right) . \tag{2.1}$$

It is easily checked that $\phi(t,x) = \varphi\left(x+t\right) + \psi\left(x-t\right)$ solves $\phi_{tt} - \phi_{xx} = 0$ for any C^2 functions φ and ψ. To realize the given data we need

$$\psi\left(x\right) + \varphi\left(x\right) = f\left(x\right) \tag{2.2}$$

$$-\psi'\left(x\right) + \varphi'\left(x\right) = g\left(x\right) \tag{2.3}$$

Differentiating (2.2) and inverting the linear system we obtain

$$\psi' = \frac{f'}{2} - \frac{g}{2} \quad \text{and} \quad \varphi' = \frac{f'}{2} + \frac{g}{2} \tag{2.4}$$

Choosing a G with $G' = g$ we obtain the expressions

$$\psi = \frac{f}{2} - \frac{G}{2} + a \quad \text{and} \quad \varphi = \frac{f}{2} + \frac{G}{2} + b \tag{2.5}$$

On the other hand, we have $a + b = 0$ by (2.2) such that

$$\phi\left(x,t\right) = \frac{f\left(x-t\right) + f\left(x+t\right)}{2} + \frac{1}{2}\left(G\left(x+t\right) - G\left(x-t\right)\right)$$

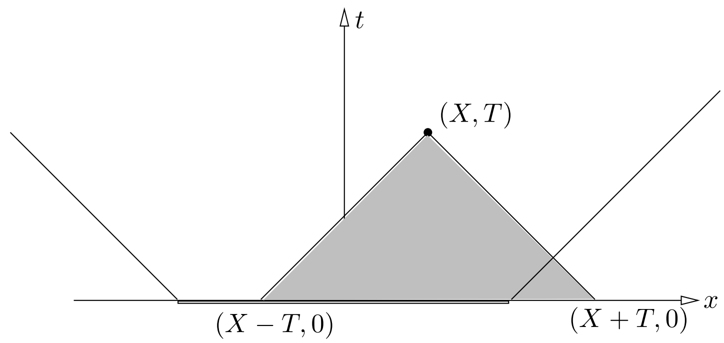
which we can write as

$$\phi\left(x,t\right) = \frac{f\left(x-t\right) + f\left(x+t\right)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g\left(s\right) ds \tag{2.6}$$

From (2.6) we immediately deduce the existence part of

Proposition 2.1.1. *For any given $f \in C^2\left(\mathbb{R}\right)$ and $g \in C^1\left(\mathbb{R}\right)$, there is a unique solution $\phi \in C^2\left(\mathbb{R}_t \times \mathbb{R}_x\right)$ of the Cauchy problem (2.1).*

Before we conclude the uniqueness, let us also interpret the formula (2.6) geometrically and explain the concept of domain of dependence (and domain of influence):



The domain of dependence of a set $K \subset \mathbb{R}$ is the set $\{(t,x) \mid \text{dist}(x, \mathbb{R} \setminus K) > t\}$ in spacetime. (It is the set of points that cannot be reached from $\mathbb{R} \setminus K$ by lines moving at most at speed one, so points in $\mathbb{R} \setminus K$ cannot influence the solution in the domain of dependence or put differently, the solution in the domain of dependence is entirely determined by the Cauchy data on K.)

We now establish the uniqueness. In view of the linearity of the equation, it suffices to show that the Cauchy problem with zero data has only the zero solution in the class of C^2 solutions (why?). We will in fact show that if the Cauchy data is zero on the interval $[a,b]$, then the solution vanishes identically in its domain of dependence $D_{[a,b]} = \{(t,x) \mid -b \leq t - x \leq -a \quad , \quad a \leq t + x \leq b\}$.

This in turn is immediate after introducing so-called null coordinates $u = t - x$, $v = t + x$ in which the wave equation reads $\partial_u \partial_v \phi = 0$. For zero Cauchy data we have in particular $\partial_u \phi(0,x) = 0$ and $\partial_v \phi(0,x) = 0$ (why?). This means that by ODE theory we have both $\partial_u \phi = 0$ and $\partial_v \phi = 0$ in the region D, hence $\phi = 0$ in D since ϕ vanishes on $[a,b]$.

2.2 Dimension $3+1$: Kirchhoff's formula

The dimension $3+1$ is of course the physically interesting one. Other dimensions will be discussed on the first example sheet. We consider the Cauchy problem

$$\phi_{tt} - \Delta\phi = 0 \quad , \quad \phi(0,x) = f(x) \quad , \quad \phi_t(0,x) = g(x) \,. \tag{2.7}$$

For $h(x) = h(x_1, \ldots, x_n)$ continuous from $\mathbb{R}^n \to \mathbb{R}$ we define

$$M_h(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} h(y) \, dS_y \,, \tag{2.8}$$

the average over a sphere of radius r around x (ω_n denoting the area element of the unit-sphere in n dimensions). Writing $y = x + r\xi$ with $|\xi| = 1$ we have

$$M_h(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r\xi) \, dS_\xi \,.$$

We can extend $M_h(x,r)$ to an *even* function defined for all real r (change $\xi \to -\xi$). Observe also that $h \in C^k(\mathbb{R}^n)$ implies $M_h \in C^k\left(\mathbb{R}^{n+1}\right)$.

Now, for $h \in C^2(\mathbb{R}^n)$ we find using the divergence theorem the identity

$$\left[\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right] M_h(x,r) = \Delta_x M_h(x,r) \tag{2.9}$$

This is known as the *Darboux* equation. Note the "initial conditions", $M_h(x,0) = h(x)$ and $\partial_r M_h(x,r)\Big|_{r=0} = 0$ since M_h is even in r.

To derive (2.9), note

$$\begin{aligned}
\partial_r M_h(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} \xi\cdot\nabla_y h(x+r\xi)\, dS_\xi = \frac{1}{\omega_n r}\int_{|\xi|=1} \xi\cdot\nabla_\xi h(x+r\xi)\, dS_\xi \\
= \frac{1}{\omega_n r}\int_{|\xi|<1} \Delta_\xi h(x+r\xi)\, d\xi = \frac{r}{\omega_n}\int_{|\xi|<1} \Delta_x h(x+r\xi)\, d\xi \\
= \frac{r}{\omega_n}\Delta_x \int_0^1 d\rho\rho^{n-1}\int_{|\xi|=1} h(x+r\rho\xi)\, dS_\xi = r\Delta_x \int_0^1 d\rho\rho^{n-1}\ M_h(x,r\rho)
\end{aligned}$$

and after a further change of variables

$$\partial_r M_h(x,r) = r^{-n+1}\Delta_x \int_0^r d\rho\ \rho^{n-1} M_h(x,\rho)\ , \tag{2.10}$$

which yields

$$\partial_r\left(r^{n-1}\partial_r M_h(x,r)\right) = \Delta_x\left(r^{n-1}M_h(x,r)\right) \tag{2.11}$$

and hence (2.9).

The idea to solve (2.7) is to write down an equation for the spherical means of ϕ. This will be a $1+1$ dimensional PDE which we can solve explicitly. Conversely, we shall be able to recover the solution from its spherical means.

We define

$$M_\phi(x,r,t) = \frac{1}{\omega_n}\int_{|\xi|=1} \phi(t,x+r\xi)\, dS_\xi \tag{2.12}$$

Note that $\phi(t,x) = M_\phi(x,0,t)$, recovering ϕ from its means. Now by the wave equation and the Darboux equation we know that we have

$$\partial_t^2 M_\phi = \Delta_x M_\phi = \left[\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right] M_\phi\,.$$

Let us restrict to $n=3$. In this case we have

$$\frac{\partial^2}{\partial t^2}(rM_\phi) = \frac{\partial^2}{\partial r^2}(rM_\phi)\ .$$

Hence we have that rM_ϕ satisfies the $1+1$ dimensional wave equation with initial values:

$$rM_\phi = rM_f(x,r) \quad \text{and} \quad \partial_t(rM_\phi) = rM_g(x,r) \quad \text{at } t=0 \tag{2.13}$$

and hence d'Alembert's formula from Section 2.1 applies. Therefore,

$$rM_\phi(x,r,t) = \frac{1}{2}\left[(r+t)M_f(x,r+t) + (r-t)M_f(x,r-t)\right] + \frac{1}{2}\int_{r-t}^{r+t} \rho M_g(x,\rho)\, d\rho\,. \tag{2.14}$$

Dividing by r and taking the limit $r\to 0$ (Exercise – use that $M_g(x,r)$ is even!) we finally find

$$\phi(t,x) = tM_g(x,t) + \partial_t(tM_f(x,t)) = \frac{1}{4\pi t}\int_{|y-x|=t} g(y)\, dS_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=t} f(y)\, dS_y\right)\ . \tag{2.15}$$

We conclude:

Proposition 2.2.1. *Any solution ϕ of the initial value problem (2.7) which is C^2 for $t\geq 0$ in $n=3$ space dimensions is given by formula (2.15) and is hence unique. Conversely, given $f\in C^3(\mathbb{R}^3)$ and $g\in C^2(\mathbb{R}^3)$ the $\phi(t,x)$ defined by the above formula is C^2 and satisfies (2.7).*

Note the loss of regularity in the statement! To make this loss more manifest, we compute

$$\partial_t \left(tM_f\left(x,t\right)\right) = M_f\left(x,t\right) + t\partial_t \left(\frac{1}{\omega_n}\int_{|\xi|=1} f\left(x+t\xi\right) dS_\xi\right)$$
$$= M_f\left(x,t\right) + t\frac{1}{\omega_n}\int_{|\xi|=1} Df\left(x+ct\xi\right)\cdot \frac{y-x}{t}\, dS_\xi \tag{2.16}$$

and obtain

$$\phi\left(t,x\right) = \frac{1}{4\pi t^2}\int_{|y-x|=t}\left(tg\left(y\right)+f\left(y\right)+\sum_i f_{y_i}\left(y_i-x_i\right)\right)dS_y\,. \tag{2.17}$$

Besides the aforementioned loss of regularity, which you will examine more closely in the exercises, many more things can be read off from the above formula (2.15). One is the *strong Huygens' principle*. The solution at a point $(x, t > 0)$ depends only on the "data" of the surface $S(x,t)$ the intersection of the past light cone with the hypersurface $t = 0$. (Draw a picture!) It does not depend on the values inside the ball $B(x,t)$. This sharp propagation of signals is special for the wave equation in odd spatial dimensions (with the exception $n = 1$). In even dimensions one only has the weak Huygens' principle (value at (t,x) depends on the values in the entire ball $B(x,t)$) and for most hyperbolic equations one also only has the weak form. Conversely, the data near a point y on the initial hyperplane $t = 0$ only influence the solution at points (t,x) near the cone $|x-y| = t$ emanating from y.

Finally, the formula (2.15) allows us to show that the solution decays in time (we already know it spreads over larger and larger regions of space).

Proposition 2.2.2. *For $f, g \in C_0^\infty(\mathbb{R}^3)$ and ϕ the unique solution of (2.7) there exists a constant $C = C(f,g)$ such that*

$$\sup_x |\phi(t,x)| \leq \frac{C}{1+t}$$

holds for all $t \geq 0$.

Proof. We have the formula

$$\phi\left(t,x\right) = \frac{1}{4\pi t^2}\int_{|y-x|=t}\left(tg\left(y\right)+f\left(y\right)+\sum_i f_{y_i}\left(y_i-x_i\right)\right)dS_y\,, \tag{2.18}$$

hence

$$|\phi\left(t,x\right)| \leq \frac{1}{4\pi t}\int_{|y-x|=t}\left(|g\left(y\right)|+\frac{1}{t}|f\left(y\right)|+\sum_i |\partial_{y_i} f|\right)dS_y\,. \tag{2.19}$$

For $t \in [0,1]$ we use that $\int_{|y-x|=t} 1 dS_y = 4\pi t^2$ hence $|\phi(t,x)| \leq \sup_y |g(y)| + \sup_y |f(y)| + \sum_{i=1}^3 \sup_y |\partial_{y_i} f(y)|$ and the right hand side is bounded by a constant C from the assumption of compact support (the size depending on the size of the support). For larger t note that in view of the compact support of f and g we have $\text{Area}\left(\{|y-x| = t\} \cap \text{supp } f \cap \text{supp } g\right) \leq C$, so the integration is over a uniformly (in $t > 1$ and x) bounded set. The estimate now follows as before taking out the supremum of the integrand. □

The compact support assumption can be dropped and, as we shall see, replaced by sufficiently fast decay towards infinity. However, some condition on the decay has to be imposed:

Exercise 1. *Construct smooth solutions of (2.7) with $|f| \to 0$, $|g| \to 0$ as $|x| \to \infty$ but such that the corresponding solution satisfies $\sup_x |\phi(t,x)| \to \infty$ as $t \to \infty$. [Hint: Construct spherically symmetric solutions using the setup of Problem 2 from the first example sheet.]*

An important goal is to establish decay estimates like the one of Proposition 2.2.2 for more general (possibly non-linear) wave equations. This will require more stable methods than the explicit solution formula derived above and we will develop these during the course.

2.3 Remark on general dimensions

There are analogues of Proposition 2.2.1 in all dimensions with similar proofs (i.e. deriving an explicit representation formula). You will discuss the case $n = 2$ on the first example sheet. In general one has

- $n \geq 3$ odd: $f \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$, $g \in C^{\frac{n+1}{2}}(\mathbb{R}^n) \implies$ Cauchy problem has a unique solution $u \in C^2\left(\mathbb{R}^{1+n}\right)$,
- $n \geq 2$ even: $f \in C^{\frac{n+4}{2}}(\mathbb{R}^n)$, $g \in C^{\frac{n+2}{2}}(\mathbb{R}^n) \implies$ Cauchy problem has a unique solution $u \in C^2\left(\mathbb{R}^{1+n}\right)$.

From the generalised representation formulae one also derives the analogue of Proposition 2.2.2 in general dimensions. Here the estimates get replaced by $\sup_x |\phi(t,x)| \leq \frac{C}{(1+t)^{\frac{n-1}{2}}}$ in dimension n. See also Section 2.6.3 below.

2.4 Duhamel's Principle

Consider the inhomogeneous wave equation with trivial data

$$\phi_{tt} - \Delta\phi = f(t,x) \quad \text{ in } (0,\infty)\times\mathbb{R}^n \quad , \quad \phi(0,x) = 0 \ \ , \ \ \phi_t(0,x) = 0\,. \tag{2.20}$$

We define $\tilde{\phi} = \tilde{\phi}(t,x;s)$ for $s \geq 0$ to be the solution of

$$\tilde{\phi}_{tt}(\cdot;s) - \Delta\tilde{\phi}(\cdot;s) = 0 \qquad \text{in } (s,\infty)\times\mathbb{R}^n\,, \tag{2.21}$$

$$\tilde{\phi}(t=s,x;s) = 0 \ \ , \ \ \tilde{\phi}_t(t=s,x;s) = f(s,x) \qquad \text{for } \{t=s\}\times\mathbb{R}^n\,. \tag{2.22}$$

Now set

$$\phi(t,x) := \int_0^t \tilde{\phi}(t,x;s)\,ds \quad \text{for} \quad x \in \mathbb{R}^n \ \ \textit{and} \ \ t \geq 0\,. \tag{2.23}$$

We claim this is a solution of the problem (2.20) and prove it explicitly for $n = 3$:

Proposition 2.4.1. *Let $n = 3$ and $f \in C^2(\mathbb{R}^n \times [0,\infty))$. Then ϕ defined by (2.23) solves (2.20).*

Proof. Note first that by our well-posedness result for the homogeneous wave equation, the $\tilde{\phi}(t,x;s)$ are well-defined and C^2 in all its arguments for $0 \leq s \leq t$ (why?). Hence $\phi(t,x)$ is $C^2([0,\infty)\times\mathbb{R}^n)$. Clearly, the trivial initial conditions of (2.20) are also satisfied. To see that it also solves the inhomogenous wave equation we compute

$$\phi_t(t,x) = \tilde{\phi}(t,x;t) + \int_0^t \tilde{\phi}_t(t,x;s)\,ds = \int_0^t \tilde{\phi}_t(t,x;s)\,ds\,, \tag{2.24}$$

$$\phi_{tt}(t,x) = \tilde{\phi}_t(t,x;t) + \int_0^t \tilde{\phi}_{tt}(t,x;s)\,ds = f(t,x) + \int_0^t \tilde{\phi}_{tt}(t,x;s)\,ds\,. \tag{2.25}$$

Combining this with

$$\Delta\phi(x,t) = \int_0^t \Delta\tilde{\phi}(x,t;s)\,ds \tag{2.26}$$

yields the result. □

The identical argument works in any dimension and the statement of the proposition holds with appropriate regularity assumptions on f for all n. However recall we have only explicitly proven an existence result for the homogeneous problem for $n = 1$ and $n = 3$, which is why I have only claimed it for $n = 3$. In the $n = 1$ case we note that the solution to (2.21) is given explicitly by

$$\tilde{\phi}(t,x;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(s,\bar{x})d\bar{x} \tag{2.27}$$

and hence the solution to the inhomogeneous problem (2.20) by

$$\phi(t,x) = \frac{1}{2}\int_0^t ds \int_{x-(t-s)}^{(x+(t-s)} f(s,\bar{x})d\bar{x}\,. \tag{2.28}$$

This is precisely an integral over the past light cone of the inhomogeneity f.

Finally, if the data in (2.20) is non-trivial, we just add to (2.28) the unique solution of the homogeneous problem arising from this data. By linearity the sum provides a solution to the inhomogeneous problem assuming the prescribed data and by energy estimates (or otherwise) there can only be one solution to this problem.

2.5 The energy estimate

Let us assume that we have a classical C^2 solution of $\Box\phi = 0$ on $[0,T]\times\mathbb{R}^n$ with "data" $\phi\,(0,x) = f\,(x)$ and $\partial_t\phi\,(0,x) = g\,(x)$. Multiplying the wave equation by $-\partial_t\phi$ yields

$$0 = -\Box\phi\cdot\partial_t\phi = \frac{1}{2}\partial_t\left(\partial_t\phi\right)^2 - \nabla_x\left(\partial_t\phi\nabla_x\phi\right) + \nabla_x\partial_t\phi\cdot\nabla_x\phi = 0\,, \tag{2.29}$$

which can be rearranged to

$$0 = \frac{1}{2}\partial_t\left[\left(\partial_t\phi\right)^2 + |\nabla_x\phi|^2\right] - \nabla_x\left(\partial_t\phi\nabla_x\phi\right)\,. \tag{2.30}$$

If we integrate this over the spacetime slab $[0,T]\times\mathbb{R}^n$, then assuming that ϕ is of compact support (or decays sufficiently rapidly near infinity) we would obtain the energy conservation law

$$\int_{t=T} d^nx\left[\left(\partial_t\phi\right)^2 + |\nabla_x\phi|^2\right] = \int_{t=0} d^nx\left[\left(\partial_t\phi\right)^2 + |\nabla_x\phi|^2\right]$$

As this works for any $\tau\leq T$ we obtain

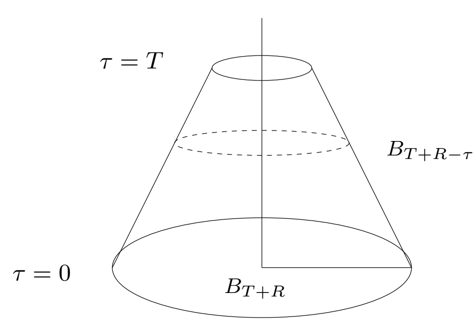
$$\|\partial_t\phi\,(\tau,\cdot)\,\|_{L^2_x} + \|\phi\,(\tau,\cdot)\,\|_{\mathring{H}^1_x} = \|g\|_{L^2} + \|f\|_{\mathring{H}^1}\,. \tag{2.31}$$

In order to derive this identity we have assumed that ϕ is C^2 and that it vanishes sufficiently rapidly near spatial infinity in order to make the boundary term arising from $\nabla_x\left(\partial_t\phi\nabla_x\phi\right)$ vanish. We will now see that we can do much better if we suitably localize the estimate.

Fix $T>0$, $R>0$ and consider a region

$$K = \bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau} \tag{2.32}$$

where $B_{R+T-\tau}$ is the ball of radius $R+T-\tau$ centred at the origin.



You may think of this region as a cut-off (at $t=0$ and $t=T$) past light cone with tip at $\left(T+R,\vec{0}\right)$. We will denote the boundary of $B_{R+T-\tau}$ in $\mathbb{R}^n$ by $S_{R+T-\tau}$ and the unit outward normal to this boundary by N.

Integrating (2.30) over the region K then yields

$$\begin{aligned} &\frac{1}{2}\int_{\{t=T\}\times B_R} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \\ &+ \int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}|\nabla_x\phi|^2 - \partial_t\phi\cdot N\phi\right] d\sigma_{S_{R+T-\tau}} \\ &= \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] . \end{aligned} \tag{2.33}$$

It is not hard to see using Cauchy-Schwarz that the integrand in the second line is non-negative. We can actually obtain something more quantitative. Let us denote the induced gradient on the spheres $S_{R+T-\tau}$ by $\not\nabla$ (i.e. the derivatives *tangent* to these $n-2$ dimensional spheres). We may decompose

$$\partial_t = N + V ,$$

where V is a derivative tangent to the wall of the cone[1] Then, from the easily verified identities

$$\begin{aligned} -\partial_t\phi N\phi &= -(N\phi)^2 - N\phi\cdot V\phi , \\ \frac{1}{2}\partial_t\phi\partial_t\phi &= \frac{1}{2}(N\phi)^2 + N\phi\cdot V\phi + \frac{1}{2}(V\phi)^2 , \\ \frac{1}{2}|\nabla_x\phi|^2 &= \frac{1}{2}(N\phi)^2 + \frac{1}{2}|\not\nabla\phi|^2 , \end{aligned}$$

we see that (2.33) becomes

$$\begin{aligned} &\frac{1}{2}\int_{\{t=T\}\times B_R} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \\ &+ \int_0^T dt \int_{\{\tau\}\times S_{R+T-\tau}} \left[\frac{1}{2}(V\phi)^2 + \frac{1}{2}|\not\nabla\phi|^2\right] d\sigma_{S_{R+T-\tau}} \\ &= \frac{1}{2}\int_{\{t=0\}\times B_{R+T}} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] . \end{aligned} \tag{2.34}$$

This identity is truly remarkable and illustrates the domain of dependence property of the wave equation. Indeed, we certainly have

$$\int_{\{t=T\}\times B_R} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \le \int_{\{t=0\}\times B_{R+T}} d^n x \left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] \tag{2.35}$$

and hence

Corollary 2.5.1. *Suppose $\phi = 0 = \partial_t\phi$ in $\{t=0\}\times B_{R+T}$. Then $\phi = 0$ in $\bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau}$.*

Corollary 2.5.2. *Two C^2 solutions ϕ and ψ in $K = \bigcup_{\tau\in[0,T]}\{\tau\}\times B_{R+T-\tau}$ that satisfy $\phi = \psi$ and $\partial_t\phi = \partial_t\psi$ on $\{t=0\}\times B_{R+T}$ have to agree in all of K.*

Let us understand a bit better the underlying geometry of this computation. The expression (2.30) is apparently a boundary term and it will induce different expressions dependent on the geometry of the boundary hypersurfaces. What is useful in the estimates is if the expressions induced are non-negative, as it was the case for the hypersurfaces of constant t and the null hypersurfaces (the wall of the truncated cone; see the remark below) discussed above. More generally, we define

Definition 2.5.3. *We will call a smooth hypersurface S spacelike if it can be represented locally as $f = 0$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth satisfying*

$$-(\partial_t f)^2 + |\nabla_x f|^2 < 0 . \tag{2.36}$$

We call it timelike if "$<$" above is replaced by "$>$" and null if "$<$" is replaced by "$=$".

[1] In polar coordinates $\partial_t = \partial_r + (\partial_t - \partial_r)$ since the wall of the cone is given by zero set of $H(t, x_1, \ldots, x_n) = t + \sqrt{x_1^2 + \ldots x_n^2} - R - T = t - T + r - R$, so that indeed $(\partial_t - \partial_r) H = 0$.

Remark 2.5.4. *For those familiar with Minkowski geometry and its inner-product* $< x, y >= -x_0 y_0 + \sum_{i=1}^n x_i y_i$ *(giving rise to a notion of timelike, spacelike and null vectors depending on the sign of* $< x, x >$*), note that the expression (2.36) is precisely the Minkowski "norm" of the gradient of* f*. Hence a hypersurface is spacelike if its normal vector is timelike etc.*

Exercise 2. *Obtain the energy estimate for two homologous spacelike hypersurfaces* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, i.e. spacelike hypersurfaces with common boundary* $\partial\mathcal{S}_1 = \partial\mathcal{S}_2$ *bounding a region. Hint: Observe that integrating (2.30) over a spacelike hypersurface produces after using (2.36) a non-negative expression controlling all derivatives.*

It is remarkable that the energy estimate (which is at the level of L^2) does *not* lose regularity: It relates the $\dot{H}^1 \times L^2$ norm for data to the same norm for the solution at any later time. We have already seen that this property does not hold at the C^k level.

2.6 The Fourier representation

2.6.1 Brief review of Fourier theory

We recall the definition of the Schwartz space

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}_0^n \ , \ \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty\} \, , \tag{2.37}$$

where we use the multi-index notation $x^\alpha = x^{\alpha_1} ... x^{\alpha_n}$ and $\partial^\beta = (\partial_{x_1})^{\beta_1} ... (\partial_{x_n})^{\beta_n}$. The space $\mathcal{S}(\mathbb{R}^n)$ is a vectorspace equipped with a countable family of semi-norms

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|.$$

A sequence (u_n) in $\mathcal{S}(\mathbb{R}^n)$ is convergent to $u \in \mathcal{S}(\mathbb{R}^n)$ iff for all α, β there holds $\|u_n - u\|_{\alpha,\beta} \to 0$ as $n \to \infty$ (you can read this as the definition of convergence in the Schwartz space). One can define a metric on $\mathcal{S}(\mathbb{R}^n)$ via

$$\rho(f, g) = \sum_{\alpha,\beta} 2^{-|\alpha|-|\beta|} \frac{\|g - f\|_{\alpha,\beta}}{1 + \|g - f\|_{\alpha,\beta}}$$

and prove that $(\mathcal{S}(\mathbb{R}^n), \rho)$ is a complete metric space. (It is a so-called Frechet space.)

Definition 2.6.1. *Given a function* $f \in \mathcal{S}(\mathbb{R}^n)$*, we define the* <u>*Fourier transform*</u> *by the formula*

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \, . \tag{2.38}$$

We will also use the notation $\mathcal{F}(f) = \hat{f}$*. We also define* <u>*the inverse Fourier transform*</u> *by the formula*

$$\check{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{+2\pi i x \cdot \xi} dx \, . \tag{2.39}$$

We will also use the notation $\mathcal{F}^{-1}(f) = \check{f}$*.*

Theorem 2.6.2. *Basic properties of the Fourier transform:*

1. *The Fourier transform maps the Schwartz space onto itself.*
2. *Plancherel: For* $f \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \, .$$

As a consequence, the Fourier transform extends to an isometric isomorphism $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$*.*

Proof. This will be discussed in the examples classes. ◻

2.6.2 Solving the wave equation

Our starting point is again the initial value problem (2.7) with $f, g \in \mathcal{S}(\mathbb{R}^n)$. We have already proved existence and uniqueness of a solution. This section provides an alternative approach and result in establishing a different type of representation formula. Taking the Fourier transformation in space and using that

$$\mathcal{F}\left(\partial_i \phi\right)(\xi) = 2\pi i \xi_i \hat{\phi}(\xi) \tag{2.40}$$

we obtain

$$-\partial_t^2 \hat{\phi}(t,\xi) - 4\pi^2 |\xi|^2 \hat{\phi}(t,\xi) = 0\,. \tag{2.41}$$

The solution to this ODE is easily computed to be

$$\hat{\phi}(t,\xi) = A(\xi)\sin(2\pi t|\xi|) + B(\xi)\cos(2\pi t|\xi|)\,. \tag{2.42}$$

From the initial conditions we obtain

$$A(\xi) = \frac{\hat{g}(\xi)}{2\pi|\xi|} \quad , \quad B(\xi) = \hat{f}(\xi)\,. \tag{2.43}$$

Summarising we have

$$\hat{\phi}(t,\xi) = \frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi t|\xi|) + \hat{f}(\xi)\cos(2\pi t|\xi|)\,. \tag{2.44}$$

This function is smooth and Schwartz in ξ, hence taking the inverse Fourier transform in x we obtain the representation formula for the (unique, smooth) solution

$$\phi(t,x) = \int_{\mathbb{R}^n} d\xi e^{2\pi i x\cdot\xi}\left[\frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi t|\xi|) + \hat{f}(\xi)\cos(2\pi t|\xi|)\right]\,. \tag{2.45}$$

Proposition 2.6.3. *Given $f, g \in \mathcal{S}(\mathbb{R}^n)$, there exists a unique solution $\phi \in C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ of the Cauchy problem (2.7).*

We will use the formula (2.45) in due course to deduce global properties of the solution. Note that it is difficult to read off the domain of dependence property from this representation formula! On the other hand we easily derive the identity (exercise):

$$|\hat{\phi}_t|^2 + 4\pi^2|\xi|^2|\hat{\phi}|^2 = |\hat{g}|^2 + 4\pi^2|\xi|^2|\hat{f}|^2\,. \tag{2.46}$$

Note that the right hand side is independent of t and that integration in ξ will produce the familiar energy estimate on constant time hypersurfaces (the Schwartz condition corresponding to the sufficiently fast decay at infinity used in the physical space derivation). Of course (2.46) is stronger in that it asserts energy conservation in Fourier space for all fixed frequencies individually! In particular, multiplying by $\left(|\xi|^2+1\right)^{s-1}$ for $s \in \mathbb{R}$ and integrating we obtain a conservation law for all $s \in \mathbb{R}$:

$$\|\phi_t\|_{H^{s-1}} + \|\nabla\phi\|_{H^{s-1}} = \|g\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}}\,.$$

2.6.3 Decay via Fourier representation

We have proven already a decay estimate from the Kirchhoff formula. In this section we shall obtain the same result from the Fourier representation. We start with a strong decay estimate, which is, however, not uniform ins space. On the other hand, it illustrates in the most elementary fashion the main decay mechanism in this picture: exploiting oscillations. We may rewrite (2.45) as

$$\phi(t,x) = \int_{\mathbb{R}^n} d\xi e^{2\pi i x\cdot\xi}\left(e^{2\pi i t|\xi|}\left(\frac{\hat{f}(\xi)}{2} + \frac{\hat{g}(\xi)}{2\pi i|\xi|}\right) + e^{-2\pi i t|\xi|}\left(\frac{\hat{f}(\xi)}{2} - \frac{\hat{g}(\xi)}{2\pi i|\xi|}\right)\right)\,. \tag{2.47}$$

Proposition 2.6.4. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and ϕ be the unique solution ϕ arising from Proposition 2.6.2. Fix $R > 0$. Then there exists a constant $C(f, g, R) > 0$ such that*

$$\sup_{x\in\mathbb{R}^n} |\phi(t,x)| \leq \frac{C}{(1+t)^{n-1}}$$

whenever $|x| \leq R$ and $t \geq 0$.

Proof. We prove the statement for $n = 3$ only. We also only consider the integral

$$I = \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)} \frac{\hat{g}(\xi)}{2\pi i|\xi|} d\xi \,,$$

the other three integrals being treated entirely analogously. It also suffices to consider $t \geq 1$ since $|\phi(t,x)| \leq C$ holds easily from the representation formula using that $f, g \in \mathcal{S}(\mathbb{R}^n)$. We next introduce polar coordinates $(|\xi|, \xi_\theta, \xi_\phi)$ on $\mathbb{R}^3_\xi$ related to the standard Euclidean coordinates by $\xi_1 = |\xi| \sin\xi_\theta \cos\xi_\phi$, $\xi_2 = |\xi|\sin\xi_\theta \sin\xi_\phi$ and $\xi_3 = |\xi|\cos\xi_\theta$. Note that the volume form is $d\xi = |\xi|^2 \sin\xi_\theta d|\xi| d\xi_\theta d\xi_\phi$. Now note the identity for any $N \in \mathbb{N} \cup \{0\}$

$$\left(\frac{1}{2\pi i t}\frac{\partial}{\partial|\xi|}\right)^N e^{2\pi i t|\xi|} = e^{2\pi i t|\xi|} \,.$$

We can use this to re-express I and to integrate by parts:

$$\begin{aligned} I &= \int_{\mathbb{R}^3} \left(\left(\frac{1}{2\pi i t}\frac{\partial}{\partial|\xi|}\right) e^{2\pi i t|\xi|}\right) e^{2\pi i x\cdot\xi} \frac{\hat{g}(\xi)}{2\pi i} |\xi| \sin\xi_\theta d|\xi| d\xi_\theta d\xi_\phi \,, \\ &= \int_{\mathbb{R}^3} \frac{1}{4\pi^2 t} e^{2\pi i t|\xi|} e^{2\pi i x\cdot\xi} \left(\hat{g}(\xi) + |\xi|\frac{\partial}{\partial|\xi|}\hat{g}(\xi) + (2\pi i x\cdot\xi)\hat{g}(\xi)\right) \sin\xi_\theta d|\xi| d\xi_\theta d\xi_\phi \,. \end{aligned} \tag{2.48}$$

Note that there are no boundary term (why?) and that we have already gained a power in t! We can integrate by parts again (exercise) to produce the estimate claimed. □

We finally prove a uniform (in x estimate):

Proposition 2.6.5. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and ϕ be the unique solution ϕ arising from Proposition 2.6.2. Then there exists a constant $C(f, g) > 0$ such that*

$$\sup_{x\in\mathbb{R}^n} |\phi(t,x)| \leq \frac{C}{(1+t)^{\frac{n-1}{2}}}$$

holds for $t \geq 0$.

Proof. We prove the statement for $n = 3$ only. As before we can assume $t \geq 1$ and we only treat the term

$$I = \int_{\mathbb{R}^3} e^{2\pi i(t|\xi|+x\cdot\xi)} \frac{\hat{g}(\xi)}{2\pi i|\xi|} d\xi \,.$$

Given $x \in \mathbb{R}^n$ we choose coordinates such that $x = (0, 0, \alpha)$ for some $\alpha \in \mathbb{R}$. We choose cyclindrical coordinates $(\rho, \xi_\theta, \xi_3)$ on $\mathbb{R}^3_\xi$ related to the Euclidean coordinates by $\rho = \sqrt{|\xi_1|^2 + |\xi_2|^2}$, $\xi_1 = \rho\cos\theta_\xi$, $\xi_2 = \rho \sin\theta_\xi$. Note with this $\partial_\rho(e^{2\pi i x\cdot\xi}) = 0$ and also

$$\frac{\partial|\xi|}{\partial\rho} = \frac{\rho}{|\xi|} \,,$$

which follows from differentiating $|\xi|^2 = \rho^2 + |\xi_3|^2$ with respect to ρ. We conclude for every $N \in \mathbb{N} \cup \{0\}$

$$\left(\frac{|\xi|}{2\pi i\rho t}\frac{\partial}{\partial\rho}\right)^N e^{2\pi i t|\xi|} = e^{2\pi i t|\xi|} \,.$$

We now integrate by parts the expression

$$I = \int_{\mathbb{R}^3} \left(\left(\frac{|\xi|}{2\pi i \rho t} \frac{\partial}{\partial \rho} \right) e^{2\pi i t|\xi|} \right) e^{2\pi i x\cdot\xi} \frac{\hat{g}(\xi)}{2\pi i|\xi|} \rho d\rho d\xi_\theta d\xi_3 \tag{2.49}$$

$$\begin{aligned} &= -\frac{1}{4\pi^2 t} \int_{\xi_3=-\infty}^{\infty} d\xi_3 \int_0^\infty d\rho \int_0^{2\pi} d\xi_\theta \left(\frac{\partial}{\partial\rho} e^{2\pi i t|\xi|} \right) e^{2\pi i x\cdot\xi} \hat{g}(\xi) \\ &= \frac{1}{4\pi^2 t} \int_{\xi_3=-\infty}^{\infty} d\xi_3 \int_0^\infty d\rho \int_0^{2\pi} d\xi_\theta \left(e^{2\pi i t|\xi|} \right) e^{2\pi i x\cdot\xi} \frac{\partial}{\partial\rho} \hat{g}(\xi) \\ &\quad + \frac{1}{2\pi t} \int_{\xi_3=-\infty}^{\infty} d\xi_3 \left(e^{2\pi i t|\xi_3|} \right) e^{2\pi i \alpha\cdot\xi_3} \hat{g}(\rho = 0, \xi_3)) \end{aligned} \tag{2.50}$$

from which we deduce $|I| \leq \frac{C}{t}$ uniformly in x. The other three integrals are treated in exactly the same fashion (Exercise). □

2.7 Analytic Solutions

By the Cauchy-Kovalevskaya theorem one easily obtains that if the data f, g in (2.7) is analytic, the solution is analytic. Analytic data is of course against the spirit of relativity and the finite speed of propagation but one can sometimes use the energy estimates in conjunction with approximation by analytic data to produce generalised solutions. See the following section.

For completeness, we give here a basic version of the Cauchy Kovalevskaya theorem. We consider the following class of PDEs for a scalar function $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$:

$$\begin{aligned} \partial_t^m \phi &= G\left(t, x, \partial_t^j \partial_x^\alpha \phi; j \leq m-1\ ,\ j + |\alpha| \leq m\right) \\ \partial_t^i \phi\left(0, x\right) &= g_i\left(x\right) \qquad i = 0, \ldots, m-1 \ \text{ near } x = \bar{x} \end{aligned} \tag{2.51}$$

with G and g_ν analytic functions.

Theorem 2.7.1. *Suppose g_j is real analytic on a neighborhood of $\bar{x} \in \mathbb{R}^n$ and that G is real analytic on a neighborhood of $(0, \bar{x}, \partial_x^\alpha g_j\left(\bar{x}\right);\ j \leq m-1, j + |\alpha| \leq m)$. Then, there exists a real analytic solution of (2.51) defined on a neighborhood of $(0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n$. The solution is unique in the class of analytic solutions, i.e. two analytic solutions u and v of (2.51) defined on a neighborhood of $(0, \bar{x})$ have to agree.*

Remark 2.7.2. *The statement generalizes to fully non-linear equations and arbitrary non-characteristic initial data hypersurfaces.*

2.8 Generalised solutions (Sobolev regularity)

The explicit solution formulae can be used in conjunction with the energy estimate to produce solutions in lower regularity. We will discuss here only the simplest case to convey the main idea, which we will meet in more elaborate circumstances later.

Proposition 2.8.1. *Let $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then there exists a unique solution ϕ to the wave equation with $\phi \in C^1\left(\mathbb{R}, L^2(\mathbb{R}^n)\right) \cap C^0\left(\mathbb{R}, H^1(\mathbb{R}^n)\right)$.*

Proof. Since functions of compact support are dense in $L^2(\mathbb{R}^n)$ we can approximate $f_n \to f$ and $g_n \to g$ where (f_n) and (g_n) are sequences of smooth functions of compact support and the convergence is in $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ respectively. Using our well-posedness theory for smooth data, we produce a sequence (ϕ_n) of smooth solutions of the wave equation on $\mathbb{R} \times \mathbb{R}^n$. The energy conservation produces, for any $T > 0$

$$\sup_{t\in[-T,T]} \left(\|\phi_m(t,\cdot) - \phi_n(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t\phi_m(t,\cdot) - \partial_t\phi_n(t,\cdot)\|_{L^2(\mathbb{R}^n)} \right) = \|f_m - f_n\|_{\dot{H}^1(\mathbb{R}^n)} + \|g_m - g_n\|_{L^2(\mathbb{R}^n)}.$$

By the fundamental theorem of calculus we also have for any $t \in [-T, T]$

$$\|\phi_m(t,\cdot) - \phi_n(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|f_m(t,\cdot) - f_n(t,\cdot)\|_{L^2(\mathbb{R}^n)} + C \cdot T \sup_{t\in[-T,T]} \|\partial_t\phi_m(t,\cdot) - \partial_t\phi_n(t,\cdot)\|_{L^2(\mathbb{R}^n)}.$$

Combining the two estimates we conclude

$$\sup_{t\in[-T,T]} \|\phi_m(t,\cdot) - \phi_n(t,\cdot)\|_{H^1(\mathbb{R}^n)} + \sup_{t\in[-T,T]} \|\partial_t\phi_m(t,\cdot) - \partial_t\phi_n(t,\cdot)\|_{L^2(\mathbb{R}^n)} \\ \le C\|f_m - f_n\|_{H^1(\mathbb{R}^n)} + C \cdot T\|g_m - g_n\|_{L^2(\mathbb{R}^n)}. \tag{2.52}$$

Now the right hand side converges to zero as $m, n \to 0$. Hence ϕ_n is a Cauchy sequence in the Banach space $C^1\left([-T, T], L^2(\mathbb{R}^n)\right) \cap C^0\left([-T, T], H^1(\mathbb{R}^n)\right)$ and converges to a unique limiting ϕ in this space with $\phi(0) = f$ and $\partial_t\phi(0) = g$. It is also easy to see that ϕ satisfies the wave equation in the sense of distributions, i.e. for any $\chi \in C_0^\infty((-T, T) \times \mathbb{R}^n)$ we have $\int_{\mathbb{R}^{1+n}} dtdx\phi\Box\chi = 0$ (indeed, write $\phi = \phi_n + (\phi - \phi_n)$ and integrate the ϕ_n-term by parts and use the convergence for the $(\phi - \phi_n)$-term).

We next claim that this is also the unique solution in $C^1\left([-T, T], L^2(\mathbb{R}^n)\right) \cap C^0\left([-T, T], H^1(\mathbb{R}^n)\right)$. For this it suffices to show that for zero initial data there is only the zero solution in the claimed regularity class. Suppose $\phi \in C^1\left([-T, T], L^2(\mathbb{R}^n)\right) \cap C^0\left([-T, T], H^1(\mathbb{R}^n)\right)$ is a solution with zero initial data. We define the mollification $\phi_n = \phi \star \chi_n$ such that for fixed $\epsilon > 0$ we have (you should check that this is possible!)

- ϕ_n is smooth and $\phi_n \to \phi$ in $C^1\left([-T + \epsilon, T - \epsilon], L^2(\mathbb{R}^n)\right) \cap C^0\left([-T + \epsilon, T - \epsilon], H^1(\mathbb{R}^n)\right)$
- $\phi_n(0) \to 0$ and $\partial_t\phi_n(0) \to 0$ in $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ respectively.
- ϕ_n satisfies the wave equation on $[-T + \epsilon, T - \epsilon] \times \mathbb{R}^n$

Since ϕ_n is smooth, we can apply the energy estimate which yields $\phi_n \to 0$ in $C^1\left([-T + \epsilon, T - \epsilon], L^2(\mathbb{R}^n)\right) \cap C^0\left([-T + \epsilon, T - \epsilon], H^1(\mathbb{R}^n)\right)$ and since $\phi_n \to \phi$ we conclude $\phi \equiv 0$.

Since the above construction works for any $T > 0$ and produces a unique solution (implying in particular that the solutions we construct for different T agree on the region where they are both defined) we deduce the existence of a unique $\phi \in C^1\left(\mathbb{R}, L^2(\mathbb{R}^n)\right) \cap C^0\left(\mathbb{R}, H^1(\mathbb{R}^n)\right)$ satisfying the initial conditions $\phi(0) = f$ and $\partial_t\phi(0) = g$ and obeying the wave equation in the sense of distributions. $\Box$

Chapter 3

Existence and Uniqueness for General Linear Wave Equations

We now turn to establishing existence and uniqueness a general class of *linear, non-constant coefficient* wave equations. In the next chapter we will use them to prove results for non-linear equations using a suitable iteration scheme.

3.1 The class of equations

We consider the following Cauchy problem for $\phi : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}$:

$$\begin{cases} \partial_\alpha \left(a^{\alpha\beta} \partial_\beta \phi \right) = F \\ \phi(0,x) = f(x) \\ \partial_t \phi(0,x) = g(x) \end{cases} \tag{3.1}$$

Here $a^{\alpha\beta}$ are the components of a symmetric $(n+1) \times (n+1)$ matrix on $\mathbb{I} \times \mathbb{R}^n$ which satisfies

$$\sum_{\alpha,\beta} |a^{\alpha\beta} - \eta^{\alpha\beta}| < \frac{1}{10} \,. \tag{3.2}$$

In (3.1) we have also employed the *Einstein summation convention*: Repeated indices are always summed over (which implies that there is an implicit $\sum_{\alpha=0}^{3} \sum_{\beta=0}^{3}$ in front of the first equation). We will also need some regularity assumptions on the $a^{\alpha\beta} : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}$ and the initial data $f, g : \mathbb{R}^n \to \mathbb{R}$. They will be stated as we go along. For the moment, let us assume that they are smooth and that f, g and F decay sufficiently fast as $|x| \to \infty$.

Remark 3.1.1. *The number $1/10 < 1$ in (3.2) is of course chosen arbitrarily. If you think about Lorentzian metrics and their associated wave operator you should keep in mind that one can always choose coordinates such that at a point the metric in these coordinates takes the form of the Minkowski metric. Since we expect solutions to the wave equation to localise (domain of dependence) the assumption (3.1) should provide no restrictions to construct local solutions.*

Remark 3.1.2. *You will see that the results of this chapter easily generalise to equations of the form $\partial_\alpha \left(a^{\alpha\beta} \partial_\beta \phi \right) + b^\gamma \partial_\gamma \phi + c\phi = F$. To keep notation clean, we restrict to (3.1) and note that incorporating the lower order terms can also be done a posteriori using a fixed point argument (see the exercise sheet).*

3.2 The energy estimate

We introduce the notation

$$|\partial \phi|^2 := (\partial_t \phi)^2 + \sum_{i=1}^{n} (\partial_i \phi)^2 \,.$$

We first assume we have a smooth solution to (3.1) and prove an estimate on the solution.

Proposition 3.2.1. *Given a classical solution ϕ to (3.1), there exists a constant $C = C(n) > 0$ such that the following energy estimate holds for any $T > 0$:*

$$\sup_{t\in[0,T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C\left(\|\nabla f\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)dt\right)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)dt\right). \tag{3.3}$$

Proof. The proof is analogous to that of the standard linear wave equation on flat space, integrating

$$\partial_t\phi\left(\partial_\alpha\left(a^{\alpha\beta}\partial_\beta\phi\right) - F\right) = 0 \tag{3.4}$$

over a spacetime slab $\int_0^T dt\int_{\mathbb{R}^n}$. Recall the Einstein summation convention. We first derive the following identities

- If the indices in (3.4) are $\alpha = \beta = 0$ we have

$$\partial_t\phi\left(\partial_t\left(a^{tt}\partial_t\phi\right)\right) = \frac{1}{2}\partial_t\left(a^{tt}(\partial_t\phi)^2\right) + \frac{1}{2}(\partial_t a^{tt})\left(\partial_t\phi\right)^2 . \tag{3.5}$$

- If one of the indices is t and the other runs from $j = 1, ..., n$ we have

$$\begin{aligned}\partial_t\phi\left(\partial_t\left(a^{tj}\partial_j\phi\right) + \partial_j\left(a^{tj}\partial_t\phi\right)\right) &= a^{tj}\partial_j\left(\partial_t\phi\right)^2 + (\partial_t a^{tj})\partial_j\phi\partial_t\phi + (\partial_j a^{tj})(\partial_t\phi)^2 \\ &= \partial_j\left(a^{tj}\left(\partial_t\phi\right)^2\right) + (\partial_t a^{tj})\partial_j\phi\partial_t\phi\,. \end{aligned} \tag{3.6}$$

- Finally, if both indices run from $i = 1, ..., n$ and $j = 1, ..., n$ we have

$$\begin{aligned}\partial_t\phi\left(\partial_i\left(a^{ij}\partial_j\phi\right)\right) &= \partial_i\left(\partial_t\phi a^{ij}\partial_j\phi\right) - a^{ij}\partial_t\partial_i\phi\partial_j\phi \\ &= \partial_i\left(\partial_t\phi a^{ij}\partial_j\phi\right) - \frac{1}{2}\partial_t\left(a^{ij}\partial_i\phi\partial_j\phi\right) + \frac{1}{2}(\partial_t a^{ij})\partial_i\phi\partial_j\phi\,. \end{aligned} \tag{3.7}$$

Integrating (3.4) over $\int_0^T dt\int_{\mathbb{R}^n}$ and using the identities above provides the identity

$$\begin{aligned}&\frac{1}{2}\int_{\{T\}\times\mathbb{R}^n}\left[-a^{tt}(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]dx \\ &\leq \frac{1}{2}\int_{\{0\}\times\mathbb{R}^n}\left[-a^{tt}(\partial_t\phi)^2 + a^{ij}\partial_i\phi\partial_j\phi\right]dx \\ &+ C\int_0^T\left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\right)dt\,. \end{aligned} \tag{3.8}$$

Remark 3.2.2. *We have ignored here all spatial boundary terms in the integration. To justify this one should integrate for $R > 4T$ over a truncated cone-type region $\bigcup_{\tau=0}^T B(0, R-2\tau)$. The boundary terms on the wall of the cone (which is spacelike, as you should check) then have a favourable sign and can be dropped resulting in a uniform estimate for the energy through $B(0, R-2T)$ from the energy through $B(0, R)$ and a spacetime term over the cone. One can then carry though the entire proof and take $R \to \infty$ in the end. [This being said, below we will only apply the statement of the proposition knowing also a priori that ϕ is compactly supported in space!] I very much recommend redoing the proof in this setting and computing the additional boundary terms!*

Using now the assumption (3.2) we deduce

$$\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(T) \leq C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\int_0^T\left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\right)dt.$$

and in fact (why?)

$$\sup_{t\in[0,T]} \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)$$
$$\leq C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\int_0^T \left(\|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\|F\|_{L^2(\mathbb{R}^n)}(t) + \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)\right) dt.$$

We finally insert the following estimate, valid for any $\delta > 0$:

$$\int_0^T \|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\|F\|_{L^2(\mathbb{R}^n)}(t)dt \leq \sup_{t\in[0,T]} \|\partial\phi\|_{L^2(\mathbb{R}^n)}(t)\int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)dt$$
$$\leq \delta \sup_{t\in[0,T]} \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t) + \frac{1}{\delta}\left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)dt\right)^2 . \tag{3.9}$$

Choosing δ small depending only on C to absorb the sup-term on the left we deduce

$$\sup_{t\in[0,T]} \|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)$$
$$\leq C\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(0) + C\left(\int_0^T \|F\|_{L^2(\mathbb{R}^n)}(t)dt\right)^2 + C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)\|\partial\phi\|^2_{L^2(\mathbb{R}^n)}(t)dt.$$

The desired estimate is now a direct application of Gronwall's Lemma proven immediately below. □

Lemma 3.2.3 (Gronwall's Lemma)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and $g : \mathbb{R} \to \mathbb{R}$ be a non-negative integrable function such that*

$$f(t) \leq A + \int_0^t f(s)g(s)ds \tag{3.10}$$

holds for some $A \geq 0$ for every $t \in [0, T]$. Then

$$f(t) \leq A\exp\left(\int_0^t g(s)ds\right) \tag{3.11}$$

holds for every $t \in [0, T]$.

Proof. We will prove the estimate for any $A > 0$ and obtain the case $A = 0$ as a limit. We compute

$$\frac{d}{dt}\left(A + \int_0^t f(s)g(s)ds\right) \leq f(s)g(s) \leq g(s)\left(A + \int_0^t f(s)g(s)ds\right) . \tag{3.12}$$

For $A > 0$ we have

$$\frac{d}{dt}\left[\log\left(A + \int_0^t f(s)g(s)ds\right)\right] = g(s) .$$

Integrating yields

$$\log\left(A + \int_0^t f(s)g(s)ds\right) = \int_0^t g(s)ds + \log A$$

and we conclude

$$f(t) \leq A + \int_0^t f(s)g(s)ds \leq A\exp\left(\int_0^t g(s)ds\right)$$

as desired. □

We actually give a second proof of Gronwall's inequality, which illustrates – in its simplest form – a technique known as the "bootstrap method" or "method of oontinuity" that we will use frequently in the course.

Proof. Insert bootstrap proof of Gronwall.... □

Corollary 3.2.4. *Any classical solution of (3.1) is unique.*

Remark 3.2.5. *A mollification argument just as the one we did for the free wave equation yields uniqueness in $C^0\left([0, T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0, T], L^2(\mathbb{R}^n)\right)$ and ϕ a distributional solution of (3.1).*

3.3 Higher order energy estimates

The next proposition shows that higher order H^k norms are also propagated. It is instructive to first consider the case of the free wave equation for which ∂_t and ∂_{x_i} of course commute trivially!

Proposition 3.3.1. *Let ϕ be a smooth solution to (3.1) and k a positive integer. Then there exists a constant $C = C(n,k) > 0$ such that the following estimate holds for any $T > 0$:*

$$\begin{aligned} &\sup_{t\in[0,T]} \|\phi\|_{H^k(\mathbb{R}^n)}(t) + \sup_{t\in[0,T]} \|\partial_t\phi\|_{H^{k-1}(\mathbb{R}^n)}(t) \\ \leq C\,(1+T)\exp\left(C\int_0^T \|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)dt\right) &\left(\|f\|_{H^k(\mathbb{R}^n)} + \|g\|_{H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)dt\right. \\ &\left. + \int_0^T \sum_{\substack{|\alpha|+|\beta|\leq k-1 \\ |\beta|\leq k-2}} \|\partial_x^\alpha a\partial\partial\partial_x^\beta\phi\|_{L^2(\mathbb{R}^n)}(t)dt + \int_0^T \sum_{|\alpha|+|\beta|\leq k-1} \|\partial\partial_x^\alpha a\partial\partial_x^\beta\phi\|_{L^2(\mathbb{R}^n)}(t)dt\right). \end{aligned}$$

Proof. Exercise. Compute the equation with ∂_x^α, put all commutator terms on the right and apply the previous inhomogeneous estimate of Proposition 3.2.1. For the L^2 term integrate by parts as in Proposition 2.8.1 (giving rise to the growth in T). □

Corollary 3.3.2. *Under the assumptions of Proposition 3.3.1 suppose in addition that up to k derivatives of a are bounded in L^∞. The we have the estimate*

$$\sup_{t\in[0,T]} \|\phi\|_{H^k(\mathbb{R}^n)}(t) + \sup_{t\in[0,T]} \|\partial_t\phi\|_{H^{k-1}(\mathbb{R}^n)}(t) \leq C\left(\|f\|_{H^k(\mathbb{R}^n)} + \|g\|_{H^{k-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{k-1}(\mathbb{R}^n)}(t)dt\right)$$

for a constant C that depends on T, the L^∞-bounds on derivatives of a and the numbers n and k.

Proof. This follows easily from Gronwall's inequality. □

3.4 Existence of solutions via Hahn-Banach

We now let $k \in \mathbb{N}$ and assume $f \in H^k(\mathbb{R}^n)$, $g \in H^{k-1}(\mathbb{R}^n)$ as well as $F \in L^1\left([0,T], H^{k-1}(\mathbb{R}^n)\right)$. For the matrix-valued function a we will assume that it is smooth and that derivatives of all orders are bounded pointwise in $[0,T]\times\mathbb{R}^n$ (the bound potentially growing with the number of derivatives).

We can state the main result of this chapter

Theorem 3.4.1. *Let $k \in \mathbb{N}$. Given $f \in H^k(\mathbb{R}^n)$, $g \in H^{k-1}(\mathbb{R}^n)$ and $F \in L^1\left([0,T], H^{k-1}(\mathbb{R}^n)\right)$, there exists a unique solution*

$$\phi \in C^0\left([0,T], H^k\right) \cap C^1\left([0,T], H^{k-1}(\mathbb{R}^n)\right)$$

solving (3.1).

The main ingredients of the proof will be the Hahn-Banach theorem and an auxiliary Lemma that we state before embarking on the proof proper.

Theorem 3.4.2 (Hahn-Banach)**.** *Let ϕ be a bounded linear functional on subspace M of a normed (real) vectorspace X. Then there exists a bounded linear functional Φ on X which is an extension of ϕ to X and has the same norm, i.e.*

$$\|\Phi\|_X = \|\phi\|_M\,,$$

where

$$\|\Phi\|_X = \sup_{x\in X,\|x\|=1} |\Phi(x)| \quad \textit{and} \quad \|\phi\|_M = \sup_{x\in M,\|x\|=1} |\phi(x)|.$$

For the following Lemma and the remainder of the proof we define the operator L by

$$L\phi := \partial_\alpha \left(a^{\alpha\beta} \partial_\beta \phi \right)$$

and its formal adjoint $L^\star$ by

$$L^\star \psi = \partial_\alpha \left(a^{\alpha\beta} \partial_\beta \psi \right) \, .$$

Lemma 3.4.3. *Suppose $\psi \in C_0^\infty \left((-\infty, T) \times \mathbb{R}^n \right)$. Then, for every $m \in \mathbb{Z}$ there exists $C = C(m, T, a) > 0$ such that the following estimate holds for every $t \in [0, T]$:*

$$\|\psi\|_{H^m(\mathbb{R}^n)}(t) \leq C \int_t^T \|L^\star \psi\|_{H^{m-1}(\mathbb{R}^n)}(s) ds \, . \tag{3.13}$$

Proof. For $m \geq 1$ the estimate follows directly from Corollary 3.3.2 (think about the solution of the problem $L^\star \psi = F = L^\star \psi$ with zero data at $t = T$, backwards in time). For $m \leq 0$ we will use an inductive argument. Suppose the desired estimate holds for $m = m_0 + 2$ and we will establish it for $m = m_0$.

Given $\psi \in C_0^\infty \left((-\infty, T) \times \mathbb{R}^n \right)$ we first define (using the Fourier transform)

$$\Psi = (1 - \Delta)^{-1} \psi \, .$$

Note that while Ψ is not necessarily of compact support in space, it is still Schwartz in space. Since the induction assumption holds (by density) also for $\psi \in C_0^\infty \left((-\infty, T), H^{m_0+1}(\mathbb{R}^n) \right)$ and hence in particular for functions which are Schwartz in the spatial variables we have for all $t \in [0, T]$

$$\|\Psi\|_{H^{m_0+2}(\mathbb{R}^n)}(t) \leq C \int_t^T \|L^\star \Psi\|_{H^{m_0+1}(\mathbb{R}^n)}(s) ds \, . \tag{3.14}$$

We now estimate for a constant $C = C(T, a) > 0$

$$|L^\star \psi - (1 - \Delta) L^\star \Psi| = |L^\star (1 - \Delta) \Psi - (1 - \Delta) L^\star \Psi| \leq C \sum_{1 \leq |\alpha| \leq 3} |\partial^\alpha \Psi| \, .$$

Therefore,

$$\|L^\star \Psi\|_{H^{m_0+1}(\mathbb{R}^n)} \leq C \|(1 - \Delta) L^\star \Psi\|_{H^{m_0-1}(\mathbb{R}^n)} \leq C \left(\|L^\star \psi\|_{H^{m_0-1}(\mathbb{R}^n)} + \|\Psi\|_{H^{m_0+2}(\mathbb{R}^n)} \right) .$$

Integrating in $\int_0^T dt$ and combining with (3.14) we deduce

$$\begin{aligned} \|\Psi\|_{H^{m_0+2}(\mathbb{R}^n)}(t) &\leq C \int_t^T \left(\|L^\star \psi\|_{H^{m_0-1}(\mathbb{R}^n)} + \|\Psi\|_{H^{m_0+2}(\mathbb{R}^n)} \right)(s) ds \\ &\leq C \int_t^T \|L^\star \psi\|_{H^{m_0-1}(\mathbb{R}^n)} \end{aligned} \tag{3.15}$$

with the last step following from (a slight generalisation of) Gronwall's inequality and it is implict that $C = C(T, a)$ may change from line to line. Since we have $\|\psi\|_{H^{m_0}(\mathbb{R}^n)}(t) \leq C \|\Psi\|_{H^{m_0+2}(\mathbb{R}^n)}(t)$, the desired estimate follows for $m = m_0$. □

After these preparations we can prove Theorem 3.4.1.

Proof of Theorem 3.4.1. **Step 1. Existence for trivial data.** We first prove the theorem with trivial Cauchy data in (3.1), i.e. $f = 0$, $g = 0$.

We let $\psi \in C_0^\infty \left((-\infty, T) \times \mathbb{R}^n \right)$. By the Lemma above, we have the estimate

$$\|\psi\|\|_{H^{-k}(\mathbb{R}^n)}(t) \leq C \int_t^T \|L^\star \psi\|_{H^{-k+1}(\mathbb{R}^n)}(s) ds \quad \text{for alll } t \in [0, T]$$

In particular, given $\phi, \psi \in C_0^\infty \left((-\infty, T) \times \mathbb{R}^n \right)$ with $L^\star \psi = L^\star \phi$ then $\phi(t, \cdot) = \psi(t, \cdot)$ holds for all $t \in [0, T]$ so $L^\star \psi$ determines ψ uniquely in $[0, T]$.

Let M denote the image of $C_0^\infty\left((-\infty,T)\times\mathbb{R}^n\right)$ under the map $L^\star$:

$$M=\{L^\star\psi \mid \psi\in C_0^\infty\left((-\infty,T)\times\mathbb{R}^n\right)\}\subset X:=L^1\left((-\infty,T),H^{-k}(\mathbb{R}^n)\right).$$

We define a functional

$$\Phi: M\to\mathbb{R}$$

by

$$\Phi(L^\star\psi)=\langle F,\psi\rangle:=\int_0^T\int_{\mathbb{R}^n}\psi F dxdt\,.$$

We claim that this functional is bounded. Indeed, we have

$$\begin{aligned}\Big|\int_0^T\int_{\mathbb{R}^n}\psi F dxdt\Big| &\le C\left(\int_0^T\|F\|_{H^{k-1}(\mathbb{R}^n)}(t)dt\right)\sup_{t\in[0,T]}\|\psi\|_{H^{-k+1}(\mathbb{R}^n)}(t)\\ &\le C\int_0^T\|L^\star\psi\|_{H^{-k}(\mathbb{R}^n)}(s)ds\,,\end{aligned}\tag{3.16}$$

with the second line following from the assumption on F and the estimate of the Lemma. This means that Φ is a bounded linear functional on $M\subset X$ (equipped with the norm on $X=L^1\left((-\infty,T),H^{-k}(\mathbb{R}^n)\right)$).

By the Hahn-Banach theorem, there exists a $\Phi'\in\left(L^1\left((-\infty,T),H^{-k}(\mathbb{R}^n)\right)\right)^\star=L^\infty\left((-\infty,T),H^{-k}(\mathbb{R}^n)\right)$ that extends Φ to all of X and that we will from now on – abusing notation – also denote by Φ. In particular

$$\langle\Phi,L^\star\psi\rangle=\langle F,\psi\rangle\quad\text{ holds for all }\psi\in C_0^\infty\left((-\infty,T)\times\mathbb{R}^n\right).\tag{3.17}$$

We can also set $\Phi=0$ for $t<0$ as the right hand side of (3.17) is zero for ψ supported in $t<0$. Now (3.17) is precisely the statement that Φ is a distributional solution of the wave equation (with trivial data at $t=0$).[1]

Step 2. Improving the regularity. From $F\in L^1\left([0,T],H^k(\mathbb{R}^n)\right)$ we construct a sequence (F_n) with $F_n\in C_0^\infty\left([0,T)\times\mathbb{R}^n\right)$ with $F_n\to F$ in $L^1\left([0,T],H^k(\mathbb{R}^n)\right)$. Running the argument of Step 1 we produce a sequence of solutions such that

$$a^{tt}\partial_t\partial_t\Phi_n+2a^{tj}\partial_j\partial_t\Phi_n+\left(\partial_j a^{jt}\right)\partial_t\Phi_n=-(\partial_t a^{tj})\partial_j\Phi_n+\partial_i\left(a^{ij}\partial_j\Phi_n\right)-F_n\in L^\infty\left([0,T],H^{k-2}(\mathbb{R}^n)\right).$$

Using elliptic estimates on $\mathbb{R}^{1+n}$ (consider the *sum* of the operator on left hand side and the right hand side) we can show that the solution is classical (Exercise!). Therefore, the energy estimate of Corollary 3.3.2 with zero data applies to $\Phi_n-\Phi_m$ and we conclude that the Φ_n converge in $C^0\left([0,T],H^k\right)\cap C^1\left([0,T],H^{k-1}(\mathbb{R}^n)\right)$ (now for any $k\ge 1$!) as $F_n\to F$. It is easy to check that the limit is still a distributional solution. This concludes the proof for $f=g=0$. Note that the argument produces smooth solutions if F is smooth (and at least classical solutions if F is sufficiently regular, depending on the dimension).

Step 3. General data. Consider now initial data $f\in H^k(\mathbb{R}^n)$ and $g\in H^{k-1}(\mathbb{R}^n)$. Choose sequences of functions of compact support (f_n) and (g_n) converging to f in H^k and to g in H^{k-1} respectively and similarly (F_n) smooth and of compact spatial support converging to F in $L^1\left([0,T],H^{k-1}(\mathbb{R}^n)\right)$. Defining $u_n=f_n+tg_n$ (which is smooth) we produce by Steps 1+2, choosing k sufficiently large, a sequence of C^2 solutions for the problem $L\psi_n=F_n-Lu_n$ *with trivial initial data*. In other words, $L(\psi_n+u_n)=F_n$, so ψ_n+u_n satisfies the wave equation with initial data (f_n,g_n). Using once more the energy estimate, we see that ψ_n+u_n is a Cauchy sequence in the Banach space $C^0\left([0,T],H^k(\mathbb{R}^n)\right)\cap C^1\left([0,T],H^{k-1}(\mathbb{R}^n)\right)$ and hence converges to a unique limit. It is is easy to see that the limit satisfies the equation in the sense of distributions (and for higher k, classically). □

[1]Recall that Φ is a distributional solution of (3.1) with data $f\in H^1(\mathbb{R}^n)$ and $g\in L^2(\mathbb{R}^n)$ provided that

$$-\int_{t=0}dx\left(a^{tt}g+a^{tj}\partial_j f\right)\psi+\int_{t=0}dx a^{\mu t}f\partial_\mu\psi+\int_0^T dt\int_{\mathbb{R}^n}\left[\Phi\left(\partial_\mu\left(a^{\mu\nu}\partial_\nu\psi\right)\right)-F\psi\right]=0$$

holds for all $\psi\in C_0^\infty\left([0,T)\times\mathbb{R}^n\right)$. You should check that a classical (C^2) solution is always a distributional solution (easy integration by parts!). You can also check that if $\Phi\in C^1\left([0,T],H^{k-1}\right)\cap C^0\left([0,T],H^k(\mathbb{R}^n)\right)$ satisfies $\int_0^T dt\int_{\mathbb{R}^n}\left[\Phi\left(\partial_\mu\left(a^{\mu\nu}\partial_\nu\psi\right)\right)-F\psi\right]=0$ for all $\psi\in C_0^\infty\left([0,T)\times\mathbb{R}^n\right)$ then $\Phi(t=0,x)=0$ and $\partial_t\Phi(t=0,x)=0$ (construct appropriate ψ!).

3.5 Alternative approaches

While we will not pursue this further, it is worth mentioning that there are other approaches to prove Theorem 3.4.1. One idea is to localise the theorem to small cones of uniform size and then paste things together afterwards using domain of dependence and uniqueness. For the localised problem, the coefficients of the PDE are almost constant and we can solve a constant coefficient problem with smooth data (using Fourier techniques, for instance) and small error on the right hand side (coming from the variation of the coefficients over the small cone), which can be absorbed in the energy estimate after integration by parts.

Another related approach is to approximate both the data and the coefficients in the PDE by analytic functions, apply the Cauchy-Kovalevskaya theorem to construct a sequence of analytic solutions which then, using energy estimates, converge in a small cone to a solution in the space $C^0\left([0,T], H^k(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{k-1}(\mathbb{R}^n)\right)$.[2]

Whatever the approach is, you should not that the energy estimate is the crucial ingredient in all of them!

[2]You also need to use the propagation of analyticity for hyperbolic equations to make this approach work globally.

Chapter 4

Local Wellposedness for Non-linear Wave Equations

4.1 The class of equations

We finally consider non-linear wave equations! The equations we would like to look at is the following class of quasilinear wave equations for $\phi : I \times \mathbb{R}^n \to \mathbb{R}$:

$$\begin{cases} \partial_\alpha \left(a^{\alpha\beta}(\phi)\partial_\beta \phi \right) = F(\phi, \partial\phi) \\ \phi(0,x) = f(x) \\ \partial_t \phi(0,x) = g(x) \end{cases} \tag{4.1}$$

with data $f \in H^k(\mathbb{R}^n)$ and $g \in H^{k-1}(\mathbb{R}^n)$ and $k \in \mathbb{N}$ specified later. We require $a^{\alpha\beta} = a^{\beta\alpha}$ and

$$\sum_{\alpha\beta} |a^{\alpha\beta} - m^{\alpha\beta}| \leq \frac{1}{10} \quad , \quad a^{\alpha\beta}(0) = m^{\alpha\beta} \quad , \quad F(0,0) = 0$$

and

$$a^{\alpha\beta} : \mathbb{R} \to \mathbb{R} \quad , \quad F : \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R} \quad \text{are smooth functions of their arguments.}$$

Note that these assumption imply

$$\sum_{\alpha\beta} \sum_{|\gamma| \leq N} \sup_{|x| \leq A} |\partial_x^\gamma (a^{\alpha\beta})|(x) \leq C_{A,N} \tag{4.2}$$

and

$$\sum_{|\gamma| \leq N} \sup_{|x|, \|p\| \leq A} |\partial_{x,p}^\gamma F|(x,p) \leq C_{A,N} \, . \tag{4.3}$$

We remark that the class (4.1) is general enough to allow us to deduce local-wellposedness for the Einstein equations of general relativity later. The equations of compressible fluid mechanics, however, are even more non-linear (although they can also be treated with the techniques we are about to introduce).

4.2 The main well-posedness theorem

The goal now is to prove the following theorem.

Theorem 4.2.1. *Fix a and F satisfying the assumptions above.*[1]

[1] In particular, all constants below can depend on a and F without us explicitly denoting this.

1. *(Existence and Uniqueness of local in-time solutions.) There exists*

$$T = T\left(\|f\|_{H^{n+3}(\mathbb{R}^n)}, \|g\|_{H^{n+2}(\mathbb{R}^n)}\right) > 0$$

such that there exists a (classical) solution ϕ to (4.1) with

$$\phi \in C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right),$$

which is moreover unique in this space.

2. *(Continuous dependence on the data.) Let $f \in H^{n+3}(\mathbb{R}^n)$, $g \in H^{n+2}(\mathbb{R}^n)$. Let $f^{(i)}$, $g^{(i)}$ be sequences of functions such that $f^{(i)} \to f$ in $H^{n+3}(\mathbb{R}^n)$ and $g^{(i)} \to g$ in $H^{n+2}(\mathbb{R}^n)$. Then, taking T sufficiently small, we have*

$$\sup_{t\in[0,T]} \|\phi^{(i)} - \phi\|_{H^s(\mathbb{R}^n)}(t) + \sup_{t\in[0,T]} \|\partial_t\phi^{(i)} - \partial_t\phi\|_{H^{s-1}(\mathbb{R}^n)}(t) \to 0 \tag{4.4}$$

for every $1 \le s < n+3$, where $\phi^{(i)}$ is the solution arising from data $(f^{(i)}, g^{(i)})$ and ϕ is the solution arising from data (f,g).

Remark 4.2.2. *This is the mathematical formulation of Hadamard well-posedness for (4.1) in the sense that existence, uniqueness and continuous dependence on the data holds.*

Remark 4.2.3. *Note that the regularity $C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$ already implies that the solution is classical (use the equation and Sobolev embedding).*

Remark 4.2.4. *The regularity assumptions on the data in Theorem 4.2.1 are far from optimal (and we will see opportunities for improvement in the proof). Since we will later mainly be interested in smooth solutions this is not a serious restriction.*

Proof. **Proof of the existence part by Picard iteration.** By a density argument, it suffices to assume $f, g \in \mathcal{S}(\mathbb{R}^n)$. We define a sequence of smooth functions $\phi^{(i)}$ for $i \ge 1$ as follows: We define $\phi^{(1)} = 0$ and $\phi^{(i)}$ iteratively for $i \ge 2$ as the unique solution (by Theorem 3.4.1) to

$$\begin{cases} \partial_\alpha\left(a^{\alpha\beta}(\phi^{(i-1)})\partial_\beta\phi^{(i)}\right) = F\left(\phi^{(i-1)}, \partial\phi^{(i-1)}\right) \\ \phi^{(i)}(0,x) = f(x) \\ \partial_t\phi^{(i)}(0,x) = g(x). \end{cases} \tag{4.5}$$

The logic of the existence part of the proof is now as follows. We will show that for T sufficiently small (depending only on $\|f\|_{H^{n+3}(\mathbb{R}^n)}$ and $\|g\|_{H^{n+2}(\mathbb{R}^n)}$), the sequence $(\phi^{(i)})$ is

(a) uniformly bounded in $C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$ and

(b) Cauchy in $C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$.

From (b) it then follows that $\phi^{(i)}$ has a limit in $C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$ and (a) together with the Banach-Alaoglu theorem (see Example Sheet 4) imply that the limit is actually in the smaller space $C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$.[2] It is also easy to see that the limit satisfies the equation.

The hard part is of course to establish (a) and (b) above. We begin with (a). Let us introduce the shorthand notation

$$\|\phi\|_{n+3}(t) := \|\phi\|_{H^{n+3}(\mathbb{R}^n)}(t) + \|\partial_t\phi\|_{H^{n+2}(\mathbb{R}^n)}(t).$$

Proof of (a). We will prove that for T sufficiently small there exists an $A > 0$ such that for all i

$$\sup_{t\in[0,T]} \|\phi^{(i-1)}\|_{n+3}(t) \le A \quad \Longrightarrow \quad \sup_{t\in[0,T]} \|\phi^{(i)}\|_{n+3}(t) \le A. \tag{4.6}$$

[2] Indeed, for fixed $t \in [0,T]$ the sequence $(\phi^{(i)}(t), \partial_t\phi^{(i)}(t)) \in H^{n+3}(\mathbb{R}^n) \times H^{n+2}(\mathbb{R}^n)$ is a bounded sequence in a Hilbert space. Banach-Alaoglu produces a subsequence converging weakly in $H^{n+3}(\mathbb{R}^n) \times H^{n+2}(\mathbb{R}^n)$. But we also know that the (sub)sequence converges strongly in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. By the uniqueness of limits, the two limits must agree. This leaves showing the continuity in time, i.e. showing that the solution is actually in $C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$ rather than just $\phi \in L^\infty\left([0,T], H^{n+3}(\mathbb{R}^n)\right)$ and $\partial_t\phi \in L^\infty\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$. We leave this to the exercise sheet.

We will choose T and A explicitly below and now make the inductive assumption

$$\sup_{t\in[0,T]} \|\phi^{(i-1)}\|_{n+3}(t) \leq A\,. \tag{4.7}$$

Note that (4.7) implies from Sobolev embedding (Example Sheet 1) for any ϕ the pointwise bounds

$$\sum_{|\alpha|\leq\lfloor n+3-\frac{n}{2}-\frac{1}{2}\rfloor} \|\partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) + \sum_{|\alpha|\leq\lfloor n+2-\frac{n}{2}-\frac{1}{2}\rfloor} \|\partial_t \partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) \leq C\|\phi\|_{n+3}(t) \tag{4.8}$$

for all $t \in [0,T]$, where C is a constant depending only on the dimension n and $\lfloor q \rfloor$ is the largest integer smaller or equal to q. Note that for $\phi = \phi^{(i-1)}$ the right hand side of (4.8) is bounded by $C \cdot A$ by (4.7).

To prove the bound (4.7) for $\phi^{(i)}$ we use the energy estimates of Proposition 3.3.1 applied to the linear inhomogeneous system (4.5). As we are commuting $n+2$ times to estimate $n+3$ derivatives, this requires bounds on the following expressions:

$$\begin{aligned}
\alpha &= \int_0^T \sum_{0\leq|\alpha|\leq n+2} \|\partial_x^\alpha F(\phi^{(i-1)}, \partial\phi^{(i-1)})\|_{L^2(\mathbb{R}^n)}(t)dt\,, \\
\beta &= \int_0^T \sum_{0\leq|\gamma|+|\sigma|\leq n+2} \sum_{\alpha,\beta} \|\partial\partial_x^\gamma(a^{\alpha\beta}(\phi^{(i-1)}))\partial\partial_x^\sigma\phi^{(i)}\|_{L^2(\mathbb{R}^n)}(t)dt\,, \\
\gamma &= \int_0^T \sum_{\substack{0\leq|\gamma|+|\sigma|\leq n+2\\ |\sigma|\leq n+1}} \sum_{\alpha,\beta} \|\partial_x^\gamma(a^{\alpha\beta}(\phi^{(i-1)}))\partial\partial\partial_x^\sigma\phi^{(i)}\|_{L^2(\mathbb{R}^n)}(t)dt\,, \\
\delta &= \int_0^T \sum_{\alpha,\beta} \|\partial(a^{\alpha\beta}(\phi^{(i-1)}))\|_{L^\infty(\mathbb{R}^n)}(t)dt\,.
\end{aligned}$$

We begin by estimating α. We claim

$$\sum_{0\leq|\alpha|\leq n+2} \|\partial_x^\alpha F(\phi^{(i-1)}, \partial\phi^{(i-1)})\|_{L^2(\mathbb{R}^n)}(t) \leq C \cdot C_{n+2,A} \cdot (1+A)^{n+2} \leq C_A\,, \tag{4.9}$$

where C is a uniform constant depending only on the dimension n, $C_{n+2,A}$ is the constant in (4.3) and A is the constant in the induction assumption (4.7). The claim (4.9) is a consequence of the chain rule, Sobolev embedding and (4.3). Indeed, we have[3]

$$\begin{aligned}
\sum_{0\leq|\alpha|\leq n+2} \|\partial_x^\alpha F(\phi^{(i-1)}, \partial\phi^{(i-1)})\|_{L^2(\mathbb{R}^n)}(t) \leq C_{n+2,A} \Bigg(&\sum_{0\leq|\alpha|\leq n+2} \|\partial\partial_x^\alpha \phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \\
&+ \sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+2} \|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\partial\partial_x^{\alpha_2}\phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \\
&+\text{cubic} + ... + \text{terms of order } (n+2)\Bigg).
\end{aligned}$$

Note that the first (linear) term in the round bracket is bounded by A from (4.7). For the second (quadratic) term we note that we have $|\alpha_1| \leq \frac{n+2}{2}$ or $|\alpha_2| \leq \frac{n+2}{2}$ (or both) for each term in the sum. Wlog let the inequality hold for α_1. Then we have

$$\|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\partial\partial_x^{\alpha_2}\phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \leq \|\partial\partial_x^{\alpha_1}\phi^{(i-1)}\|_{L^\infty(\mathbb{R}^n)}(t) \cdot \|\partial\partial_x^{\alpha_2}\phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \leq CA^2 \tag{4.10}$$

[3]Note that the constant $C_{n+2,A}$ in front comes from applying (4.3) to estimate the derivatives of F that appear when applying the chain rule. Indeed, by (4.8) we have $|\phi| + |\partial\phi| \leq CA$ and hence $F(\phi^{(i-1)}, \partial_\phi^{(i-1)})$ and derivatives of F are only ever evaluated on a ball of radius $C \cdot A$.

because we can apply (4.8) to the term in L^∞ in view of $|\alpha_1| \leq \frac{n+2}{2} \leq \lfloor \frac{n}{2} + \frac{3}{2} \rfloor$, and estimate the resulting L^2-terms by (4.7).

It is clear that the cubic and higher order terms can be estimated analogously, putting the top-order term in L^2 and the lower order terms in L^∞ using (4.8). This yields C^2A^3 for the cubic term, C^3A^4 for the quartic one etc. The claim (4.9) has hence been established.

For β and γ we claim

$$\sum_{0 \leq |\gamma|+|\sigma| \leq n+2} \sum_{\alpha,\beta} \|\partial \partial_x^\gamma (a^{\alpha\beta}(\phi^{(i-1)})) \partial \partial_x^\sigma \phi^{(i)}\|_{L^2(\mathbb{R}^n)}(t) dt \leq C_A \|\phi^{(i)}\|_{n+3}(t) \tag{4.11}$$

and

$$\sum_{\substack{0 \leq |\gamma|+|\sigma| \leq n+2 \\ |\sigma| \leq n+1}} \sum_{\alpha,\beta} \|\partial_x^\gamma (a^{\alpha\beta}(\phi^{(i-1)})) \partial\partial \partial_x^\sigma \phi^{(i)}\|_{L^2(\mathbb{R}^n)}(t) dt \leq C_A \|\phi^{(i)}\|_{n+3}(t). \tag{4.12}$$

We leave establishing these claims as an exercise (Example Sheet 4). The mechanics is entirely identical to the term from α: Put the top order term in L^∞ (and estimate it by some higher order L^2 norm using (4.8)) and use the induction assumption (4.7) for all L^2-norms involving $\phi^{(i-1)}$. The only slightly tricky bit is that in (4.12) one needs to replace the two ∂_t-derivatives that could appear in $\partial\partial$ using the non-linear wave equation. This will result in terms exhibiting at most one ∂_t-derivative, which are the ones for which (4.8) applies.

Finally, we easily see that $\delta \leq C_A T$ so that in summary, the estimate from Proposition 3.3.1 reads:

$$\sup_{t \in [0,T]} \|\phi^{(i)}\|_{n+3}(t) \leq C \left(\|f\|_{H^{n+3}(\mathbb{R}^n)} + \|g\|_{H^{n+2}(\mathbb{R}^n)} + C_A \int_0^T \|\phi^{(i)}\|_{n+3}(t) dt + C_A T \right) \exp(C_A T). \tag{4.13}$$

Choosing $T > 0$ such that $CC_AT \leq \frac{1}{2}$, $\exp(C_AT) \leq 2$ and $C_AT \leq \|f\|_{H^{n+3}(\mathbb{R}^n)} + \|g\|_{H^{n+2}(\mathbb{R}^n)}$ (for trivial data $f = g = 0$, clearly $\phi = 0$ is a solution of (4.1) so (global) existence is immediate) we conclude

$$\sup_{t \in [0,T]} \|\phi^{(i)}\|_{n+3}(t) \leq 8C \left(\|f\|_{H^{n+3}(\mathbb{R}^n)} + \|g\|_{H^{n+2}(\mathbb{R}^n)} \right). \tag{4.14}$$

We therefore set $A = 8C \left(\|f\|_{H^{n+3}(\mathbb{R}^n)} + \|g\|_{H^{n+2}(\mathbb{R}^n)} \right)$. Choosing A and T as above we have therefore shown the induction step (4.6) and hence the uniform boundedness of the sequence $\phi^{(i)}$ has been established.

Proof of (b). To show the sequence $\phi^{(i)}$ is Cauchy in $C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$, we look at the equation for $\phi^{(i)} - \phi^{(i-1)}$ with $i \geq 3$. We have

$$\begin{aligned} \partial_\alpha \left(a^{\alpha\beta}(\phi^{(i-1)}) \partial_\beta (\phi^{(i)} - \phi^{(i-1)}) \right) = & - \partial_\alpha \left(\left(a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}) \right) \partial_\beta \phi^{(i-1)} \right) \\ & + F(\phi^{(i-1)}, \partial\phi^{(i-1)}) - F(\phi^{(i-2)}, \partial\phi^{(i-2)}). \end{aligned} \tag{4.15}$$

By the mean value theorem and (4.2) on the one hand and the uniform pointwise bound on up to two derivatives of $\phi^{(i-2)}$ obtained in part (a) on the other, we deduce

$$\left| - \partial_\alpha \left(\left(a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}) \right) \partial_\beta \phi^{(i-1)} \right) \right| \leq C |\partial(\phi^{(i-1)} - \phi^{(i-2)})|$$

for a constant independent of i and T. We immediately infer

$$\left\| - \partial_\alpha \left(\left(a^{\alpha\beta}(\phi^{(i-1)}) - a^{\alpha\beta}(\phi^{(i-2)}) \right) \partial_\beta \phi^{(i-1)} \right) \right\|_{L^2(\mathbb{R}^n)}(t) \leq C \|\partial(\phi^{(i-1)} - \phi^{(i-2)})\|_{L^2(\mathbb{R}^n)}(t).$$

A similar application of the mean value theorem and (4.3) provides the bound

$$\begin{aligned} & \|F(\phi^{(i-1)}, \partial\phi^{(i-1)}) - F(\phi^{(i-2)}, \partial\phi^{(i-2)})\|_{L^2(\mathbb{R}^n)}(t) \\ \leq & C \left(\|\phi^{(i-1)} - \phi^{(i-2)}\|_{H^1(\mathbb{R}^n)}(t) + \|\partial_t \phi^{(i-1)} - \partial_t \phi^{(i-2)}\|_{L^2(\mathbb{R}^n)}(t) \right). \end{aligned}$$

We apply Proposition 3.3.1, or more precisely Corollary 3.3.2 where we use that

$$\|\partial a^{\alpha\beta}(\phi^{(i-1)})\|_{L^\infty(\mathbb{R}^n)} \leq C$$

is uniformly bounded (independently of i), to deduce (in view of the data agreeing for all $\phi^{(i)}$ with $i \geq 2$), the estimate

$$\begin{aligned} &\sup_{t\in[0,T]} \|\phi^{(i)} - \phi^{(i-1)}\|_{H^1(\mathbb{R}^n)} + \sup_{t\in[0,T]} \|\partial_t\phi^{(i)} - \partial_t\phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \\ &\leq CT \left(\sup_{t\in[0,T]} \|\phi^{(i-1)} - \phi^{(i-2)}\|_{H^1(\mathbb{R}^n)} + \sup_{t\in[0,T]} \|\partial_t\phi^{(i-1)} - \partial_t\phi^{(i-2)}\|_{L^2(\mathbb{R}^n)}(t) \right). \end{aligned} \tag{4.16}$$

Making T smaller if necessary, we obtain for $i \geq 3$

$$\begin{aligned} &\sup_{t\in[0,T]} \|\phi^{(i)} - \phi^{(i-1)}\|_{H^1(\mathbb{R}^n)} + \sup_{t\in[0,T]} \|\partial_t\phi^{(i)} - \partial_t\phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \\ &\leq \frac{1}{2} \left(\sup_{t\in[0,T]} \|\phi^{(i-1)} - \phi^{(i-2)}\|_{H^1(\mathbb{R}^n)} + \sup_{t\in[0,T]} \|\partial_t\phi^{(i-1)} - \partial_t\phi^{(i-2)}\|_{L^2(\mathbb{R}^n)}(t) \right). \end{aligned} \tag{4.17}$$

It follows that

$$\sup_{t\in[0,T]} \|\phi^{(i)} - \phi^{(i-1)}\|_{H^1(\mathbb{R}^n)} + \sup_{t\in[0,T]} \|\partial_t\phi^{(i)} - \partial_t\phi^{(i-1)}\|_{L^2(\mathbb{R}^n)}(t) \leq C \cdot 2^{-i+2}, \tag{4.18}$$

from which the Cauchy property easily follows. This finishes the proof of the existence part. The uniqueness part of the first statement of Theorem 4.2.1 can be deduced once we have proven the continuous dependence on the data, part 2. of the theorem to which we now turn.

Proof of continuous dependence on the data. The difference $\phi^{(i)} - \phi$ satisfies

$$\partial_\alpha \left(a^{\alpha\beta}(\phi^{(i)}) \partial_\beta(\phi^{(i)} - \phi) \right) = -\partial_\alpha \left(\left(a^{\alpha\beta}(\phi^{(i)}) - a^{\alpha\beta}(\phi) \right) \partial_\beta \phi \right) + F\left(\phi^{(i)}, \partial\phi^{(i)}\right) - F\left(\phi, \partial\phi\right). \tag{4.19}$$

Applying the energy estimate we deduce for all $t \in [0,T]$ (with the constant C independent of i and T)

$$\begin{aligned} &\sup_{s\in[0,t]} \|\phi^{(i)} - \phi\|_{H^1(\mathbb{R}^n)}(s) + \sup_{s\in[0,t]} \|\partial_t\phi^{(i)} - \partial_t\phi\|_{L^2(\mathbb{R}^n)}(s) \\ &\leq C \Bigg\{ (\|f^{(i)} - f\|_{H^1(\mathbb{R}^n)} + \|g^{(i)} - g\|_{L^2(\mathbb{R}^n)} \\ &\quad + \int_0^t \left\| \partial_\alpha \left(\left(a^{\alpha\beta}(\phi^{(i)}) - a^{\alpha\beta}(\phi) \right) \partial_\beta\phi \right) \right\|_{L^2(\mathbb{R}^n)} (s) ds \\ &\quad + \int_0^t \left\| F\left(\phi^{(i)}, \partial\phi^{(i)}\right) - F\left(\phi, \partial\phi\right) \right\|_{L^2(\mathbb{R}^n)} (s) ds \Bigg\}. \end{aligned} \tag{4.20}$$

Applying the mean value theorem and the estimates as in the first part of the proof we deduce from Gronwall's inequality the estimate

$$\sup_{s\in[0,t]} \|\phi^{(i)} - \phi\|_{H^1(\mathbb{R}^n)}(s) + \sup_{s\in[0,t]} \|\partial_t\phi^{(i)} - \partial_t\phi\|_{L^2(\mathbb{R}^n)}(s) \leq C(T) \left((\|f^{(i)} - f\|_{H^1(\mathbb{R}^n)} + \|g^{(i)} - g\|_{L^2(\mathbb{R}^n)} \right)$$

for a constant $C(T)$ depending (exponentially) on T. Since for all $t \in [0,T]$ the right hand side goes to zero as $i \to \infty$ we conclude that the left hand side goes to zero.

Using that $\sup_{t\in[0,T]} \|\phi^{(i)} - \phi\|_{H^{n+3}(\mathbb{R}^n)}(t) + \sup_{t\in[0,T]} \|\partial_t\phi^{(i)} - \partial_t\phi\|_{H^{n+2}(\mathbb{R}^n)}(t) \leq C$ is uniformly bounded (by the triangle inequality and the results from the first part of the theorem), we can now use the interpolation inequality from Exercise Sheet 4 to deduce also

$$\sup_{t\in[0,T]} \|\phi^{(i)} - \phi\|_{H^s(\mathbb{R}^n)}(t) + \sup_{t\in[0,T]} \|\partial_t\phi^{(i)} - \partial_t\phi\|_{H^{s-1}(\mathbb{R}^n)}(t) \to 0 \quad \text{for all } 1 \leq s < n+3$$

as desired. This concludes the proof of the theorem. □

Remark 4.2.5. *As the proof shows, we can of course replace H^{n+3} by H^s and H^{n+2} by H^{s-1} everywhere in Theorem 4.2.1 provided $s \geq n+3$. Of course the time T might a priori get smaller as we increase s.*

Remark 4.2.6. *Note the loss of derivatives in the contraction argument. From (4.15) one proves estimate on first derivatives of the differences but there are second derivatives of the $\phi^{(i)}$ appearing on the right hand side for which one needs to have established uniform a-priori bounds.*

4.3 Persistence of regularity

The next proposition can be paraphrased by saying that as long as we have a solution $\phi \in C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$ all higher regularity of the initial data is propagated.

Theorem 4.3.1. *(Persistence of regularity.) Let $f \in H^{n+3}(\mathbb{R}^n)$ and $g \in H^{n+2}(\mathbb{R}^n)$ in (4.1). Let*

$$T_\star := \sup\{T > 0 \mid \text{there exists a unique } C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right) \text{ solution}\} > 0$$

be the maximal time of existence.[4] *Then*

1. *If $f \in H^m(\mathbb{R}^n)$, $g \in H^{m-1}(\mathbb{R}^n)$ for $m \geq n+4$ then the solution ϕ is in $C^0([0,T], H^m(\mathbb{R}^n)) \cap C^1\left([0,T], H^{m-1}(\mathbb{R}^n)\right)$ for every $T < T_\star$.*
2. *If $f \in \cap_{m=1}^\infty H^m(\mathbb{R}^n)$, $g \in \cap_{m=1}^\infty H^{m-1}(\mathbb{R}^n)$, then the solution is smooth on $[0, T_\star) \times \mathbb{R}^n$.*

Proof. Note that the second statement will follow from the first and the Sobolev embedding theorem so we will only prove the first statement. The proof is by induction on m. Assume the conclusion holds for some $m - 1 \geq n + 3$ (hence $m \geq n + 4$), so in particular

$$\|\phi\|_{H^{m-1}(\mathbb{R}^n)}(t) + \|\partial_t \phi\|_{H^{m-2}(\mathbb{R}^n)}(t) \leq D \quad \text{holds for some constant } D \text{ and all } t \in [0, T_\star).$$

We establish the conclusion for m. We use the energy estimate to estimate the H^m norm of the solution. For fixed $0 < T < T_\star$

$$\begin{aligned}\sup_{t\in[0,T)} \|\phi\|_{H^m(\mathbb{R}^n)}(t) &+ \sup_{t\in[0,T)} \|\partial_t \phi\|_{H^{m-1}(\mathbb{R}^n)}(t) \\ &\leq C\Bigg(\|f\|_{H^m(\mathbb{R}^n)} + \|g\|_{H^{m-1}(\mathbb{R}^n)} + \int_0^T \|F\|_{H^{m-1}(\mathbb{R}^n)}(t) \\ + \int_0^T \sum_{\substack{|\alpha|+|\beta|\leq m-1 \\ |\beta| \leq m-2}} \|\partial_x^\alpha a \partial\partial_x^\beta \phi\|_{L^2(\mathbb{R}^n)}(t)dt &+ \int_0^T \sum_{|\alpha|+|\beta|\leq m-1} \|\partial \partial_x^\alpha a \partial \partial_x^\beta \phi\|_{L^2(\mathbb{R}^n)}(t)dt \Bigg). \end{aligned} \tag{4.21}$$

Our goal is to show that the last three terms can be estimated *linearly* in the top order derivatives, which will then allow us to apply Gronwall's lemma to bound the top order norm. The key here is again the Sobolev embedding theorem.

Analogous to what we did in the well-posedness theorem, we estimate

$$\begin{aligned}&\sum_{0\leq|\alpha|\leq m-1} \|\partial_x^\alpha F(\phi, \partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \\ &\leq C\Bigg(\sum_{0\leq|\alpha|\leq m-1} \|\partial_x^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{0\leq|\alpha_1|+|\alpha_2|\leq m-1} \|\partial\partial_x^{\alpha_1}\phi \partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) \\ &\qquad + \text{cubic terms} + \ldots + \text{terms of order } (m-1)\Bigg). \end{aligned} \tag{4.22}$$

[4] Note that indeed $T_\star > 0$ by Theorem 4.2.1.

We focus on the quadratic term. Wlog $|\alpha_1| \leq \frac{m-1}{2}$. By Sobolev embedding we have

$$\|\partial_x^\alpha \partial\phi\|_{L^\infty(\mathbb{R}^n)}(t) \leq \|\partial\phi\|_{H^k(\mathbb{R}^n)}(t) \quad \text{for } k = \lfloor |\alpha| + n/2 + 1/2 \rfloor.$$

By the induction assumption, the right hand side is uniformly bounded on $[0, T_\star)$ for $k = m-2$. It follows that the left hand side is uniformly bounded for $|\alpha_1| \leq \frac{m-1}{2}$ since $\frac{m-1}{2} + \frac{n}{2} + \frac{1}{2} \leq m-2$ for $m \geq n+4$. We can therefore estimate the quadratic term as

$$\begin{aligned}\sum_{0\leq|\alpha_1|+|\alpha_2|\leq m-1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) &\leq C\left(\sum_{|\alpha_1|\leq\frac{m-1}{2}} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)\left(\sum_{|\alpha_2|\leq m-1} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\right) \\ &\leq C\cdot D\cdot\left(\sum_{|\alpha_2|\leq m-1} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\right) \qquad (4.23)\end{aligned}$$

where the constant D is from the induction assumption. This is indeed linear in the top order term! Of course cubic and higher order terms can be handled similarly (top order term in L^2, all lower order terms in L^∞). The terms involving a can be handled similarly. Therefore we deduce from (4.21) the estimate

$$\begin{aligned}&\sup_{t\in[0,T)} \|\phi\|_{H^m(\mathbb{R}^n)}(t) + \sup_{t\in[0,T)} \|\partial_t\phi\|_{H^{m-1}(\mathbb{R}^n)}(t) \\ &\leq C\left(\|f\|_{H^m(\mathbb{R}^n)} + \|g\|_{H^{m-1}(\mathbb{R}^n)} + D\int_0^T \left(\|\phi\|_{H^m(\mathbb{R}^n)}(t) + \|\partial_t\phi\|_{H^{m-1}(\mathbb{R}^n)}(t)\right) dt\right). \qquad (4.24)\end{aligned}$$

Gronwall's inequality shows a uniform bound in terms on T, D and C on the norm on the left hand side. □

Remark 4.3.2. *Strictly speaking we cheated a bit in the above proof: A priori we do not know whether writing down (4.21) makes sense because we do not know at this point whether the solution is in* $H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)$ *for* <u>*all*</u> $0 < T < T_\star$. *However, we do know from the well-posedness theorem (see Remark 4.2.5) that what we wrote down makes sense at least for all* $0 < T < T_{max}$ *for some* $0 < T_{max} < T_\star$, *including the conclusion*

$$\sup_{t\in[0,T)} \|\phi\|_{H^m(\mathbb{R}^n)}(t) + \sup_{t\in[0,T)} \|\partial_t\phi\|_{H^{m-1}(\mathbb{R}^n)}(t) \leq C\left(\|f\|_{H^m(\mathbb{R}^n)} + \|g\|_{H^{m-1}(\mathbb{R}^n)}\right)\exp(CDT_\star). \qquad (4.25)$$

Define now the set

$$A = \{T \in [0, T_\star) \;\mid\; \phi \in C^0([0,T], H^m(\mathbb{R}^n)) \cap C^1\left([0,T], H^{m-1}(\mathbb{R}^n)\right) \quad \textit{satisfying} \quad (4.25)\}.$$

By the above $\frac{T_{max}}{2} \in A$. *Assume for contradiction that* $A \neq [0, T_\star)$ *and let* $T^\star$ *be the smallest* T *with* $T^\star \notin A$. *Then, since the uniform bound (4.25) holds for all* $T = T^\star - \epsilon$ *with* $\epsilon > 0$ *we can, using the well-posedness theorem in* $H^m(\mathbb{R}^n)\times H^{m-1}(\mathbb{R}^n)$, *extend the solution past* $T^\star$ *as a* $C^0([0, T^\star+\epsilon], H^m(\mathbb{R}^n)) \cap C^1\left([0, T^\star+\epsilon], H^{m-1}(\mathbb{R}^n)\right)$ *solution for some* ϵ *with* $T^\star + \epsilon \leq T_\star$. *The extended solution will (by running the energy estimate again – now everything is defined!) satisfy the uniform bound (4.25) for any* $T < T^\star + \epsilon$. *Contradiction. Hence* $A = [0, T_\star)$.

4.4 Breakdown criteria

It is in general hard to determine the maximal time of existence $T_\star$ for a given initial data set. The next theorem provides criteria that signal the breakdown of the solution.

Theorem 4.4.1 (Breakdown criteria)**.** *Let* $f \in H^{n+3}(\mathbb{R}^n)$, $g \in H^{n+2}(\mathbb{R}^n)$ *and* ϕ *be the solution of (4.1) arising from Theorem 4.2.1. Let* $T_\star$ *be the maximal time of existence. If* $T_\star < \infty$, *then all of the following holds:*

1. $$\liminf_{t\to T_\star} \left(\|\phi\|_{H^{n+3}(\mathbb{R}^n)}(t) + \|\partial_t\phi\|_{H^{n+2}(\mathbb{R}^n)}(t)\right) = \infty,$$

2.

$$\limsup_{t\to T_\star}\left(\sum_{|\alpha|\le\lfloor\frac{n+2}{2}\rfloor}\|\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)+\sum_{|\alpha|\le\lfloor\frac{n}{2}\rfloor}\|\partial_t\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)=\infty\,,$$

3.

$$\limsup_{t\to T_\star}\left(\sum_{|\alpha|\le 1}\|\partial^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)+\|\partial_t\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)=\infty\,.$$

Remark 4.4.2. *Note that the last statement implies the second.*

Proof. The strategy to prove the three statements is always the same. We assume the condition does not hold, which yields some estimates on the solution which are in turn strong enough to apply the energy estimates to estimate the top order energy of the solution.

Proof of 1. Assume the criterion does not hold. Then there exists a sequence (t_m) of times with $t_m\to T_\star$ such that along that sequence

$$\|\phi\|_{H^{n+3}(\mathbb{R}^n)}(t_m)+\|\partial_t\phi\|_{H^{n+2}(\mathbb{R}^n)}(t_m)\le C$$

holds for a constant C uniformly in n. Let $\tau(C)>0$ be the time of existence promised by Theorem 4.2.1 for any initial data with $\|f\|_{H^{n+3}(\mathbb{R}^n)}+\|g\|_{H^{n+2}(\mathbb{R}^n)}\le C$. Choose t_M such that $T_\star-t_M<\frac{\tau}{2}$ and apply Theorem 4.2.1 with data at $t=t_M$. This extends the solution to $\left[0,T_\star+\frac{\tau}{2}\right)$ and we hence obtain the desired contradiction to $T_\star$ being the maximal time of existence.

Proof of 2. Assume for contradiction that $T_\star<\infty$ and

$$\limsup_{t\to T_\star}\left(\sum_{|\alpha|\le\lfloor\frac{n+2}{2}\rfloor}\|\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)+\sum_{|\alpha|\le\lfloor\frac{n}{2}\rfloor}\|\partial_t\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)<\infty\,.$$

It follows that

$$\sup_{t\in[0,T_\star)}\left(\sum_{|\alpha|\le\lfloor\frac{n+2}{2}\rfloor}\|\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)+\sum_{|\alpha|\le\lfloor\frac{n}{2}\rfloor}\|\partial_t\partial_x^\alpha\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)\le D.$$

We will use this estimate to show that the $H^{n+3}(\mathbb{R}^n)\times H^{n+2}(\mathbb{R}^n)$ norm of the solution remains bounded as $t\to T_\star$ reducing the problem to the case treated in 1.

We start by showing **boundedness of the $H^{n+1}(\mathbb{R}^n)\times H^n(\mathbb{R}^n)$ norm.** The energy estimate for ϕ implies

$$\begin{aligned}\sup_{t\in[0,T)}\|\phi\|_{H^{k+1}(\mathbb{R}^n)}+\sup_{t\in[0,T)}\|\partial_t\phi\|_{H^k(\mathbb{R}^n)}\le C\Bigg(\|f\|_{H^{k+1}(\mathbb{R}^n)}+\|g\|_{H^k(\mathbb{R}^n)}+\int_0^T\|F\|_{H^k}(t)dt\\
+\int_0^T\Big(\sum_{|\gamma|+|\sigma|\le k}\sum_{\alpha,\beta}\|\partial\partial_x^\gamma\left(a^{\alpha\beta}(\phi)\right)\partial\partial_x^\sigma\phi\|_{L^2(\mathbb{R}^n)}+\sum_{\substack{|\gamma|+|\sigma|\le k\\|\sigma|\le k-1}}\sum_{\alpha,\beta}\|\partial_x^\gamma\left(a^{\alpha\beta}(\phi)\right)\partial\partial\partial_x^\sigma\phi\|_{L^2(\mathbb{R}^n)}\Big)dt\Bigg)\\
\times\exp\left(C\int_0^T\|\partial a\|_{L^\infty(\mathbb{R}^n)}(t)dt\right)\end{aligned}\tag{4.26}$$

for all $T\le T_\star$, which we apply with $k=n$. We again focus on the terms involving F, the terms involving a can be treated entirely analogously. We have

$$\begin{aligned}&\sum_{|\alpha|\le k}\|\partial_x^\alpha F(\phi,\partial\phi)\|_{L^2(\mathbb{R}^n)}(t)\\
&\le C\left(\sum_{0\le|\alpha|\le k}\|\partial\partial_x^\alpha\phi\|_{L^2(\mathbb{R}^n)}(t)+\sum_{0\le|\alpha_1|+|\alpha_2|\le k}\|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)+\text{cubic}+\ldots+\text{order }k\right)\,.\end{aligned}$$

For the quadratic term we have wlog $|\alpha_1| \leq \lfloor \frac{k}{2} \rfloor$. Since we are assuming $k = n$ we have $|\alpha_1| \leq \lfloor \frac{n}{2} \rfloor$. Therefore, the assumption yields

$$\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C\left(\sum_{|\alpha_1|\leq\lfloor\frac{n}{2}\rfloor} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)\left(\sum_{|\alpha_2|\leq n} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\right)$$
$$\leq CD \sum_{|\alpha_2|\leq n} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\,. \tag{4.27}$$

For the cubic and higher order terms we can of course perform the analogous estimate (top order term in L^2, lower order terms in L^∞) and control

$$\sum_{|\alpha|\leq n} \|\partial_x^\alpha F(\phi,\partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \leq C\,(1+D)^n \sum_{|\alpha_2|\leq n} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\,.$$

The same estimate can be established for all terms involving $a^{\alpha\beta}$ in (4.26) producing

$$\sup_{t\in[0,T)} \|\phi\|_{H^{n+1}(\mathbb{R}^n)} + \sup_{t\in[0,T)} \|\partial_t\phi\|_{H^n(\mathbb{R}^n)}$$
$$\leq C\,(T_\star, D)\left(\|f\|_{H^{n+1}(\mathbb{R}^n)} + \|g\|_{H^n(\mathbb{R}^n)} + \int_0^T C\,(1+D)^n \sum_{|\alpha_2|\leq n} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)dt\right). \tag{4.28}$$

Applying Gronwall's Lemma yields

$$\sup_{t\in[0,T)} \|\phi\|_{H^{n+1}(\mathbb{R}^n)} + \sup_{t\in[0,T)} \|\partial_t\phi\|_{H^n(\mathbb{R}^n)} \leq C\,(T_\star, D)\left(\|f\|_{H^{n+1}(\mathbb{R}^n)} + \|g\|_{H^n(\mathbb{R}^n)}\right)$$

as desired. We next show **boundedness of the $H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)$ norm.** We apply (4.26) with $k = n+1$. Suppose first that n is odd. Then in the quadratic term

$$\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) \tag{4.29}$$

we either have $|\alpha_1| \leq \lfloor \frac{n}{2} \rfloor$ or $|\alpha_2| \leq \lfloor \frac{n}{2} \rfloor$ or $|\alpha_1| = |\alpha_2| = \frac{n+1}{2}$. We can therefore estimate

$$\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) \tag{4.30}$$

by

$$\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C\left(\sum_{|\alpha_1|\leq\lfloor\frac{n}{2}\rfloor} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)\left(\sum_{|\alpha_2|\leq n+1} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\right)$$
$$+ C\left(\sum_{|\alpha_1|=\frac{n+1}{2}} \|\partial\partial_x^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^n)}(t)\right)\left(\sum_{|\alpha_2|=\frac{n+1}{2}} \|\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t)\right).$$

The first factor in the first sum on the right is bounded by D and the second factor in the second sum is bounded by the $H^{n+2}(\mathbb{R}^n) \times H^{n+1}(\mathbb{R}^n)$ of ϕ (that we just established boundedness for in the previous step) in view of $\frac{n+1}{2} + 1 \leq n+2$. For the first factor in the second sum we can use Sobolev embedding since $\frac{n+1}{2} + \frac{n}{2} + \frac{1}{2} \leq n+1$. This yields in total

$$\sum_{0\leq|\alpha_1|+|\alpha_2|\leq n+1} \|\partial\partial_x^{\alpha_1}\phi\partial\partial_x^{\alpha_2}\phi\|_{L^2(\mathbb{R}^n)}(t) \leq C\,(T_\star, D)\left(\sum_{|\alpha|\leq n+1} \|\partial\partial_x^{\alpha}\phi\|_{L^2(\mathbb{R}^n)}(t)\right). \tag{4.31}$$

The case of n even is much easier because now we either have $|\alpha_1| \leq \lfloor \frac{n}{2} \rfloor$ or $|\alpha_2| \leq \lfloor \frac{n}{2} \rfloor$ and the estimate (4.31) is readily established. Since again higher order terms can be treated in the same manner and since the same argument works for the terms involving $a^{\alpha\beta}$ in (4.26) we conclude, from Gronwall's inequality

$$\sup_{t\in[0,T)} \|\phi\|_{H^{n+2}(\mathbb{R}^n)} + \sup_{t\in[0,T)} \|\partial_t\phi\|_{H^{n+1}(\mathbb{R}^n)} \leq C\left(T_\star, D\right) \Big(\|f\|_{H^{n+2}(\mathbb{R}^n)} + \|g\|_{H^{n+1}(\mathbb{R}^n)} \Big) .$$

We finally show **boundedness of the $H^{n+3}(\mathbb{R}^n) \times H^{n+2}(\mathbb{R}^n)$ norm.** We leave the details to the reader.

Proof of (3). Assuming the condition does not hold we would have

$$\sup_{t\in[0,T_\star)} \left(\sum_{|\alpha|\leq 1} \|\partial_x^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) + \|\partial_t\phi\|_{L^\infty(\mathbb{R}^n)}(t) \right) \leq D. \tag{4.32}$$

It is easy to see (apply (4.26) with $k = 0$ and use $F(\phi, \partial\phi) \leq C_D\left(|\phi| + |\partial\phi|\right)$ following in turn from the assumption $F(0,0) = 0$ and (4.3)) that this implies a uniform bound for the $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ norm of the solution. We now want to apply (4.26) for higher k. Note that for $1 \leq |\alpha| \leq n+1$ we have that $\partial_x^\alpha F(\phi, \partial\phi)$ is a finite sum of terms of the form

$$\left[\frac{\partial^{A+|B|}F}{(\partial\phi)^A(\partial p_0)^{B_0}(\partial p_1)^{B_1}\dots(\partial p_n)^{B_n}}(\phi, p = \partial\phi)\right] \left(\prod_{j=1}^{A} \partial_x^{\alpha_j}\phi \prod_{j=A+1}^{A+|B|} \partial\partial_x^{\alpha_j}\phi \right)$$

with $1 \leq A + |B| \leq |\alpha|$ and $\alpha_1 + \alpha_2 + \dots \alpha_{A+|B|} = \alpha$. [Read this as follows: $A \geq 0$ and $B = (B_0, B_1, ..., B_n)$ with $|B| = B_0 + B_1 + \dots + B_n$ determine how many times (and how) F itself gets hit by the derivative, which corresponds to the number of terms the product in the round bracket has in total. For each such factor in the product, the tuples α_j can distribute in various ways but their sum must always be the tuple α that is applied in total. We are using the convention that the product equals one if the set of allowed j is the empty set, that all α_j that appear are non-trivial and that the $\partial \in \{\partial_t, \partial_{x_1}, ...\partial_{x_n}\}$ that appears in the second product may be different for each factor.]

Now the term in the square bracket is uniformly bounded in L^∞ by (4.32) and (4.3). We therefore deduce applying Hölder's inequality the estimate

$$\begin{aligned} \|\partial_x^\alpha F(\phi,\partial\phi)\|_{L^2(\mathbb{R}^n)}(t) &\leq C_D \Big\| \prod_{j=1}^{A} \partial_x^{\alpha_j}\phi \prod_{j=A+1}^{A+|B|} \partial\partial_x^{\alpha_j}\phi \Big\|_{L^2(\mathbb{R}^n)}(t) \\ &\leq C_D \prod_{j=1}^{A} \|\partial_x^{\alpha_j}\phi\|_{L^{\frac{2}{\lambda_j}}(\mathbb{R}^n)}(t) \prod_{j=A+1}^{A+|B|} \|\partial\partial_x^{\alpha_j}\phi\|_{L^{\frac{2}{\lambda_j}}(\mathbb{R}^n)}(t) \end{aligned} \tag{4.33}$$

provided that $\sum_{j=1}^{A+|B|} \lambda_j = 1$.[5] We will in fact choose $\lambda_j = \frac{|\alpha_j|}{|\alpha|}$ so that this condition is manifestly satisfied.

The key now is to apply the Gargliardo-Nirenberg interpolation estimate from Example Sheet 5, which implies that

$$\|\partial_x^{\alpha_j}\phi\|_{L^{\frac{2}{\lambda_j}}(\mathbb{R}^n)}(t) \leq C\left(\|\phi\|_{L^\infty(\mathbb{R}^n)}\right)^{\frac{|\alpha|-|\alpha_j|}{|\alpha_j|}} \left(\|\phi\|_{H^{|\alpha|}(\mathbb{R}^n)}\right)^{\frac{|\alpha_j|}{|\alpha|}} \leq C_D\left(\|\partial_t\phi\|_{H^{|\alpha|}(\mathbb{R}^n)} + \|\phi\|_{H^{|\alpha|+1}(\mathbb{R}^n)}\right)^{\frac{|\alpha_j|}{|\alpha|}} ,$$

$$\|\partial_x^{\alpha_j}\partial\phi\|_{L^{\frac{2}{\lambda_j}}(\mathbb{R}^n)}(t) \leq C\left(\|\partial\phi\|_{L^\infty(\mathbb{R}^n)}\right)^{\frac{|\alpha|-|\alpha_j|}{|\alpha_j|}} \left(\|\partial\phi\|_{H^{|\alpha|}(\mathbb{R}^n)}\right)^{\frac{|\alpha_j|}{|\alpha|}} \leq C_D\left(\|\partial_t\phi\|_{H^{|\alpha|}(\mathbb{R}^n)} + \|\phi\|_{H^{|\alpha|+1}(\mathbb{R}^n)}\right)^{\frac{|\alpha_j|}{|\alpha|}} .$$

It follows that

$$\|\partial_x^\alpha F(\phi,\partial\phi)\|_{L^2(\mathbb{R}^n)}(t) \leq C_D\left(\|\partial_t\phi\|_{H^{|\alpha|}(\mathbb{R}^n)} + \|\phi\|_{H^{|\alpha|+1}(\mathbb{R}^n)}\right) . \tag{4.34}$$

In other words, we can estimate the term involving F linearly in the top order norm. Since the terms involving $a^{\alpha\beta}$ in (4.26) can be treated entirely analogously, we conclude from Gronwall's inequality that we can bound uniformly for $t \in [0, T^\star)$ the $H^{n+3}(\mathbb{R}^n) \times H^{n+2}(\mathbb{R}^n)$ norm of the solution. The contradiction is now obtained as in the proof of 1. □

[5] It is implicit that the right hand side has to be summed over all $A+|B| \leq |\alpha|$ and all decompositions $\alpha_1 + ... + \alpha_{A+|B|} = \alpha$.

4.5 Some Examples

4.5.1 ODE blow-up

Recall the Problems from Sheet 1 and Sheet 6. In these problems both ϕ and $\partial_t \phi$ blow up.

4.5.2 Shocks

We next give a simple example in $1+1$ dimensions where derivatives blow up but the solution itself remains bounded all the way to the singularity.

Recall Burger's equation

$$\partial_t u + u \partial_x u = 0 \quad \text{with data} \quad u(t=0,x) = h(x) \text{ and } h \text{ smooth.}$$

Fix a point $(0,a)$ on the x-axis. It is easy to show that the solution is constant along the characteristic lines

$$x = h(a)t + a.$$

It follows (how?) that derivatives of u blow up in finite time (unless $h(a)$ is monotonically increasing) while u remains uniformly bounded (in fact, the L^∞_x norm of u is easily seen to be conserved in time for as long as the solution exists). Such a solution is called a shock. Note shocks will form for arbitrarily small data!

We now consider a solution of Burger's equation arising from an h which is not monotonically increasing and such that $\|h-1\|_{L^\infty} \leq \frac{1}{20}$. From the above considerations, the solution u blows up in finite time while $\|u-1\|_{L^\infty([0,T^\star)\times\mathbb{R}^n)} \leq \frac{1}{20}$ all the way to the shock discontinuity.

Set $\phi = u - 1$. Then ϕ satisfies

$$\partial_t \phi + (1+\phi)\partial_x \phi = 0$$

hence

$$\partial_t^2 \phi + (1+\phi)\partial_t \partial_x \phi + (\partial_t \phi)(\partial_x \phi) = 0$$

or, inserting the equation for $\partial_x \partial_t \phi$,

$$\partial_t^2 \phi - (1+\phi)^2 \partial_x^2 \phi = (1+\phi)\left(\partial_x \phi\right)^2 - (\partial_t \phi)(\partial_x \phi)\,.$$

So ϕ satisfies a wave equation of the type discussed in lectures and if we choose initial data $\phi(t=0,x) = h(x) - 1$, $\phi_t(t=0,x) = -h(x)h_x(x)$, then the unique solution will blow up in finite time in the sense that derivatives of ϕ blow up, while ϕ itself remains small in the sense that $\|u-1\|_{L^\infty([0,T^\star)\times\mathbb{R}^n)} \leq \frac{1}{20}$ holds.

Chapter 5

Global existence and regularity for subcritical equations

Having established a satisfactory local theory for non-linear wave equations and having seen some examples of what might happen globally, we now turn to investigating more systematically the global behaviour of non-linear wave equations. We start with the following class of *semi-linear* equations for $\phi : I \times \mathbb{R}^n \to \mathbb{R}$:

$$\begin{cases} \Box\phi = \pm|\phi|^{p-1}\phi \\ \phi(0,x) = f(x) \\ \partial_t\phi(0,x) = g(x) \end{cases} \tag{5.1}$$

for integer $p > 1$. We will start by investigating the case $n = 3$, $p = 3$ and the plus sign in (5.1). We will then introduce an important concept, that of *scaling*.

We will then say something about the general case but postpone a more detailed study to a later point, once we have proven the famous Strichartz estimates, which play a fundamental role.

5.1 The equation $\Box\phi = \phi^3$ in $3 + 1$ dimensions

Let us consider the following semi-linear equation:

$$\begin{cases} \Box\phi = |\phi|^2\phi \\ \phi(0,x) = f(x) \\ \partial_t\phi(0,x) = g(x) \end{cases} \tag{5.2}$$

Proposition 5.1.1. *As long as the solution of (5.2) remains sufficiently regular, we have the energy conservation law*

$$E(t) = E(0) \tag{5.3}$$

for

$$E(t) := \int_{\mathbb{R}^3} \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}\sum_{i=1}^{3}(\partial_i\phi)^2 + \frac{1}{4}|\phi|^4\right](t,x)dx\,.$$

We can now prove

Theorem 5.1.2. *Assume (f, g) are smooth with $f \in H^6(\mathbb{R}^3)$ and $g \in H^5(\mathbb{R}^3)$. Then there exists a global in time smooth solution of (5.2).*

Remark 5.1.3. *The assumption of smoothness is merely for convenience. Theorem 4.2.1 already implies that the solution is classical which is in particular sufficient for Proposition 5.1.1 to hold.*

Proof. From Theorem 4.2.1 and the persistence of regularity we infer the existence of a (unique) local-in time solution of (5.2). By Theorem 4.4.1 it suffices to show that for every T we have the bound

$$\sum_{|\alpha|\leq 1} \|\partial_x^\alpha \phi\|_{L^\infty([0,T]\times\mathbb{R}^3)} + \|\partial_t \phi\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq C(T)\,.$$

By Sobolev embedding this is implied if we can show

$$\sup_{t\in[0,T]} \|\phi\|_{H^3(\mathbb{R}^3)}(t) + \sup_{t\in[0,T]} \|\partial_t \phi\|_{H^2(\mathbb{R}^3)}(t) \leq C(T)\,.$$

Note that the $\dot{H}^1 \times L^2$-norm is uniformly bounded by the conservation law and we can use the (fundamental theorem of calculus) argument from Proposition 2.8.1 to estimate the L^2-norm of ϕ itself by a constant growing linearly in T. We next control the H^2-norm from the energy estimate.

We have

$$\sup_{t\in[0,T]} \|\phi\|_{H^2(\mathbb{R}^3)} + \sup_{t\in[0,T)} \|\partial_t \phi\|_{H^1(\mathbb{R}^3)} \leq C\left(\|f\|_{H^2(\mathbb{R}^3)} + \|g\|_{H^1(\mathbb{R}^3)} + \int_0^T \|F\|_{H^1(\mathbb{R}^3)}(t)dt \right). \tag{5.4}$$

We note using Hölder and the Sobolev embedding $\|\phi\|_{L^6(\mathbb{R}^3)} \leq C\|\phi\|_{\dot{H}^1(\mathbb{R}^3)}$ from Example Sheet 6

$$\begin{aligned}
\|F\|_{H^1(\mathbb{R}^3)}(t) &\leq \|\phi\phi\phi\|_{L^2(\mathbb{R}^3)}(t) + 3\sum_{i=1}^3 \|\phi\phi\partial_{x_i}\phi\|_{L^2(\mathbb{R}^3)}(t) \\
&\leq \|\phi\|^3_{L^6(\mathbb{R}^3)}(t) + 3\sum_{i=1}^3 \|\phi\|_{L^6(\mathbb{R}^3)}\|\phi\|_{L^6(\mathbb{R}^3)}\|\partial_{x_i}\phi\|_{L^6(\mathbb{R}^3)}(t) \\
&\leq C\|\phi\|^3_{\dot{H}^1(\mathbb{R}^3)}(t) + C\|\phi\|^2_{\dot{H}^1(\mathbb{R}^3)}\|\partial_x\phi\|_{\dot{H}^1(\mathbb{R}^3)}(t) \\
&\leq CE^3 + CE^2\|\phi\|_{H^2(\mathbb{R}^3)}(t).
\end{aligned} \tag{5.5}$$

Inserting this into the energy estimate and applying Gronwall's inequality we infer

$$\sup_{t\in[0,T]} \|\phi\|_{H^2(\mathbb{R}^3)}(t) + \sup_{t\in[0,T]} \|\partial_t \phi\|_{H^1(\mathbb{R}^3)}(t) \leq C(T)\,.$$

We finally estimate the $H^3 \times H^2$-norm. We have

$$\sup_{t\in[0,T]} \|\phi\|_{H^3(\mathbb{R}^3)} + \sup_{t\in[0,T)} \|\partial_t \phi\|_{H^2(\mathbb{R}^3)} \leq C\left(\|f\|_{H^3(\mathbb{R}^3)} + \|g\|_{H^2(\mathbb{R}^3)} + \int_0^T \|F\|_{H^2(\mathbb{R}^3)}(t)dt \right). \tag{5.6}$$

We note using Hölder and the Sobolev embedding $\|\phi\|_{L^6(\mathbb{R}^3)} \leq C\|\phi\|_{\dot{H}^1(\mathbb{R}^3)}$ from Example Sheet 6

$$\begin{aligned}
\|F\|_{H^2(\mathbb{R}^3)}(t) &\leq C(T) + \|F\|_{\dot{H}^2(\mathbb{R}^3)}(t) \\
&\leq C\sum_{i,j} \|\phi\phi\partial_{x_i}\partial_{x_j}\phi\|_{L^2(\mathbb{R}^3)}(t) + C\sum_{i,j} \|\phi\partial_{x_i}\phi\partial_{x_j}\phi\|_{L^2(\mathbb{R}^3)}(t) \\
&\leq C\|\phi\|^2_{L^\infty(\mathbb{R}^3)}\|\phi\|_{H^2(\mathbb{R}^3)}(t) + C\sum_{i=1}^3 \|\phi\|_{L^6(\mathbb{R}^3)}\|\partial_{x_i}\phi\|^2_{L^6(\mathbb{R}^3)}(t) \\
&\leq C\|\phi\|^3_{H^2(\mathbb{R}^3)}(t) + CE\|\phi\|^2_{H^2(\mathbb{R}^3)}(t) \\
&\leq C(T).
\end{aligned} \tag{5.7}$$

Inserting this into the energy estimate and applying Gronwall's inequality we infer

$$\sup_{t\in[0,T]} \|\phi\|_{H^3(\mathbb{R}^3)}(t) + \sup_{t\in[0,T]} \|\partial_t \phi\|_{H^2(\mathbb{R}^3)}(t) \leq C(T)$$

as desired. The proof is complete. ☐

Remark 5.1.4. *An alternative to prove this theorem is to prove a stronger local theorem for (5.2). Indeed you can convince yourself that Picard iteration closes in* $C^0\left([0,T]\times H^1(\mathbb{R}^3)\right)\cap C^1\left([0,T]\times L^2(\mathbb{R}^3)\right)$ *(since the problem is semi-linear we can prove boundedness and convergence in the same space) and that the time* T *of existence depends only on the size of* $\|f\|_{\dot{H}^1(\mathbb{R}^3)}$ *and* $\|g\|_{L^2(\mathbb{R}^3)}$*. Moreover, higher regularity is propagated. By the energy conservation law the* $\dot{H}(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ *norm of the solution is conserved so the assumption* $T_\star<\infty$ *can be quickly lead to a contradiction by applying the stronger well-posedness theorem suffciently close to* $T_\star$*.*

5.2 Scaling

Motivated by the introductory example in the previous section we introduce now a heuristic principle that gives us some intuition on what to expect for non-linear equations of the type (5.1) with the plus sign on the right hand side, i.e. the problems that admit a coercive energy that is conserved in time.[1]

The key observation is that the problem (5.1) has a conserved coercive energy and a scaling symmetry. More secifically, if $\phi(t,x)$ is a solution of (5.1) in dimension $1+3$ (that is $n=3$) then

$$\phi_\lambda(t,x):=\lambda^{\frac{2}{p-1}}\phi\left(\lambda t,\lambda x\right)$$

is also a solution. Now the energy scales as

$$\begin{aligned}E[\phi_\lambda]\,(t)&=\int_{\mathbb{R}^3}\lambda^{\frac{2(p+1)}{p-1}}\left[\frac{1}{2}|\partial_t\phi|^2+\frac{1}{2}\sum_{i=1}^3(\partial_i\phi)^2+\frac{1}{4}|\phi|^4\right](\lambda t,\lambda x)\lambda^{-3}d^3(\lambda x)\\&=\lambda^{-3+\frac{2(p+1)}{p-1}}E[\phi](\lambda t)=\lambda^{-3+\frac{2(p+1)}{p-1}}E[\phi](t)\,.\end{aligned}\tag{5.8}$$

In other words, if $1\le p<5$ (energy subcritical case) then the energy grows to infinity as $\lambda\to\infty$. For $p=5$ (energy critical case) the energy is invariant under scaling and for $p>5$ the energy goes to zero as $\lambda\to\infty$ (energy supercritical). This can be interpreted as follows: For $p<5$, going to smaller scales, i.e. sending $\lambda\to\infty$ (which you can view as trying to concentrate a given piece of solution to a smaller region of spacetime), is registered by the energy growing. The conservation law (energy globally bounded!) therefore prevents this mechanism of concentrating the solution and hence a potential blow-up mechanism of the equation. This heuristic is indeed correct in that we shall be able to prove existence of a global solution for $1\le p<5$ (in the defocussing case, i.e. when there is a coercive conserved energy corresponding to the plus sign in (5.1)). We already proved this for $p=3$. You can convince yourself that for $1\le p\le 3$ the same techniques will work to establish global existence. For $3<p<5$ (subcritical) we will need the so-called Strichartz estimates that we state below (but only prove later in the course). In the supercritical case only small data global existence has been established, the large data is still not well understood. In the critical case, global existence still holds but requires additional techniques (Morawetz estimates). For small data global existence in the critical case, see Example Sheet 7.

5.3 Strichartz estimates

We consider the solution of the linear Cauchy problem in $\mathbb{R}^{1+n}$ for $n\ge 2$:

$$\Box\phi=F\qquad,\qquad\phi(0,x)=f(x)\quad,\quad\partial_t\phi(0,x)=g(x)\,.\tag{5.9}$$

We assume that $f,g:\mathbb{R}^n\to\mathbb{R}$ are Schwartz and that $F:\mathbb{R}^{1+n}\to\mathbb{R}$ is also Schwartz. As we have seen a couple of times, by density we can then infer statements in lower regularity from the estimates that we prove.

Theorem 5.3.1. *The solution of (9.35) satisfies the estimate*

$$\|\phi\|_{L^q_tL^r_x}\le C_{q,r}\left(\|f\|_{\dot{H}^s}+\|g\|_{\dot{H}^{s-1}}+\|F\|_{L^{q'}_tL^{r'}_x}\right)\,,\tag{5.10}$$

[1] The plus sign is also known as the *defocussing* case, the minus sign as the *focussing* case for reasons that will become clear below.

where $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$ *provided* (q, r) *is wave admissible, i.e.*

$$2 \leq q \leq \infty \quad , \quad 2 \leq r < \infty \quad , \quad \frac{2}{q} \leq \frac{n-1}{2}\left(1 - \frac{2}{r}\right) \tag{5.11}$$

and the scaling condition

$$\frac{n}{r} + \frac{1}{q} = \frac{n}{r'} + \frac{1}{q'} - 2 \tag{5.12}$$

holds. The constant $C_{q,r}$ *depends only on* q *and* r*.*

We will postpone the complicated proof (which requires some Littlewood Paley theory) to a later point of the course in order not to destroy the flow. However, some remarks are already in order.

1. If the numerology is confusing (due to the number of parameters appearing!), think of $n = 3$ and $s = 1$ for now, which is the case of interest to us. Then $q = \frac{2r}{r-6}$ and $\frac{2}{q} + \frac{2}{r} \leq 1$, the latter being automatically satisfied for $r \geq 2$ and $q \geq 2$. The quantities (q', r') then have to be determined from (9.15). You can derive the conditions $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$ and (9.15) from scaling considerations (see Exercise Sheet 7). Finally, you might recognize the case $r = 6$, $q = \infty$ from earlier.

2. The estimate involves the "mixed" spaces $L^q_t L^r_x$. These are defined as follows. For $1 \leq r < \infty$, $1 \leq q < \infty$ we define $L^q_t L^r_x$ to be the space of (equivalence classes of) measurable functions on $\mathbb{R}_t \times \mathbb{R}^n_x$ such that the norm

$$\|u\|_{L^q_t L^r_x} = \left(\int_{\mathbb{R}} dt \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \tag{5.13}$$

   is finite. An obvious analogous definition can be made for the cases where $p = \infty$ or $r = \infty$ (or both).

3. The mixed spaces and the Strichartz estimate are useful because they

   - capture the decay (i.e. the dispersion) in time of the solution in an averaged (integrated) sense.
   - "gain regularity" in that they allow us to estimate the solution in a higher L^p space spatially at the cost of averaging in time (of course this is also intimately connected to dispersion!)

   Recall that in our iteration scheme for local wellposedness, we were always "naive" about the inhomogeneous term, in the sense of applying Sobolev embedding in *space* using the time integration to apply Gronwall. The Strichartz estimate gives us a chance of exploiting the integration in time in a more refined way to relate the inhomogeneous term back to the energy norm ($s = 1$). Note in this context that the Strichartz estimate (9.36) can also be localised in time i.e. with all time integrations in the mixed norms performed over $[0, T]$ for some $T > 0$ (why?) instead of all of $\mathbb{R}$. This will be useful below.

4. You can establish the following facts (see Analysis review problems)

   (a) The $L^q_t L^r_x$ are Banach spaces.

   (b) Smooth functions of compact support, $C^\infty_0(\mathbb{R}_t \times \mathbb{R}^n_x)$ are dense in $L^q_t L^r_x$.

   (c) The dual of $L^q_t L^r_x$ can be identified with $L^{q'}_t L^{r'}_x$ where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.

5. The wave admissibility condition will be discussed later.

5.4 The equation $\Box\phi = |\phi|^{p-1}\phi$ with $3 < p < 5$ in $3+1$ dimensions

We now use the Strichartz estimate and the conservation law to establish the following:

Theorem 5.4.1. *Let $3 < p < 5$ and consider (5.1) with the plus sign. Assume f and g are smooth with $f \in H^1(\mathbb{R}^3)$ and $g \in L^2(\mathbb{R}^3)$. Then there exists a unique global in time solution of (5.2).*

The key to prove the above theorem is to establish a local-wellposedness theorem in a norm which is at the level of the conserved energy (cf. with Remark 5.1.4). Note that for the local theorem below, we do not need an assumption on the sign of the non-linearity.

Proposition 5.4.2. *Let $3 < p < 5$, consider (5.1) and assume $f \in \dot{H}^1(\mathbb{R}^3)$ and $g \in L^2(\mathbb{R}^3)$. Then there exists a $T > 0$ for which (5.1) has a unique solution*

$$\phi \in C^0([0,T], \dot{H}^1(\mathbb{R}^3)) \cap C^1([0,T], L^2(\mathbb{R}^3))) \cap L^{\frac{2p}{p-3}}([0,T], L^{2p}(\mathbb{R}^3))).$$

Moreover, the time T is proportional to the size of the initial data, i.e.

$$T \sim \left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} \right)^{-\lambda} \qquad \text{for some } \lambda > 0.$$

Proof of Theorem 5.4.1. A local solution exists by Proposition 5.4.2. Moreover, the conservation of energy identity holds for the this solution, i.e.

$$E(t) := \int_{\mathbb{R}^3} \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}\sum_{i=1}^{3}(\partial_i \phi)^2 + \frac{1}{p+1}|\phi|^{p+1} \right](t,x)dx$$

is conserved in time for as long as the solution exists.[2] Consequently, $\|\phi\|_{\dot{H}^1(\mathbb{R}^3)} + \|\partial_t\phi\|_{L^2(\mathbb{R}^3)} \leq \mathcal{D}_0$ is uniformly bounded by initial data for as long as the solution exists. Let $\tau := C\mathcal{D}_0^{-\lambda}$ be the time of existence associated with the data of size $\mathcal{E}_0$. Assume for contradiction that the solution only existed up to to time $0 < T_\star < \infty$. Applying Proposition 5.4.2 from time $T_\star - \frac{\tau}{2}$ extends the solution passed $T_\star$ producing the desired contradiction. □

Proof of Proposition 5.4.2. We set up a contraction mapping argument.[3] We define the Banach space

$$X := \left\{ \phi \in C^0([0,T], \dot{H}^1(\mathbb{R}^3)) \cap C^1([0,T], L^2(\mathbb{R}^3)) \cap L^{\frac{2p}{p-3}}([0,T], L^{2p}(\mathbb{R}^3)) \right\}$$

equipped with the norm

$$\|\phi\|_X := \sup_{t\in[0,T]} \|\partial_t\phi\|_{L^2(\mathbb{R}^3)} + \sup_{t\in[0,T]} \|\partial_x\phi\|_{L^2(\mathbb{R}^3)} + \|\phi\|_{L^{\frac{2p}{p-3}}([0,T],L^{2p}(\mathbb{R}^3))}.$$

We define the closed ball of radius R, $X_R \subset X$ by

$$X_R = \left\{ u \in X \mid \|\phi\|_X \leq R := 2(C_p+1)E_0 \right\},$$

where C_p is the constant appearing in the Strichartz estimate and $E_0 := \|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)}$.

We define a map $\Phi : X_R \to X_R$ by $\psi \mapsto \phi$ where ϕ is defined as the unique solution to the inhomogeneous linear wave equation

$$\Box\phi = \pm|\psi|^{p-1}\psi \tag{5.14}$$

[2] This requires some justification as the *derivation* of the energy identity (integration by parts) assumes higher regularity. This can be provided as follows: Using persistence of regularity one deduces that the local solution of Proposition 5.4.2 is in fact classical if f and g are assumed to be smooth (see Exercise Sheet 7. Warning: The non-linearity F is not smooth but only C^3 in general. One can still commute at least three times though, which is sufficient to show the solution is classical). One can then use an approximation argument to show that the energy conservation law also holds for the less regular solution.

[3] This is of course equivalent to the iterative argument seen before, the iteration appearing in the proof of Banach's fixed point theorem.

with data (f,g). We have to check that this is well defined (cf. Theorem 3.4.1), in particular that the right hand side is in $L^1\left([0,T],L^2(\mathbb{R}^3)\right)$ and that the solution ϕ again lives in X_R as claimed. From Hölder's inequality we have

$$\begin{aligned}\||\psi|^{p-1}\psi\|_{L^1([0,T],L^2(\mathbb{R}^3))} &= \int_0^T dt \||\psi|^{p-1}\psi\|_{L^2(\mathbb{R}^3)} = \int_0^T dt \left(\|\psi\|_{L^{2p}(\mathbb{R}^3)}\right)^p \cdot 1 \\ &\leq \left(\int_0^T dt \left(\|\psi\|_{L^{2p}(\mathbb{R}^3)}\right)^{p\tilde{p}}\right)^{1/\tilde{p}} T^{1/\tilde{q}} \leq \left(\|\psi\|_{L^{\frac{2p}{p-3}}([0,T],L^{2p}(\mathbb{R}^3))}\right)^p \cdot T^{\frac{5-p}{2}}, \end{aligned} \tag{5.15}$$

where we have set $\tilde{p} = \frac{2}{p-3}$ and $\tilde{q} = \frac{2}{5-p}$. This shows that indeed the right hand side of (5.14) is in the correct space for Theorem 3.4.1 to apply and the associated energy estimate and the Strichartz estimate (with $q'=1$, $r'=2$, satisfying $\frac{3}{r}+\frac{1}{q} = \frac{3}{r'}+\frac{1}{q'}-2$ since $r=2p$ and $q=\frac{2p}{p-3}$) gives

$$\begin{aligned}\|\phi\|_X &\leq E_0 + \||\psi|^{p-1}\psi\|_{L^1([0,T],L^2(\mathbb{R}^3))} + C_p\left(E_0 + \||\psi|^{p-1}\psi\|_{L^1([0,T],L^2(\mathbb{R}^3))}\right) \\ &\leq (C_p+1)\left(E_0 + R^p \cdot T^{\frac{5-p}{2}}\right), \end{aligned} \tag{5.16}$$

which means $u \in X_R$ as long as we choose T small enough so that $T^{\frac{5-p}{2}} R^p = T^{\frac{5-p}{2}}((2C_p+1)E_0)^p \leq E_0$ holds (Condition 1).

We next show that the map Φ is a contraction. We note that $\phi_1 = \Phi(\psi_1)$ and $\phi_2 = \Phi(\psi_2)$ satisfies the linear equation

$$\Box(\phi_1-\phi_2) = |\psi_1|^{p-1}\psi_1 - |\psi_2|^{p-1}\psi_2$$

with trivial data. We apply the energy estimate and using that

$$\begin{aligned}\||\psi_1|^{p-1}\psi_1 - |\psi_2|^{p-1}\psi_2\|_{L^1([0,T],L^2(\mathbb{R}^3))} &\leq \||\psi_1-\psi_2|\left(|\psi_1|^{p-1}+|\psi_2|^{p-1}\right)\|_{L^1([0,T],L^2(\mathbb{R}^3))} \\ &\leq \|\psi_1-\psi_2\|_{L^p([0,T],L^{2p}(\mathbb{R}^3))} \cdot \||\psi_1|^{p-1}+|\psi_2|^{p-1}\|_{L^{\frac{p}{p-1}}([0,T],L^{\frac{2p}{p-1}}(\mathbb{R}^3))} \\ &\leq \|\psi_1-\psi_2\|_{L^p([0,T],L^{2p}(\mathbb{R}^3))}\left(\|\psi_1\|^{p-1}_{L^p([0,T],L^{2p}(\mathbb{R}^3))} + \|\psi_2\|^{p-1}_{L^p([0,T],L^{2p}(\mathbb{R}^3))}\right) \\ &\leq T^{\frac{5-p}{2}}\|\psi_1-\psi_2\|_{L^{\frac{2p}{p-3}}([0,T],L^{2p}(\mathbb{R}^3))}\left(\|\psi_1\|^{p-1}_{L^{\frac{2p}{p-3}}([0,T],L^{2p}(\mathbb{R}^3))} + \|\psi_2\|^{p-1}_{L^{\frac{2p}{p-3}}([0,T],L^{2p}(\mathbb{R}^3))}\right) \\ &\leq T^{\frac{5-p}{2}}\|\psi_1-\psi_2\|_X\left(\|\psi_1\|_X^{p-1}+\|\psi_2\|_X^{p-1}\right) \\ &\leq 2R^{p-1}T^{\frac{5-p}{2}}\|\psi_1-\psi_2\|_X \end{aligned} \tag{5.17}$$

we deduce $\|\Phi(\psi_1)-\Phi(\psi_2)\|_X =$

$$\begin{aligned}\|\phi_1-\phi_2\|_X &\leq \||\psi_1|^{p-1}\psi_1 - |\psi_2|^{p-1}\psi_2\|_{L^1([0,T],L^2(\mathbb{R}^3))} + C_p\left(\||\psi_1|^{p-1}\psi_1 - |\psi_2|^{p-1}\psi_2\|_{L^1([0,T],L^2(\mathbb{R}^3))}\right) \\ &\leq 2(C_p+1)R^{p-1}T^{\frac{5-p}{2}}\|\psi_1-\psi_2\|_X \end{aligned} \tag{5.18}$$

Hence Φ is a contraction provided $2(C_p+1)R^{p-1}T^{\frac{5-p}{2}} < 1$ (Condition 2). It is clear that Conditions 1 and 2 hold if $T \leq C\cdot(E_0)^{-2\frac{p-1}{5-p}}$ for some easily computed constant C depending on p. Banach's fixed point argument now guarantees that Φ has a unique fixed point which is the solution to the non-linear wave equation. □

Chapter 6

The Klainerman-Sobolev inequality

In this chapter we will briefly return to the analysis of the *linear* wave equation. The reason is that we would like to develop a robust way to measure the dispersion of the wave equation.[1] Here robust simply means that our techniques for the linear equation should eventually be applicable to non-linear (ideally, quasi-linear) problems. For instance, we would like to avoid explicit representation formulae and the Fourier transform (which were the two methods we used to derive decay for the linear equation so far!). The techniques should also depend as little as possible on the fact that the wave operator is exactly the wave operator of Minkowski space, i.e. $a^{\mu\nu} = \eta^{\mu\nu}$ because in the non-linear case the $a^{\mu\nu}(\phi)$ are dynamical (in fact, depend on the solution).

6.1 The commuting vectorfields

Let us define the following sets of vectorfields on Minkowski space $\mathbb{R}^{1+n}$ expressed in the standard coordinates $(t, x_1, x_2, \ldots, x_n)$:

$$\begin{aligned}
\mathcal{X}_P &= \{\partial_t, \partial_i, \Omega_{ij} := x_i\partial_j - x_j\partial_i, \Omega_{0i} = t\partial_i + x_i\partial_t\}\,, \\
\mathcal{X} &= \mathcal{X} \cup \{S = t\partial_t + \sum_{i=1}^{n} x_i\partial_i\}\,.
\end{aligned} \tag{6.1}$$

The vectorfields of $\mathcal{X}_P$ generate the so-called *Poincaré group* of isometries of Minkowski space, consisting of translations, spatial rotations and Lorentz boosts (of course traditionally and most importantly $n = 3$). A pedestrian way to understand this geometric statement is the following: If $\Gamma \in \mathcal{X}_P$ and $\phi_s : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ denotes the corresponding 1-parameter family of diffeomorphisms generated by Γ (hence $d\phi_s|_p : T_p\mathbb{R}^4 \to T_{\phi_s(p)}\mathbb{R}^4$), then

$$\eta_p(v, w) = \left((\phi_s)^\star \eta\right)\big|_p (v, w) = \eta\big|_{\phi_s(p)} \left(d\phi_s|_p v, d\phi_s|_p w\right) \tag{6.2}$$

holds, where the last equality is the definition of the pull-back of the metric η along ϕ_s and the first equality is the defining property of an isometry. (Exercise: Check explicitly that (6.2) indeed holds for all $\Gamma \in \mathcal{X}_P$. Does it hold for the scaling vectorfield S above?)

Lemma 6.1.1. *If* $\Box\phi = 0$*, then*

$$\Box(\Gamma\phi) = 0$$

holds for all $\Gamma \in \mathcal{X}$*.*

Proof. Direct computation. In particular one has $[\Box, \Gamma] = 0$ for $\Gamma \in \mathcal{X}_P$ and $[\Box, S] = 2\Box$. □

[1] We have already seen the Strichartz estimates. However, their proof (as we will see later) depends heavily on Fourier analysis and hence the fact that we are dealing with a constant coefficient equation.

In view of the above Lemma, we refer to the vectorfields from $\mathcal{X}$ as the commuting vectorfields (of the wave operator). We also remark that more generally, the wave operator associated with a general Lorentzian metric, $\Box_g$, defined at the beginning of the notes will commute with vectorfields generating the isometries of the metric g.

For the following, let us fix an arbitrary ordering of the Γ of $\mathcal{X}$, i.e.

$$\Gamma_1 = \partial_t, \Gamma_2 = \partial_{x_1}, \Gamma_3 = ..., \Gamma_k = S \qquad \text{where } k := 2n + 2 + \frac{n(n-1)}{2}.$$

This follows as there are $n+1$ translations, $\frac{n(n-1)}{2}$ rotations, n boosts and one scaling vector. For $\alpha = (\alpha_1, ..., \alpha_k)$ a multi-index, we will use the notation

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \dots \Gamma_k^{\alpha_k}$$

Note that $\sum_{|\alpha| \leq k} |\Gamma^\alpha \phi|$ controls up to k derivatives in *any* order. This is because the Γ from $\mathcal{X}$ form a Lie-Algebra[2]:

$$[\Gamma_i, \Gamma_j] = \sum_k c_{ij}^k \Gamma_k \, ,$$

where the c_{ij}^k are constants, holds for $\Gamma_i \in \mathcal{X}$.

Proposition 6.1.2. *Given smooth compactly supported initial data for the linear wave equation $\Box \phi = 0$ in $(n+1)$-dimensions, we have*

$$\sum_{|\alpha| \leq k} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) = \sum_{|\alpha| \leq k} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) \, . \tag{6.3}$$

Proof. This is an immediate consequence of Lemma 6.1.1 and the standard energy conservation law for the wave equation. □

6.2 Statement and consequences of the Klainerman-Sobolev inequality

We now state the main result of this section, the famous Klainerman-Sobolev inequality. The idea is quite simple: We would like to exploit the weights in t and r appearing in the commuting vectorfields to prove *weighted* Sobolev estimates.

Theorem 6.2.1. *There exists a constant C depending only on the dimension n such that the following estimate holds for all $\phi \in H^{\lfloor \frac{n+2}{2} \rfloor}(\mathbb{R}^n)$ for $t \geq 0$:*

$$\sup_x \left(1+t+r\right)^{\frac{n-1}{2}} \left(1+|t-r|\right)^{\frac{1}{2}} |\phi|(t,x) \leq C \sum_{|\alpha| \leq \lfloor \frac{n+2}{2} \rfloor} \|\Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) \, . \tag{6.4}$$

Note that this is a purely a statement about functions and does not involve anything about wave equations. From the theorem we immediately deduce

Corollary 6.2.2. *Given smooth data of compact support for the wave equation in $(n+1)$-dimensions, we have the following estimate for a constant C depending only on n*

$$\sup_x \sum_{|\alpha|=1} |\partial^\alpha \phi|(t,x) \leq \frac{C}{(1+t)^{\frac{n-1}{2}}} \sum_{|\alpha| \leq \lfloor \frac{n+2}{2} \rfloor} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) \qquad \textit{holds for } t \geq 0. \tag{6.5}$$

Proof. Apply Theorem 6.2.1 with $\partial \phi$ instead of ϕ and use the energy conservation of Proposition 6.1.2 on the right hand side. □

[2]The vectorfields from $\mathcal{X}_P$ also form a Lie-Algebra on their own, the Poincaré algebra generating the Poincaré group of special relativity discussed above.

Note this is the sharp decay rate we obtained for the solution from the explicit representation formula. However, the estimate (6.5) now also exhibits a precise norm on the initial data whose boundedness implies the decay rate. (Recall that for the representation formula we obtained (6.5) without the norm on the right hand side and C dependending on the size of the support of the data.)

Away from the light cone (i.e. away from $t = |x|$) we can obtain an improved decay rate

Corollary 6.2.3. *Given smooth data of compact support for the wave equation in $(n+1)$-dimensions, the following decay estimate holds in the region $S_t = \{x \mid |x| < (1-\epsilon)t \quad or \quad |x| > (1+\epsilon)t\}$ for a constant C depending only on n and ϵ:*

$$\sup_{x \in S_t} \sum_{|\alpha|=1} |\partial^\alpha \phi|(t,x) \leq \frac{C}{(1+t)^{\frac{n}{2}}} \sum_{|\alpha| \leq \lfloor \frac{n+2}{2} \rfloor} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) \qquad \textit{holds for } t \geq 0. \tag{6.6}$$

Proof. Note that $|t - r| \geq \epsilon t$ in the region S_t. □

What can we say about ϕ itself? Integrating in the u-direction (where $u = t - r$) we obtain the following (crude) estimate:

Corollary 6.2.4. *Given smooth data of compact support for the wave equation in $(n+1)$-dimensions, we have the following estimate for a constant $C_\star$ depending only on n and the size of the support of the data*

$$\sup_x (1+t+r)^{\frac{n-1}{2}} (1+|t-r|)^{-\frac{1}{2}} |\phi|(t,x) \leq C_\star \sum_{|\alpha| \leq \lfloor \frac{n+2}{2} \rfloor} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) \,. \tag{6.7}$$

Proof. Let the data be supported in the ball $B(0,R)$. Fix a $(t,|x|,\theta)$ and set $u = t - |x|$ and $v = t + |x|$. We have (draw a picture!)

$$\phi(t,|x|,\theta) = \phi(0, t+|x|, \theta) + \int_{-t-|x|}^{t-|x|} \partial_u \phi\left(\bar{u}, v = t+|x|, \theta\right) d\bar{u}$$

If $v = t + |x| > R$, the first term on the right is zero and the estimate follows by inserting the bound on $\partial_u \phi$ following from (6.4). For $v \leq R$ the first term can be estimated by the $\dot{H}^1 \times \dot{H}^2$ Sobolev embedding from Example Sheet 6 (weights in R can be absorbed into the constant $C_\star$). □

Remark 6.2.5. *Note that this rate is worse than what we proved from the explicit representation formula, $|\phi| \leq \frac{1}{1+t}$ for $t \geq 0$. However, using an additional vectorfield (the conformal vectorfield) one can establish the sharp rate and eliminate the dependence on the size of the support in the constant at the cost of introducing stronger weighted norms on the data. See Example Sheet 8.*

We finally describe how the Klainerman-Sobolev inequality allows us to prove that some derivatives of ϕ decay better than others. (This fact is not immediate from the explicit representation formulae!) To simplify the proof, we introduce some notation. We recall $u = t - r$ and $v = t + r$ where $r = |x|$.

We have

$$(\partial_t \phi)^2 + \sum_{i=1}^n (\partial_i \phi)^2 = (\partial_t \phi)^2 + (\partial_r \phi)^2 + |\nabla\!\!\!/ \phi|^2 = 2(\partial_u \phi)^2 + 2(\partial_v \phi)^2 + |\nabla\!\!\!/ \phi|^2 \,,$$

where we recall that $\nabla\!\!\!/$ is the gradient induced on the spheres of constant r with norm satisfying

$$|\nabla\!\!\!/ \phi|^2 = \frac{1}{2} \sum_{i,j=1}^n \left(\frac{x_i}{r}\partial_j \phi - \frac{x_j}{r}\partial_i \phi\right)^2 \,. \tag{6.8}$$

We finally define the notion of "good derivative", $\overline{\partial}$, which are derivatives *tangent* to the outgoing line cones, i.e. either ∂_v or a derivative $\nabla\!\!\!/$ tangent to the spheres foliating the outgoing cone. In particular, we denote

$$|\overline{\partial} \phi|^2 = (\partial_v \phi)^2 + |\nabla\!\!\!/ \phi|^2$$

Corollary 6.2.6. *Given smooth data of compact support for the wave equation in $(n+1)$-dimensions, we have the following estimate for a constant $C_\star$ depending only on n and the size of the support of the data*

$$\sup_x (1+t+r)^{\frac{n+1}{2}} (1+|t-r|)^{-\frac{1}{2}} |\overline{\partial}\phi|(t,x) \leq C_\star \sum_{|\alpha|\leq \lfloor \frac{n+4}{2} \rfloor} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) \tag{6.9}$$

for all $t \geq 0$.

Proof. It suffices to establish the bound

$$|\overline{\partial}\phi| \leq C \frac{\sum_{|\alpha|=1} |\Gamma^\alpha \phi|}{1+t+r}, \tag{6.10}$$

as we can then apply Corollary 6.2.4. To prove (6.10) note that it suffices to prove this for $t+r > 1$ as for $t+r \leq 1$ the result follows since the Γ's contain in particular ∂_t and all ∂_{x_i}. Now note that

$$\partial_v = \frac{S + \sum_{i=1}^n \frac{x_i}{r} \Omega_{0i}}{2(t+r)} \tag{6.11}$$

and that

$$|\nabla\!\!\!/ \phi| \leq \frac{1}{r} \sum_{i,j=1}^n |\Omega_{ij}\phi| \leq \frac{1}{t} \sum_{i=1}^n |\Omega_{0i}\phi| \tag{6.12}$$

with the first inequality following from (6.8) and the second from the readily verified identity

$$\Omega_{ij} = \frac{1}{t}\left(x_i \Omega_{0j} - x_j \Omega_{0i}\right) .$$

The result follows. □

6.3 Proof of the Klainerman-Sobolev inequality

We will prove (6.4) replacing $\sum_{|\alpha|\leq \lfloor \frac{n+2}{2} \rfloor}$ by $\sum_{|\alpha|\leq \lfloor \frac{n+3}{2} \rfloor}$ on the right hand side of the inequality. For n even this gives the desired result. For a proof of the sharp result when n is odd, consult the textbooks of Sogge or Hörmander. The loss arising from the additional commutation on the right hand side is unimportant for the applications we are going to see.

Be begin with two Lemmas. The first one expresses the fact that replacing ∂ derivatives by Γ's gains us a factor of $|t-r|$ (pointwise):

Lemma 6.3.1. *There exists a constant $C = C(n,k) > 0$ such that for $t \geq 0$ we have*

$$\sum_{|\alpha|=k} |\partial^\alpha \phi| \leq \frac{C}{|t-r|^k} \sum_{|\alpha|\leq k} |\Gamma^\alpha \phi| \tag{6.13}$$

for every C^k function $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$.

Proof. Apply iteratively the easily verified identities

$$\partial_i = \frac{-\sum_j x_j \Omega_{ij} + t\Omega_{0i} - x_i S}{(t-r)(t+r)} \quad \text{and} \quad \partial_t = \frac{tS - x_i \Omega_{0i}}{(t-r)(t+r)} .$$

□

The second Lemma is an interpolation estimate:

Lemma 6.3.2. *For $k \in \mathbb{N}$ and $k > \frac{n}{2}$, there exists $C = C(n,k) > 0$ such that the following holds for all functions $\phi \in H^k(\mathbb{R}^n)$.*

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C \|\phi\|_{L^2(\mathbb{R}^n)}^{\frac{2k-n}{2k}} \left(\sum_{|\alpha|=k} \|\partial_x^\alpha \phi\|_{L^2(\mathbb{R}^n)} \right)^{\frac{n}{2k}} \tag{6.14}$$

Proof. Note that by the standard Sobolev embedding and a simple interpolation

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C \|\phi\|_{H^k(\mathbb{R}^n)} \leq C \left(\|\phi\|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|=k} \|\partial_x^\alpha \phi\|_{L^2(\mathbb{R}^n)} \right) .$$

Clearly, the above estimate is also satisfied for the rescaled function $\phi_\lambda(x) := \phi(\lambda x)$ for any $\lambda \neq 0$. Hence

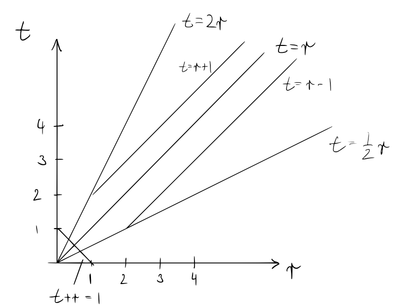
$$\|\phi_\lambda\|_{L^\infty(\mathbb{R}^n)} \leq C \left(\|\phi_\lambda\|_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|=k} \|\partial_x^\alpha \phi_\lambda\|_{L^2(\mathbb{R}^n)} \right) ,$$

which can be written as

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C \left(\lambda^{-\frac{n}{2}} \|\phi\|_{L^2(\mathbb{R}^n)} + \lambda^{-\frac{n}{2}+k} \sum_{|\alpha|=k} \|\partial_x^\alpha \phi\|_{L^2(\mathbb{R}^n)} \right) .$$

Choose now $\lambda = \left(\sum_{|\alpha|=k} \|\partial_x^\alpha \phi\|_{L^2(\mathbb{R}^n)} \right)^{-\frac{1}{k}} \left(\|\phi\|_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{k}}$. (Note that this is well-defined for $\phi \in H^k(\mathbb{R}^n)$ non-trivial. For ϕ identically zero (6.14) of course holds trivially.) □

With the two Lemmas at hand we can embark on the proof proper. By standard Sobolev embedding we can assume $t + r > 1$ throughout. Here is a picture of the different regions to be used below:



Step 1. The region $\{r \leq \frac{t}{2}\}$. The idea is to use Lemma 6.3.1 to exchange ∂ by Γ which gains a power of t in this region.

We let $\chi\left(\frac{r}{t}\right)$ be a smooth cut-off function such that $\chi(x) = 1$ for $x \leq \frac{1}{2}$ and $\chi(x) = 0$ for $x \geq \frac{3}{4}$. Now using that $\left|\partial^\alpha \chi\left(\frac{r}{t}\right)\right| \leq \frac{C}{t^{|\alpha|}}$ (Exercise. – Note that $t \sim r$ in the region where χ' is supported). We deduce

$$\sum_{|\alpha|=k} \|\partial^\alpha (\chi\phi)\|_{L^2(\mathbb{R}^n)} \leq C \sum_{|\alpha|\leq k} t^{-k+|\alpha|} \left(\int_{\{r \leq \frac{3}{4}t\}} |\partial^\alpha \phi|^2(x) dx \right)^{\frac{1}{2}} \leq C \sum_{|\alpha|\leq k} t^{-k} \left(\int_{\{r \leq \frac{3}{4}t\}} |\Gamma^\alpha \phi|^2(x) dx \right)^{\frac{1}{2}}$$

with the second inequality following from Lemma 6.3.1. Applying now Lemma 6.3.2 for $\chi\phi$ instead of ϕ and inserting the estimate just proven yields the desired bound in the region $\{r \leq \frac{t}{2}\}$. Note in particular that in this region $(1+t+r)^{\frac{n-1}{2}} (1+|t-r|)^{\frac{1}{2}} \leq C t^{\frac{n}{2}}$.

Step 2. The region $\{r \geq 2t\}$. Exercise. This is entirely analogous to Step 1!

Step 3a. The region $\{\frac{t}{2} \leq r \leq 2t\} \cap \{t-r \geq 1\}$. In this region (and the region of Step 3b) we have to be more clever as the gain in $|t-r|^{-1}$ from Lemma 6.3.1 is not sufficient in this region. The idea is to exploit that different directions gain different powers: We gain a power of $\frac{1}{r}$ (note $\frac{1}{r} \sim \frac{1}{t} \sim \frac{1}{t+r}$ in this region!) from replacing the angular directions (essentially we are using that the Ω_{ij} have norm r) and accept the "bad" gain of $\frac{1}{|t-r|}$ from Lemma 6.3.1 when replacing the radial directions. The details are as follows.

Let $\chi\left(\frac{r}{t}\right)$ be a smooth cut-off function such that $\chi(x) = 1$ if $\frac{1}{2} \leq x \leq 2$ and $\chi(x) = 0$ if $x < \frac{1}{4}$ or $x > 4$. We write $x = (r, \theta)$ where $\theta \in \mathbb{S}^{n-1}$. We denote by $d\sigma_\theta$ the standard measure on the unit sphere $\mathbb{S}^{n-1}$. Note that the volume form on $\mathbb{R}^n$ can be written in polar coordinates as $d^n x = r^{n-1} dr d\sigma_\theta$.

In the following, we will use the Sobolev embedding on the sphere $\mathbb{S}^{n-1}$ (see Exercise Sheet 8):

$$\sup_{\mathbb{S}^{n-1}} |\phi| \leq C \left(\sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \int_{\mathbb{S}^{n-1}} |\Omega_{ij}^\alpha \phi|^2(\theta) d\sigma_\theta \right)^{\frac{1}{2}} . \tag{6.15}$$

Using the above inequality, the fundamental theorem of calculus, Lemma 6.3.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\chi\phi|(t,x) &\leq C \sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \left(\int_{\mathbb{S}^{n-1}} |\chi|^2 |\Omega_{ij}^\alpha \phi|^2(\theta) d\sigma_\theta \right)^{\frac{1}{2}} \\
&\leq C \sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \left(\int_{\mathbb{S}^{n-1}} \left[\int_{\frac{t}{4}}^{r} |\partial_r (\chi \Omega_{ij}^\alpha \phi)|(r', \theta) dr' \right]^2 d\sigma_\theta \right)^{\frac{1}{2}} \\
&\leq C \sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \sum_{|\beta| \leq 1} \left(\int_{\mathbb{S}^{n-1}} \left[\int_{\frac{t}{4}}^{r} \frac{1}{|t-r|} |\Gamma^\beta (\chi \Omega_{ij}^\alpha \phi)|(r', \theta) dr' \right]^2 d\sigma_\theta \right)^{\frac{1}{2}} \\
&\leq C \sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \sum_{|\beta| \leq 1} \left(\int_{\mathbb{S}^{n-1}} \left[\int_{\frac{t}{4}}^{r} |\Gamma^\beta (\chi \Omega_{ij}^\alpha \phi)|^2(r', \theta) dr' \right] \left[\int_{\frac{t}{4}}^{r} \frac{1}{|t-r|^2} dr' \right] d\sigma_\theta \right)^{\frac{1}{2}} \\
&\leq \frac{C}{|t-r|^{\frac{1}{2}} t^{\frac{n-1}{2}}} \sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \sum_{|\beta| \leq 1} \left(\int_{\mathbb{S}^{n-1}} \int_{\frac{t}{4}}^{r} |\Gamma^\beta (\chi \Omega_{ij}^\alpha \phi)|^2(r', \theta) (r')^{n-1} dr' d\sigma_\theta \right)^{\frac{1}{2}} \\
&\leq \frac{C}{|t-r|^{\frac{1}{2}} |t+r|^{\frac{n-1}{2}}} \|\Gamma^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) ,
\end{aligned} \tag{6.16}$$

which concludes the proof for this region. Here in the last step we have used $\sum_{|\alpha|=1} \left|\Gamma^\alpha \left(\chi\left(\frac{r}{t}\right)\right)\right| \leq C$, an estimate that can be easily verified.

Step 3b. The region $\{\frac{t}{2} \leq r \leq 2t\} \cap \{t-r \leq -1\}$. Exercise. This is entirely analogus to Step 3a, now integrating from $r = 4t$ inwards.

Step 4. The region $\{\frac{t}{2} \leq r \leq 2t\} \cap \{|t-r| \leq 1\}$. Note that here it suffices to prove the estimate (6.4) with the weight $r^{\frac{n-1}{2}}$ on the left. With (t, r) fixed with $|t-r| \leq 1$, this follows from Sobolev embedding on the spheres as in Step 3 and using the fundamental theorem of calculus in the r-direction on $t = const.$ (Exercise. This is essentially identical to Step 3a, except that now the r-integration is over an interval of length 2.).

Chapter 7

Small data global existence and regularity

In this chapter, we will....

7.1 Wave maps

We introduce here the wave map equation and give some heuristic motivation of its origin. We first note that the wellposedness theory we developed for scalar equations $\phi : I \times \mathbb{R}^n \to \mathbb{R}$ generalises with straightforward modifications to systems of non-linear wave equations of the form

$$a^{ij}(\phi)\partial_i\partial_j\phi = F(\phi, \partial\phi)$$

where now $\phi : I \times \mathbb{R}^n \to \mathbb{R}^m$ is vector-valued. We can simply add the energy estimates resulting from each component of ϕ. A more interesting case arises if we consider ϕ with values in an arbitrary Riemannian manifold. What is the natural wave equation to impose in this context?

We will look at the specific case when the target manifold is the m-dimensional unit-sphere which we think of as embedded into $\mathbb{R}^{m+1}$, i.e. we look at maps $\phi : I \times \mathbb{R}^n \to \mathbb{S}^m = \{y \in \mathbb{R}^{m+1} \mid \|y\| = 1\}$. The wave map equation for such ϕ is (we let T denote the transpose of a vector in $\mathbb{R}^{m+1}$)

$$\begin{cases} \Box\phi = \phi\left(\partial_t\phi^T\partial_t\phi - \sum_{i=1}^n \partial_i\phi^T\partial_i\phi\right) \\ \phi(0,x) = \phi_0 \\ \partial_t\phi(0,x) = \phi_1 \end{cases}, \tag{7.1}$$

where the data also satisfies $\|\phi_0\| = 1$ and $\phi_1^T\phi_0 = 0$. The above equation can be derived (Exercise!) as the equation satisfied by the critical points of the (constrained) Lagrangian

$$\mathcal{L}[\phi] = \int_{\mathbb{R}^{n+1}} d^n x \left[-\partial_t\phi^T\partial_t\phi + \sum_{i=1}^n \partial_i\phi^T\partial_i\phi + \lambda\left(\|\phi\|^2 - 1\right)\right]. \tag{7.2}$$

In this way we see that (7.1) is the natural generalisation of the standard wave equation if we are imposing the values to lie in the unit sphere. On Example Sheet 8 you will show explicitly that solutions to (7.1) indeed satisfy $\phi^T(t)\phi(t) = 1$ for all times as claimed.

Remark 7.1.1. *When we talk about data of compact support for (7.1) we always mean "equal to a constant (with norm 1) map outside a ball of radius R", i.e. there exists a $y \in \mathbb{S}^m$ such that $\phi_0 - y$ and ϕ_1 are compactly supported.*

You will easily check the following two claims

- The energy

$$E[\phi](t) = \frac{1}{2}\int_{\mathbb{R}^n}\left[\partial_t\phi^T\partial_t\phi + \sum_{i=1}^{n}\partial_i\phi^T\partial_i\phi\right](t,x)d^nx \tag{7.3}$$

is conserved in time for sufficiently smooth solutions to (7.1)

- The equation (7.1) admits the following scaling invariance: If $\phi(t,x)$ is a solution of (7.1), then so is $\phi_\lambda(t,x) := \phi(\lambda t, \lambda x)$. Unser the above scaling, the energy scales like

$$E[\phi_\lambda] = \lambda^{-n+2}E[\phi]\,.$$

Therefore, the case $n = 1$ one is subcritical (and we will show global existence), the case $n = 2$ is critical and the case $n \geq 3$ is supercritical. In the latter two cases, one can show small data data global existence (we'll do so for $n = 4$ and $n = 3$) and also the existence of blow-up solutions for large data. (In fact, much more detailed statements can be made due to work in the last two decades.)

7.2 Small data global existence and regularity in $4+1$ dimensions

Theorem 7.2.1. *Let $k \geq 7$. Consider the wave map equation (7.1) with initial data (ϕ_0, ϕ_1) smooth and of compact support in the ball $B(0,R)$ (cf. Remark 7.1.1) and satisfying in addition $\|\phi_0\| = 1$ and $\phi_1^T\phi_0 = 0$. Then, for every $R > 0$ there exists $\epsilon = \epsilon(R) > 0$ sufficiently small with the following property: If*

$$\sum_{|\alpha|\leq k}\|\partial\partial^\alpha\phi_0\|_{L^2(\mathbb{R}^4)} + \|\partial^\alpha\phi_1\|_{L^2(\mathbb{R}^4)} < \epsilon\,, \tag{7.4}$$

then the unique solution to (7.1) remains smooth for all times.

Proof. The regularity of the data implies existence of a local (smooth) solution by Theorem 4.2.1 and propagation of regularity.[1] Let $R > 0$ be fixed. The proof is by a bootstrap argument. Let $T_\star$ be the maximal time such that the bound

$$\sup_{[0,T]}\sum_{|\alpha|\leq k}\|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}(t) \leq \epsilon^{\frac{3}{4}} \tag{7.5}$$

holds for all $T < T_\star$. Note that choosing ϵ sufficiently small, this holds for $t = 0$ as $\epsilon \ll \epsilon^{\frac{3}{4}}$. Our goal is to use the equation to show that then (7.5) actually holds with the right hand side replaced by $\frac{1}{2}\epsilon^{\frac{3}{4}}$. By continuity, this will imply that the bound (7.5) also holds for $T < T_\star + \delta$ for some $\delta > 0$. It follows hat $T_\star = \infty$ since the assumption $T_\star < \infty$ immediately leads to a contradiction.

Step 1: Lower order weighted L^∞-decay bounds from Klainerman-Sobolev. Applying the Klainerman-Sobolev inequality to (7.5) we deduce (note we are in spatial dimension $n = 4$ and that $k \geq 6$)

$$\sum_{|\alpha|\leq\frac{k}{2}}\|\,(1+t+r)^{\frac{3}{2}}\,\partial\Gamma^\alpha\phi\|_{L^\infty(\mathbb{R}^4)} \leq C\epsilon^{\frac{3}{4}}\,. \tag{7.6}$$

Using the trivial bound

$$\sum_{1\leq|\alpha|\leq\frac{k}{2}+1}|\Gamma^\alpha\phi| \leq C\,(1+t+r)\sum_{|\alpha|\leq\frac{k}{2}}|\partial\Gamma^\alpha\phi|$$

this implies

$$\sum_{1\leq|\alpha|\leq\frac{k}{2}+1}\|\,(1+t+r)^{\frac{1}{2}}\,\Gamma^\alpha\phi\|_{L^\infty(\mathbb{R}^4)} \leq C\epsilon^{\frac{3}{4}}\,. \tag{7.7}$$

[1] Recall that by our previous remarks, the theorems holds with straightforward notational changes also for vector valued ϕ.

Note (7.6) and (7.7) also hold (trivially) replacing $1+t+r$ by $t+1$.

Step 2. Closing the energy estimates

For some constant C depending on k (but not on T!) we have the commuted energy estimate

$$\sup_{t\in[0,T]} \sum_{|\alpha|\leq k} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^4)}(t)$$
$$\leq C\left[\sum_{|\alpha|\leq k} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^4)}(0) + \int_0^T \sum_{|\alpha|\leq k} \left\|\Gamma^\alpha\left(\phi\left(\partial_t\phi^T\partial_t\phi - \sum_{i=1}^n \partial_i\phi^T\partial_i\phi\right)\right)\right\|_{L^2(\mathbb{R}^4)}(t)dt\right]. \tag{7.8}$$

We estimate the non-linear term as (note $\|\phi\|=1$)

$$\sum_{|\alpha|\leq k}\left|\Gamma^\alpha\left(\phi\left(\partial_t\phi^T\partial_t\phi - \sum_{i=1}^n \partial_i\phi^T\partial_i\phi\right)\right)\right|$$
$$\leq C\left(\sum_{|\alpha_1|+|\alpha_2|\leq k} |\partial\Gamma^{\alpha_1}\phi|\ |\partial\Gamma^{\alpha_2}\phi| + \sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\leq k\\ |\alpha_1|\geq 1}} |\Gamma^{\alpha_1}\phi|\ |\partial\Gamma^{\alpha_2}\phi|\ |\partial\Gamma^{\alpha_3}\phi|\right). \tag{7.9}$$

For the first tern, either $|\alpha_1|\leq\frac{k}{2}$ or $|\alpha_2|\leq\frac{k}{2}$ (or both). Wlog let is be α_1. Then we can apply (7.6) and estimate

$$\sum_{|\alpha_1|+|\alpha_2|\leq k}\int_0^T \||\partial\Gamma^{\alpha_1}\phi|\ |\partial\Gamma^{\alpha_2}\phi|\|_{L^2(\mathbb{R}^4)}(t)dt$$
$$\leq C\int_0^T\left(\sum_{|\alpha_1|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_2|\leq k}\|\partial\Gamma^{\alpha_2}\phi\|_{L^2(\mathbb{R}^4)}\right)(t)dt$$
$$\leq C\epsilon^{\frac{3}{4}}\int_0^T\frac{\left(\sum_{|\alpha_2|\leq k}\|\partial\Gamma^{\alpha_2}\phi\|_{L^2(\mathbb{R}^4)}\right)(t)}{(1+t)^{\frac{3}{2}}}dt \leq C\epsilon^{\frac{3}{2}} \ll \epsilon. \tag{7.10}$$

The key here is that the weighted L^∞ estimate decays like $\frac{1}{(1+t)^{\frac{3}{2}}}$ which is integrable in time. Note we cannot close (in this naive fashion) if that rate was only $\frac{1}{1+t}$, which it would be in dimension $n=3$!

We finally look at the second, cubic term. At least two of $|\alpha_1|$, $|\alpha_2$ and $|\alpha_3$ are $\leq\frac{k}{2}$ and can hence be put in L^∞. More specifically

$$\sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\leq k\\ |\alpha_1|\geq 1}}\int_0^T\left\||\Gamma^{\alpha_1}\phi|\ |\partial\Gamma^{\alpha_2}\phi|\ |\partial\Gamma^{\alpha_3}\phi|\right\|_{L^2(\mathbb{R}^4)}(t)dt$$
$$\leq C\int_0^T\left(\sum_{1\leq|\alpha_1|\leq\frac{k}{2}}\|\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_2|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_3|\leq k}\|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)}\right)(t)dt$$
$$+C\int_0^T\left(\sum_{1\leq|\alpha_1|\leq k}\|\Gamma^{\alpha_1}\phi\|_{L^2(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_2|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_3|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_3}\phi\|_{L^\infty(\mathbb{R}^4)}\right)(t)dt$$
$$\leq C\int_0^T\left(\sum_{1\leq|\alpha_1|\leq\frac{k}{2}}\|\Gamma^{\alpha_1}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_2|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_3|\leq k}\|\partial\Gamma^{\alpha_3}\phi\|_{L^2(\mathbb{R}^4)}\right)(t)dt$$
$$+C\int_0^T\left((1+t+r)\sum_{|\alpha_1|\leq k-1}\|\partial\Gamma^{\alpha_1}\phi\|_{L^2(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_2|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_2}\phi\|_{L^\infty(\mathbb{R}^4)}\right)\left(\sum_{|\alpha_3|\leq\frac{k}{2}}\|\partial\Gamma^{\alpha_3}\phi\|_{L^\infty(\mathbb{R}^4)}\right)(t)dt$$

and hence using (7.6) and (7.7)

$$\sum_{\substack{|\alpha_1|+|\alpha_2|+|\alpha_3|\leq k\\ |\alpha_1|\geq 1}} \int_0^T \left\| |\Gamma^{\alpha_1}\phi| \; |\partial\Gamma^{\alpha_2}\phi| \; |\partial\Gamma^{\alpha_3}\phi| \right\|_{L^2(\mathbb{R}^4)}(t)dt \leq C\epsilon^{\frac{3}{2}} \int_0^T dt \frac{\sum_{|\alpha|\leq k}\|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}}{(1+t)^2} \leq C\epsilon^{\frac{9}{4}}. \quad (7.11)$$

Combining the two estimates for the error-term we deduce from (7.8)

$$\sup_{t\in[0,T]} \sum_{|\alpha|\leq k} \|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}(t) \leq C\left[\sum_{|\alpha|\leq k}\|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^4)}(0)+\epsilon^{\frac{3}{2}}\right] \leq C\epsilon \leq \frac{\epsilon^{\frac{3}{4}}}{2}\,. \quad (7.12)$$

□

7.3 Unconditional global existence and regularity for $n = 1$ (subcritical case)

Theorem 7.3.1. *Consider the wave map equation with smooth initial data (ϕ_0, ϕ_1) in $H^2(\mathbb{R}) \times H^1(\mathbb{R})$.*[2] *Then there exists a unique global in time smooth solution.*

Remark 7.3.2. *A very short proof of Theorem 7.3.1 can be found on Example Sheet 9. However, the proof we are about to give is a bit more insightful and robust.*

Proof. A unique (smooth) local in time solution exists by Theorem 4.2.1. Our breakdown criterion of Theorem 4.4.1 implies that if we can show

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha\phi\|_{L^\infty(\mathbb{R})} \leq C$$

uniformly for all $T < T_\star$, then $T_\star = \infty$ and the solution must exist globally.

We first prove the boundedness of the characteristic energy. Using null coordinates $u = t - r$, $v = t + r$ we write

$$-4\partial_u\partial_v\phi = 4\phi\partial_u\phi^T\partial_v\phi \quad , \quad 4\partial_u\partial_v\phi^T = -4\phi^T\partial_u\phi^T\partial_v\phi \quad (7.13)$$

and multiply by $\partial_u\phi + \partial_v\phi$ to deduce (note $(\partial_u + \partial_v)|\phi|^2 = 0$)

$$\partial_u\left(\partial_v\phi^T\partial_v\phi\right) + \partial_v\left(\partial_u\phi^T\partial_v\phi\right) = 0\,. \quad (7.14)$$

Fix an arbitrary point (u_0, v_0) with $u_0 + v_0 = 2T$ and integrate the above over the past light cone of (u_0, v_0) truncated at $t = 0$. This yields

$$\int_{-v_0}^{2T-v_0} \partial_u\phi^T\partial_u\phi(u,v_0)du + \int_{-u_0}^{2T-u_0} \partial_v\phi^T\partial_v\phi(u_0,v)dv \leq C\left(\|\phi_0\|^2_{H^1(\mathbb{R})} + \|\phi_1\|^2_{L^2(\mathbb{R})}\right) \quad (7.15)$$

and since (u_0, v_0) is arbitrary

$$\sup_v \int_{-v}^{2T-v} \partial_u\phi^T\partial_u\phi(u,v)du + \sup_u \int_{-u}^{2T-u} \partial_v\phi^T\partial_v\phi(u,v)dv \leq C\left(\|\phi_0\|^2_{H^1(\mathbb{R})} + \|\phi_1\|^2_{L^2(\mathbb{R})}\right) \quad (7.16)$$

We can now estimate the derivatives of ϕ in L^∞ integrating (7.13) along the characteristic directions using the L^2 bounds. Integrating in the u-direction, we have for $u + v \leq 2T$

$$|\partial_v\phi|(u,v) \leq \sup_x |\partial_x\phi_0(x)| + \sup_x |\phi_1(x)| + \int_{-v}^{u} |\partial_u\phi||\partial_v\phi|(u',v)du'\,.$$

[2] Recall this is to be interpreted that there exists a $y \in \mathbb{S}^2$ (the value of the map at infinity) such that each component of $(\phi_0 - y, \phi_1)$ is in $H^2(\mathbb{R}) \times H^1(\mathbb{R})$. See Remark 7.1.1.

Gronwall's inequality implies

$$|\partial_v \phi|(u,v) \leq \left(\sup_x |\partial_x \phi_0(x)| + \sup_x |\phi_1(x)|\right) \exp\left(\int_{-v}^{u} |\partial_u \phi|(u',v) du'\right) .$$

Using Cauchy-Schwarz and (7.16) we finally conclude

$$\sup_{0\leq u+v\leq 2T} |\partial_v \phi|(u,v) \leq \left(\sup_x |\partial_x \phi_0(x)| + \sup_x |\phi_1(x)|\right) \exp\left(CT^{\frac{1}{2}} \left(\|\phi_0\|_{H^1(\mathbb{R})} + \|\phi_1\|_{L^2(\mathbb{R})}\right)\right) .$$

It is clear that integrating in the other characteristic direction will produce the analogous bound

$$\sup_{0\leq u+v\leq 2T} |\partial_u \phi|(u,v) \leq \left(\sup_x |\partial_x \phi_0(x)| + \sup_x |\phi_1(x)|\right) \exp\left(CT^{\frac{1}{2}} \left(\|\phi_0\|_{H^1(\mathbb{R})} + \|\phi_1\|_{L^2(\mathbb{R})}\right)\right) .$$

□

7.4 The null condition

Recall that in the proof of given in the previous section, we exploited the fact that $\partial\phi$ decayed like $t^{-3/2}$. As mentioned, repeating the same proof in dimension $n = 3$ would lead to a problem since t^{-1} is not integrable.

It turns out that we can still prove small data global existence for $n = 3$ but we have to exploit more of the structure of the non-linearity, a fact to which we now turn. In the following, we restrict to $n = 3$ (although most formulae below generalise easily to higher n).

Lemma 7.4.1. *We have the following identity:*

$$\partial_t \phi \partial_t \psi - \sum_{i=1}^{3} \partial_i \phi \partial_i \psi = 2\partial_u \phi \partial_v \psi + 2\partial_v \phi \partial_u \psi - \nabla\!\!\!/ \phi \cdot \nabla\!\!\!/ \psi \tag{7.17}$$

where $\nabla\!\!\!/ \phi$ *denotes the* 3*-dimensional vector with components*[3]

$$\nabla\!\!\!/_i \phi = \partial_i \phi - \frac{x_i}{r}\partial_r \phi = -\sum_{j=1}^{3} \frac{x_j}{r^2} \Omega_{ij} \phi \tag{7.18}$$

and the dot product $\cdot$ *denotes the standard inner-product in* $\mathbb{R}^3$*. In addition, we have*

$$\nabla\!\!\!/ \phi \cdot \nabla\!\!\!/ \psi = \frac{1}{2r^2} \sum_{i,j=1}^{3} \Omega_{ij}\phi \Omega_{ij}\psi \, .$$

Proof. We compute

$$\sum_{i=1}^{3} \nabla\!\!\!/_i \phi \nabla\!\!\!/_i \psi = \sum_{i=1}^{3} \partial_i \phi \partial_i \psi - 2\partial_r \phi \partial_r \psi + \partial_r \phi \partial_r \psi = \sum_{i=1}^{3} \partial_i \phi \partial_i \psi - \partial_r \phi \partial_r \psi \tag{7.19}$$

and

$$\partial_t \phi \partial_t \psi - \partial_r \phi \partial_r \psi = \frac{1}{2}\left(\partial_t \phi + \partial_r \phi\right)\left(\partial_t \psi - \partial_r \psi\right) + \frac{1}{2}\left(\partial_t \phi - \partial_r \phi\right)\left(\partial_t \psi + \partial_r \psi\right) . \tag{7.20}$$

Combining the two identities yields the desired identity. For the last identity, note

$$\begin{aligned}\sum_{i=1}^{3} \nabla\!\!\!/_i \phi \nabla\!\!\!/_i \psi = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{x_j x_k}{r^4} \Omega_{ij}\phi \Omega_{ik}\psi &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \frac{x_j x_k}{r^4} \left(x_i \partial_j \phi - x_j \partial_i \phi\right)\left(x_i \partial_k \psi - x_k \partial_i \psi\right) \\ &= \sum_{i=1}^{3} \partial_i \phi \partial_i \psi - \sum_{i=1}^{3}\sum_{k=1}^{3} \frac{x_i x_k}{r^2} \partial_i \phi \partial_k \psi \end{aligned} \tag{7.21}$$

[3] Note that this is indeed the projection of the spatial gradient to the sphere of radius r, namely subtracting from ∂_i its projection to ∂_r.

and

$$\frac{1}{2r^2}\sum_{i,j=1}^{3}\Omega_{ij}\phi\Omega_{ij}\psi = \frac{1}{2r^2}\sum_{i=1}^{3}\sum_{i,j=1}^{3}\left(x_i\partial_j\phi - x_j\partial_i\phi\right)\left(x_i\partial_j\phi - x_j\partial_i\phi\right) = \sum_{i=1}^{3}\partial_i\phi\partial_i\psi - \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{x_i x_k}{r^2}\partial_i\phi\partial_k\psi.$$

□

This already looks good for the non-linearity appearing in the wave map equation as the Lemma tells us that in the quadratic expression at least one of the factors is a $\overline{\partial}$ (i.e. ∂_v or $\not\nabla$) derivative, which has better decay in L^∞ according to Corollary 6.2.6. The next proposition shows that "good derivatives" also decay better in the L^2-sense:

Proposition 7.4.2. *Let ϕ satisfy $\Box\phi = F$. Then, for every $\delta > 0$ there exists $C = C(\delta) > 0$ such that*

$$\left(\int_0^T\int_{\mathbb{R}^3}\frac{|\overline{\partial}\phi|^2}{(1+|t-r|)^{1+\delta}}dxdt\right)^{\frac{1}{2}} \leq C\left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} + \int_0^T\|F\|_{L^2(\mathbb{R}^3)}(t)dt\right) \tag{7.22}$$

Remark 7.4.3. *The Proposition states that (up to a δ-loss) we can use the additional factor of $(1+|t-r|)^{1+\delta}$ appearing in the L^∞ Klainerman-Sobolev estimate to make the L^2-norm integrable in time. Note the estimate would hold trivially if we replaced $t-r$ by t (use the energy estimate and the fact that $t^{-1-\delta}$ is integrable in t), so the point here is precisely that the weaker factor is sufficient to make the energy containing the good derivatives integrable in time.*

Proof. Let $w : \mathbb{R} \to \mathbb{R}$ be a C^1 function to be determined. We compute using the computations in (2.29) and (2.30)

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} w(t-|x|)\partial_t\phi F dxdt &= \int_0^T\int_{\mathbb{R}^3} w(t-|x|)\partial_t\phi\Box\phi dxdt \\ &= \int_0^T\int_{\mathbb{R}^3} w(t-|x|)\left[-\frac{1}{2}\partial_t\left[(\partial_t\phi)^2 + |\nabla_x\phi|^2\right] + \nabla_x\left(\partial_t\phi\nabla_x\phi\right)\right]dxdt\,. \end{aligned} \tag{7.23}$$

Integrating the right hand side by parts yields

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} w(t-|x|)\partial_t\phi F dxdt = &-\frac{1}{2}\int_{\mathbb{R}^3} w(T-|x|)\left((\partial_t\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2\right)(T) \\ &+\frac{1}{2}\int_{\mathbb{R}^3} w(-|x|)\left((\partial_t\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2\right)(0) \\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3} w'(t-|x|)\left((\partial_t\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2 + \sum_{i=1}^{3}\frac{2x_i}{r}\partial_t\phi\partial_i\phi\right)\,. \end{aligned} \tag{7.24}$$

Using (7.19) and that $\sum_{i=1}^{3}\frac{x_i}{r}\partial_i\phi = \partial_r\phi$ as well as $\partial_t + \partial_r = 2\partial_v$ we conclude

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} w(t-|x|)\partial_t\phi F dxdt = &-\frac{1}{2}\int_{\mathbb{R}^3} w(T-|x|)\left((\partial_t\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2\right)(T) \\ &+\frac{1}{2}\int_{\mathbb{R}^3} w(-|x|)\left((\partial_t\phi)^2 + \sum_{i=1}^{3}(\partial_i\phi)^2\right)(0) \\ &+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3} w'(t-|x|)\left(4(\partial_v\phi)^2 + |\not\nabla\phi|^2\right)\,. \end{aligned} \tag{7.25}$$

Now we choose w to be a bounded monotone decreasing function e.g. set $w'(s) = -\delta\left(1+|s|\right)^{-1-\delta}$ and $w(0) = 1$. (Note that this way $w > 0$ and hence the boundary terms at $t = T$ in the above also has a "good" sign.) The boundary term on $t = 0$ can be controlled by the initial energy (since w is bounded) and we deduce

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\overline{\partial}\phi|^2}{(1+|t-|x||)^{1+\delta}} dxdt \leq C \left(\|\phi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\phi_1\|^2_{L^2(\mathbb{R}^3)} + \int_0^T \left(\sup_{t\in[0,T]} \|\partial\phi\|_{L^2(\mathbb{R}^3)} \right) \|F\|_{L^2(\mathbb{R}^3)}(t) dt \right).$$

Now insert the standard energy estimate for the sup-term to produce the estimate claimed. □

Definition 7.4.4. *Let $q^{\alpha\beta}$ be constants. We say that $Q(\phi,\psi) = q^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$ satisfies the classical null condition if*

$$q^{\alpha\beta}\xi_\alpha\xi_\beta = 0 \quad \textit{whenever} \quad \eta^{\alpha\beta}\xi_\alpha\xi_\beta = 0.$$

When Q satisfies the classical null condition, we also say that Q is a classical null form.

It turns out we can completely classify all classical null-forms:

Lemma 7.4.5. *Let Q be a classical null form. Then it is a linear combination of $Q_0(\phi,\psi) := \eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$ and $Q_{\mu\nu}(\phi,\psi) = \partial_\mu\phi\partial_\nu\psi - \partial_\nu\phi\partial_\mu\psi$.*

Proof. We can decompose Q into a symmetric and anti-symmetric part, $Q(\phi,\psi) = Q_S(\phi,\psi) + Q_A(\phi,\psi)$ where $Q_S(\phi,\psi) = \frac{1}{2}\left(Q(\phi,\psi) + Q(\psi,\phi)\right)$ and $Q_A(\phi,\psi) = \frac{1}{2}\left(Q(\phi,\psi) - Q(\psi,\phi)\right)$. For $Q_A = q_A^{\mu\nu}\partial_\mu\phi\partial_\nu\psi$ we have $q_A^{\mu\nu} = -q_A^{\nu\mu}$ and hence

$$Q_A(\phi,\psi) = q_A^{\mu\nu}\partial_\mu\phi\partial_\nu\psi = \frac{1}{2}q_A^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\psi - \partial_\nu\phi\partial_\mu\psi\right)$$

Therefore, since the $q_A^{\mu\nu}$ are constants, any antisymmetric null form is indeed a linear combination of $Q_{\mu\nu}$. It now suffices to show that any symmetric null form Q_S (with components $q_S^{\mu\nu} = q_S^{\nu\mu}$) is proportional to $Q_0(\phi,\psi)$. Taking the collection of null vectors $e_0 \pm e_i$ we deduce $q_S^{00} \pm 2q_S^{0i} + q_S^{ii} = 0$ hence $q_S^{0i} = 0$ and $q_S^{ii} = -q_S^{00}$. Taking the collection of null vectors $\sqrt{2}e_0 + e_i + e_j$ we deduce $2q_S^{00} + q_S^{ii} + q_S^{jj} + q_S^{ij} = 0$, hence (by the previous) $q_S^{ij} = 0$. This establishes $q_S^{\mu\nu} = -q_S^{00}\eta^{\mu\nu}$ and hence the result. □

The next proposition shows that a null form has the desired property and maintains it under commutation.

Proposition 7.4.6. *Let Q be a classical null form. Then we have*

$$|Q(\phi,\psi)| \leq C\left(|\partial\phi||\overline{\partial}\psi| + |\overline{\partial}\phi||\partial\psi|\right). \tag{7.26}$$

Moreover, for any commuting vectorfield $\Gamma \in \{\partial_t, \partial_i, \Omega_{ij}, \Omega_{0i}, S\}$ we have

$$\Gamma\left(Q\left(\phi,\psi\right)\right) = Q\left(\Gamma\phi,\psi\right) + Q\left(\phi,\Gamma\psi\right) + \tilde{Q}\left(\phi,\psi\right) \tag{7.27}$$

for some classical null-form $\tilde{Q}$.

Proof. In view of the previous Lemma, it suffices to prove the results for the null-forms $Q_{\mu\nu}$ and Q_0. For Q_0 the first statement follows from Lemma 7.4.1. For the antisymmetric $Q_{\mu\nu}$ if follows similarly by considering

$$\partial_t\phi\partial_i\psi - \partial_i\phi\partial_t\psi = \partial_t\phi\left(\nabla\!\!\!/_i\psi + \frac{x_i}{r}(\partial_r\psi + \partial_t\psi)\right) - \left(\nabla\!\!\!/_i\phi + \frac{x_i}{r}(\partial_r\phi + \partial_t\phi)\right)\partial_t\psi$$

where we have inserted (7.18) and used the antisymmetry. The expression $\partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi$ can be treated similarly inserting (7.18).

For the commuted formula and $Q_0(\phi,\psi)$ we write $\Gamma = \Gamma^\alpha\partial_\alpha$. Note this abuses our previous notation – during this proof Γ^α denotes *the components* of the vectorfield Γ (*not* a collection of different Γ's). We compute

$$\begin{aligned}\Gamma^\alpha\partial_\alpha\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\psi\right) &= \eta^{\mu\nu}\partial_\mu\left(\Gamma^\alpha\partial_\alpha\phi\right)\partial_\nu\psi + \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\left(\Gamma^\alpha\partial_\alpha\psi\right) - \eta^{\mu\nu}\left(\partial_\mu\Gamma^\alpha\partial_\alpha\phi\partial_\nu\psi + \partial_\nu\Gamma^\alpha\partial_\mu\phi\partial_\alpha\psi\right)\\ &= \eta^{\mu\nu}\partial_\mu\left(\Gamma^\alpha\partial_\alpha\phi\right)\partial_\nu\psi + \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\left(\Gamma^\alpha\partial_\alpha\psi\right) - \left(\eta^{\mu\alpha}\partial_\mu\Gamma^\beta + \eta^{\beta\nu}\partial_\nu\Gamma^\alpha\right)\partial_\beta\phi\partial_\alpha\psi\end{aligned}$$

For Γ an isometry of Minkowski space (i.e. all Γ except the scaling vectorfield S), we have $\eta^{\mu\alpha}\partial_\mu\Gamma^\beta + \eta^{\beta\nu}\partial_\nu\Gamma^\alpha = 0$ (Exercise. – For those with geometric background: This is the (component) formula for the Lie-derivative $\mathcal{L}_\Gamma\eta$ written in Cartesian coordinates.) The scaling vectorfield S satisfies $\mathcal{L}_S\eta = 2\eta$ (it is a conformal isometry) which indeed produces a null-form as desired.

Proving the commuted formula for $Q_{\mu\nu}(\phi,\psi)$ is left to the reader (Example Sheet 10). □

7.5 Global existence and regularity for in $3+1$-dimensions

Theorem 7.5.1. *Let $n=3$. Consider the wave map equation (7.1) with initial data (ϕ_0,ϕ_1) smooth and of compact support in the ball $B(0,R)$ (cf. Remark 7.1.1) and satisfying in addition $\|\phi_0\|=1$ and $\phi_1^T\phi_0=0$. Then, for every $R>0$ there exists $\epsilon=\epsilon(R)>0$ sufficiently small with the following property: If*

$$\sum_{|\alpha|\le 7}\|\partial\partial^\alpha\phi_0\|_{L^2(\mathbb{R}^3)}+\|\partial^\alpha\phi_1\|_{L^2(\mathbb{R}^3)}<\epsilon\,, \tag{7.28}$$

then the unique solution to (7.1) remains smooth for all times.

Remark 7.5.2. *The following proof works replacing $|\alpha|\le 7$ by $|\alpha|\le 5$ if we use the full strength of the Klainerman-Sobolev inequality and not only the slightly weaker version that we actually proved.*

Proof. We will prove this by a bootstrap argument just as in the proof of Theorem 7.2.1. Fix R and $\delta\in(0,1)$. Below we will write C to denote a constant depending on δ and R (both fixed) but not on ϵ and T and allow this constant to change from line to line. Eventually ϵ will be chosen small enough to beat the constant C.

Let $T_\star$ be the maximal time such that the bound

$$\sup_{[0,T]}\sum_{|\alpha|\le 7}\|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^3)}(t)+\sum_{|\alpha|\le 7}\left(\int_0^T\int_{\mathbb{R}^3}\frac{|\overline{\partial}\Gamma^\alpha\phi|^2}{(1+|t-|x||)^{1+\delta}}dxdt\right)^{\frac{1}{2}}\le\epsilon^{\frac{3}{4}} \tag{7.29}$$

holds for all $T<T_\star$. Note that choosing ϵ sufficiently small, this holds for $T=0$ as $\epsilon\ll\epsilon^{\frac{3}{4}}$. Our goal is to use the equation to show that then (7.29) actually holds with the right hand side replaced by $\frac{1}{2}\epsilon^{\frac{3}{4}}$. By continuity, this will imply that the bound (7.29) also holds for $T<T_\star+\delta$ for some $\delta>0$. It follows hat $T_\star=\infty$ since the assumption $T_\star<\infty$ immediately leads to a contradiction.

Step 1: Lower order weighted L^∞-decay bounds from Klainerman-Sobolev. Applying the Klainerman-Sobolev inequality to (7.5) we deduce (note we are in spatial dimension $n=3$ and that the weaker version we proved actually suffices)

$$\sum_{|\alpha|\le 4}\|\,(1+t+r)\,(1+|t-r|)^{\frac{1}{2}}\,\partial\Gamma^\alpha\phi\|_{L^\infty(\mathbb{R}^3)}\le C\epsilon^{\frac{3}{4}}\,. \tag{7.30}$$

Integrating from initial data in the ∂_u-direction and using the decay in u we deduce this implies

$$\sum_{1\le|\alpha|\le 4}\|\,(1+t+r)^{\frac{1}{2}}\,\Gamma^\alpha\phi\|_{L^\infty(\mathbb{R}^3)}\le C\epsilon^{\frac{3}{4}}\,. \tag{7.31}$$

Combining this with (6.10) we infer in addition

$$\sum_{|\alpha|\le 3}\|\,(1+t+r)^{\frac{3}{2}}\,\overline{\partial}\Gamma^\alpha\phi\|_{L^\infty(\mathbb{R}^3)}\le C\epsilon^{\frac{3}{4}}\,. \tag{7.32}$$

Note (7.30)–(7.32) also hold (trivially) replacing $1+t+r$ by $t+1$.

Step 2. Closing the energy estimates

We have from adding the commuted energy estimate and the estimate of Proposition 7.4.2

$$\sup_{t\in[0,T]}\sum_{|\alpha|\le 7}\|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^3)}(t)+\sum_{|\alpha|\le 7}\left(\int_0^T\int_{\mathbb{R}^3}\frac{|\overline{\partial}\Gamma^\alpha\phi|^2}{(1+|t-|x||)^{1+\delta}}dxdt\right)^{\frac{1}{2}}$$
$$\le C\left[\sum_{|\alpha|\le 7}\|\partial\Gamma^\alpha\phi\|_{L^2(\mathbb{R}^3)}(0)+\int_0^T\sum_{|\alpha|\le 7}\left\|\Gamma^\alpha\left(\phi\left(\partial_t\phi^T\partial_t\phi-\sum_{i=1}^n\partial_i\phi^T\partial_i\phi\right)\right)\right\|_{L^2(\mathbb{R}^3)}(t)dt\right]\,. \tag{7.33}$$

For the non-linear term we observe

$$\int_0^T \sum_{|\alpha|\leq 7} \left\| \Gamma^\alpha \left(\phi \left(\partial_t \phi^T \partial_t \phi - \sum_{i=1}^n \partial_i \phi^T \partial_i \phi \right) \right) \right\|_{L^2(\mathbb{R}^3)} (t) dt$$

$$\leq \int_0^T \left(\sum_{|\alpha_1|\leq 3, |\alpha_2|\leq 7, |\alpha_3|\leq 3} \| |\Gamma^{\alpha_1}\phi| |\partial \Gamma^{\alpha_2}\phi| |\overline{\partial} \Gamma^{\alpha_3}\phi| \|_{L^2(\mathbb{R}^3)} \right) (t) dt$$

$$+ \int_0^T \left(\sum_{|\alpha_1|\leq 3, |\alpha_2|\leq 3, |\alpha_3|\leq 7} \| |\Gamma^{\alpha_1}\phi| |\partial \Gamma^{\alpha_2}\phi| |\overline{\partial} \Gamma^{\alpha_3}\phi| \|_{L^2(\mathbb{R}^3)} \right) (t) dt$$

$$+ \int_0^T \left(\sum_{1\leq |\alpha_1|\leq 7, |\alpha_2|\leq 3, |\alpha_3|\leq 3} \| |\Gamma^{\alpha_1}\phi| |\partial \Gamma^{\alpha_2}\phi| |\overline{\partial} \Gamma^{\alpha_3}\phi| \|_{L^2(\mathbb{R}^3)} \right) (t) dt \, . \tag{7.34}$$

Denoting the terms on the right hand side by 1, 2 and 3 respectively, we see that we can estimate 1 by putting the α_1 and α_3 terms in $L^\infty(\mathbb{R}^n)$ using (7.32) and (7.31) respectively (and $|\phi| = 1$ if $|\alpha_1| = 0$)as well as the main bootstrap assumption for the α_3-term

$$\text{Term } 1 \leq C \int_0^T \frac{\epsilon^{\frac{3}{2}}}{(1+t)^2} dt \leq C\epsilon^{\frac{3}{2}} \, .$$

$$\text{Term } 2 \leq C \sum_{|\alpha_2|\leq 3, |\alpha_3|\leq 7} \int_0^T \left(\int_{\mathbb{R}^3} |\overline{\partial} \Gamma^{\alpha_3} \phi \partial \Gamma^{\alpha_2} \phi|^2 dx \right)^{\frac{1}{2}} dt$$

$$\leq C \sum_{|\alpha_2|\leq 3, |\alpha_3|\leq 7} \int_0^T \left[\left(\int_{\mathbb{R}^3} \frac{|\overline{\partial}\Gamma^{\alpha_3}\phi|^2}{(1+|t-r|)^{1+\delta}} dx \right)^{\frac{1}{2}} \left(\sup_x \left((1+|t-r|)^{\frac{1+\delta}{2}} |\partial \Gamma^{\alpha_2}\phi|(t,x) \right) \right) \right] dt$$

$$\leq C \left[\sum_{|\alpha_3|\leq 7} \int_0^T \int_{\mathbb{R}^3} \frac{|\overline{\partial}\Gamma^{\alpha_3}\phi|^2}{(1+|t-r|)^{1+\delta}} dx dt \right]^{\frac{1}{2}} \left[\sum_{|\alpha_2|\leq 3} \left(\int_0^T \left(\sup_x \left\{ (1+|t-r|)^{\frac{1+\delta}{2}} |\partial \Gamma^{\alpha_2}\phi|(t,x) \right\} \right)^2 dt \right) \right]^{\frac{1}{2}}$$

$$\leq C\epsilon^{\frac{3}{4}} \left(\int_0^T \frac{\epsilon^{\frac{3}{2}}}{(1+t)^{2-\delta}} dt \right)^{\frac{1}{2}} \leq C\epsilon^{\frac{3}{2}} \, ,$$

where we have used Cauchy-Schwarz in going to the third line and (7.30) in going to the last line. Finally, we have (using the trivial bound $|\Gamma\phi| \leq (1+t+r)|\partial\psi|$).

Term 3

$$\leq \int_0^T \sum_{|\alpha_1|\leq 6, |\alpha_2|\leq 3, |\alpha_3|\leq 3} \frac{1}{(1+t)^{\frac{3}{2}}} \| |\partial \Gamma^{\alpha_1}\phi| \|_{L^2(\mathbb{R}^3)} \| (1+t+r) \partial \Gamma^{\alpha_2}\phi \|_{L^\infty(\mathbb{R}^3)} \| |(1+t+r)^{\frac{3}{2}} \overline{\partial} \Gamma^{\alpha_3}\phi| \|_{L^\infty(\mathbb{R}^3)} (t) dt$$

$$\leq C\epsilon^{\frac{9}{4}}$$

Since the initial data term in (7.33) can be bounded by ϵ, we deduce using the estimates for Terms 1, 2 and 3 the estimate

$$\sup_{t\in[0,T]} \sum_{|\alpha|\leq 7} \|\partial \Gamma^\alpha \phi\|_{L^2(\mathbb{R}^3)}(t) + \sum_{|\alpha|\leq 7} \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\overline{\partial}\Gamma^\alpha \phi|^2}{(1+|t-|x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \leq C(\epsilon + \epsilon^{\frac{3}{2}} + \epsilon^{\frac{9}{4}}) \, . \tag{7.35}$$

We can now easily choose ϵ sufficiently small (depending only on the C that appears above) such that the right hand side is smaller than $\frac{1}{2}\epsilon^{\frac{3}{4}}$ as desired. □

Remark 7.5.3. *As a final remark we note that this small data global existence proof will work for any non-linearity satisfying the null condition!*

7.6 The weak null condition

The reader's intuition might benefit from studying Section 8.1 before this section.

We now give an example that failure of the null condition does not mean that global existence for small data must automatically fail. The example below is the simplest version of the so-called *weak null-condition* which appears most famously in the (harmonic gauge) proof of the stability of Minkowski space in general relativity due to Lindblad and Rodnianski.

We consider (in dimension $1+3$) the coupled system

$$\begin{cases} \Box\phi = Q_0(\psi,\psi) \\ \Box\psi = (\partial_t\phi)^2 \end{cases} . \tag{7.36}$$

Clearly, the system does not satisfy the null condition and we might conjecture blow-up in finite time from the results of Section 8.1! However, we can prove global existence for small data exploiting the hierarchical structure of (7.36):

Theorem 7.6.1. *Consider the system (7.36) with data $(\phi_0,\phi_1,\psi_0,\psi_1)$ smooth and compactly supported in a ball $B(0,R)$ of radius R. Then for every R there exists an $\epsilon>0$ with the following property: If*

$$\sum_{|\alpha|\leq 5} \|\partial\partial^\alpha(\phi_0,\psi_0)\|_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + \|\partial^\alpha(\phi_1,\psi_1)\|_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} < \epsilon\,, \tag{7.37}$$

then the solution to (7.36) exists globally and is smooth.

Proof. We run the usual bootstrap argument. This time we will allow ourselves to use the full strength of (6.4) although we only proved a slightly weaker version, cf. Remark 7.5.2. If we replace $|\alpha|\leq 5$ by $|\alpha|\leq 7$ in the statement, the weaker version would suffice for the proof.

Let $0<T_\star$ be the supremum of times such that the following estimates hold for all $T<T_\star$

$$\sup_{t\in[0,T]}\sum_{|\alpha|\leq 5}\|\partial\Gamma^\alpha\psi\|_{L^2(\mathbb{R}^3)}(t) + \sum_{|\alpha|\leq 5}\left(\int_0^T\int_{\mathbb{R}^3}\frac{|\overline{\partial}\Gamma^\alpha\psi|^2}{(1+|t-|x||)^{1+\delta}}dxdt\right)^{\frac{1}{2}} \leq \epsilon^{\frac{3}{4}}\,(1+T)^{\frac{1}{10}}\,. \tag{7.38}$$

Note that we are only making a bootstrap assumption on ψ. The idea is to use this to prove energy estimates for ϕ and hen plug these into the ψ equation to estimate ϕ (hence requiring no bootstrap assumptions on ϕ).

From the Klainerman-Sobolev inequality (6.4) we infer that (7.38) implies the following estimates

$$\sup_{t\in[0,T]}\sum_{|\alpha|\leq 3}\|(1+t)\,(1+|t-r|)^{\frac{1}{2}}\,\partial\Gamma^\alpha\psi\|_{L^\infty(\mathbb{R}^3)}(t) \leq C\epsilon^{\frac{3}{4}}(1+T)^{\frac{1}{10}} \tag{7.39}$$

and

$$\sup_{t\in[0,T]}\sum_{|\alpha|\leq 2}(1+t)^{\frac{3}{2}}\|\overline{\partial}\Gamma^\alpha\psi\|_{L^\infty(\mathbb{R}^3)} \leq C\epsilon^{\frac{3}{4}}(1+T)^{\frac{1}{10}}\,, \tag{7.40}$$

where (7.40) follows from (7.39), (6.10) and integration of (7.39) in the u-direction from initial data (as seen before in the other proofs).[4]

We now apply the energy estimates for ϕ which requires control of

$$\sum_{|\alpha|\leq 5}\int_0^T \|\Gamma^\alpha Q_0(\psi,\psi)\|_{L^2(\mathbb{R}^3)}(t)dt\,. \tag{7.41}$$

There will be a term when the majority of $\Gamma's$'s fall on the good derivative in the null form and one where the majority falls on the bad derivative. For the latter, we can estimate

$$\int_0^T\left(\sum_{|\alpha|\leq 5}\|\partial\Gamma^\alpha\psi\|_{L^2(\mathbb{R}^3)}\right)\left(\sum_{|\beta|\leq 2}\|\overline{\partial}\Gamma^\beta\psi\|_{L^\infty(\mathbb{R}^3)}\right)(t)dt \leq C\int_0^T\frac{\epsilon^{\frac{3}{2}}}{(1+t)^{\frac{3}{2}-\frac{1}{5}}} \leq C\epsilon^{\frac{3}{2}}\,, \tag{7.42}$$

[4]Note in particular that (7.40) holds removing the sup and replacing t by T, which is what we will use later.

where we have used (7.38) for the top-order term and (7.40) for the lower order term (and the footnote).

For the former term we use a dyadic decomposition. We define (draw a picture!)

$$T_0 = 0 \quad , \quad T_i = 2^{i-1} \ \text{ for } i = 1, 2, ..., \lfloor \log_2 T \rfloor \quad \text{and} \quad T_{\lfloor \log_2 T \rfloor + 1} = T \, .$$

We then estimate

$$\begin{aligned}
&\int_0^T \left(\sum_{|\alpha| \le 5, |\beta| \le 2} \| \overline{\partial} \Gamma^\alpha \psi \partial \Gamma^\beta \psi \|_{L^2(\mathbb{R}^3)} \right)(t) dt \\
&\le C \sum_{t=0}^{\lfloor \log_2 T \rfloor} \sum_{|\alpha| \le 5, |\beta| \le 2} \int_{T_i}^{T_{i+1}} \left[\int_{\mathbb{R}^3} \frac{|\overline{\partial} \Gamma^\alpha \psi|^2}{(1 + |t - |x||)^{1+\delta}} dx \right]^{\frac{1}{2}} \left[\sup_x \left((1 + |t - r|)^{\frac{1+\delta}{2}} |\partial \Gamma^\beta \psi|(t, x) \right) \right] dt \\
&\le C \sum_{t=0}^{\lfloor \log_2 T \rfloor} \left[\sum_{|\alpha| \le 5} \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} \frac{|\overline{\partial} \Gamma^\alpha \psi|^2}{(1 + |t - |x||)^{1+\delta}} dx dt \right]^{\frac{1}{2}} \left[\sum_{|\beta| \le 2} \left(\int_{T_i}^{T_{i+1}} \left(\sup_x \left((1 + |t - r|)^{\frac{1+\delta}{2}} |\partial \Gamma^\beta \psi|(t, x) \right) \right) dt \right)^2 \right]^{\frac{1}{2}} \\
&\le C \epsilon^{\frac{3}{2}} \sum_{t=0}^{\lfloor \log_2 T \rfloor} (1 + T_i)^{\frac{1}{10}} \left[\int_{T_i}^{T_{i+1}} \frac{dt}{(1 + t)^{2 - \frac{1}{5} - \delta}} \right]^{\frac{1}{2}} \le C \epsilon^{\frac{3}{2}}
\end{aligned}$$

for $\delta \in (0, \frac{3}{5})$. Here we have used (7.38) for the top order term and (7.39) for the lower order term.

Combining the estimates for the non-linear term the estimate for ϕ closes and we conclude

$$\sup_{t \in [0,T]} \sum_{|\alpha| \le 5} \| \partial \Gamma^\alpha \phi \|_{L^2(\mathbb{R}^3)}(t) \le C \epsilon \, . \tag{7.43}$$

The Klainerman-Sobolev inequality implies

$$\sup_{t,x} \sum_{|\alpha| \le 3} \left[(1 + t + r) (1 + |t - r|)^{\frac{1}{2}} |\partial \Gamma^\alpha \phi| \right] \le C \epsilon \, . \tag{7.44}$$

Applying finally the energy estimate for ψ and using the above estimates on ϕ for the non-linear error, we infer

$$\begin{aligned}
&\sup_{t \in [0,T]} \sum_{|\alpha| \le 5} \| \partial \Gamma^\alpha \psi \|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \le 5} \left(\int_0^T \int_{\mathbb{R}^3} \frac{|\overline{\partial} \Gamma^\alpha \psi|^2}{(1 + |t - |x||)^{1+\delta}} dx dt \right)^{\frac{1}{2}} \\
&\le C \left(\epsilon + \int_0^T \left(\sum_{|\alpha| \le 5} \| \partial \Gamma^\alpha \phi \|_{L^2(\mathbb{R}^3)}(t) \right) \left(\sum_{|\beta| \le 2} \| \partial \Gamma^\beta \phi \|_{L^\infty(\mathbb{R}^3)}(t) \right) dt \right) \\
&\le C \left(\epsilon + \epsilon^2 \int_0^T \frac{dt}{1 + t} \right) \le C \epsilon \log(2 + T) \, ,
\end{aligned} \tag{7.45}$$

which improves (7.38) and therefore finishes the proof as $T_\star = \infty$. □

Chapter 8

Finite time blow-up and singularities

8.1 Fritz John's classical blow-up theorem

We now prove the following classical result:

Theorem 8.1.1 (F. John, 1981)**.** *Let $\psi : \mathbb{R}^{3+1} \to \mathbb{R}$ be a $C^2\left(\mathbb{R}^3 \times [0,\infty)\right)$ solution of*

$$\Box\psi = -(\partial_t \psi)^2 \tag{8.1}$$

arising from smooth data of compact support. Then $\psi \equiv 0$.

The theorem can be paraphrased by saying that <u>any</u> C^2 solution arising from non-trivial data of compact support must blow up in finite time. Unfortunately, the theorem does not tell us anything about what goes wrong, i.e. about the nature of the singularity (besides what we know from the breakdown criteria).

Proof. We choose R such that $\psi(x,0) = \psi_t(x,0) = 0$ for $|x| > R$.

Step 1. We derive an equation for the spherical means. Recall that for any function $\Psi(x,t)$ we can define its spherical means

$$\overline{\Psi}(r,t) = \frac{1}{4\pi}\int_{|\xi|=1} \Psi(r\xi,t)dS(\xi)\,.$$

Here a priori $r \geq 0$ but the formula on the right makes sense for $r < 0$ as well and clearly $\overline{\Psi}(-r,t) = \overline{\Psi}(r,t)$ so we can consider $\overline{\Psi}$ as an even function of r. Turning to (8.1) we have by the Darboux identity (cf. (2.9))

$$\overline{w} := \overline{\psi}_{tt} - \overline{\psi}_{rr} - \frac{2}{r}\overline{\psi}_r = \overline{(\psi_t)^2}\,.$$

Note that $\overline{\psi}$ and $\overline{\psi}_{rr}$ as well as $\frac{2}{r}\overline{\psi}_r$ and $\overline{\psi}_t$ are even functions in r that vanish for $r > R + t$ by domain of dependence. Note also that we can write

$$(r\overline{\psi})_{tt} - (r\overline{\psi})_{rr} = r\overline{w} = r\overline{(\psi_t)^2}\,.$$

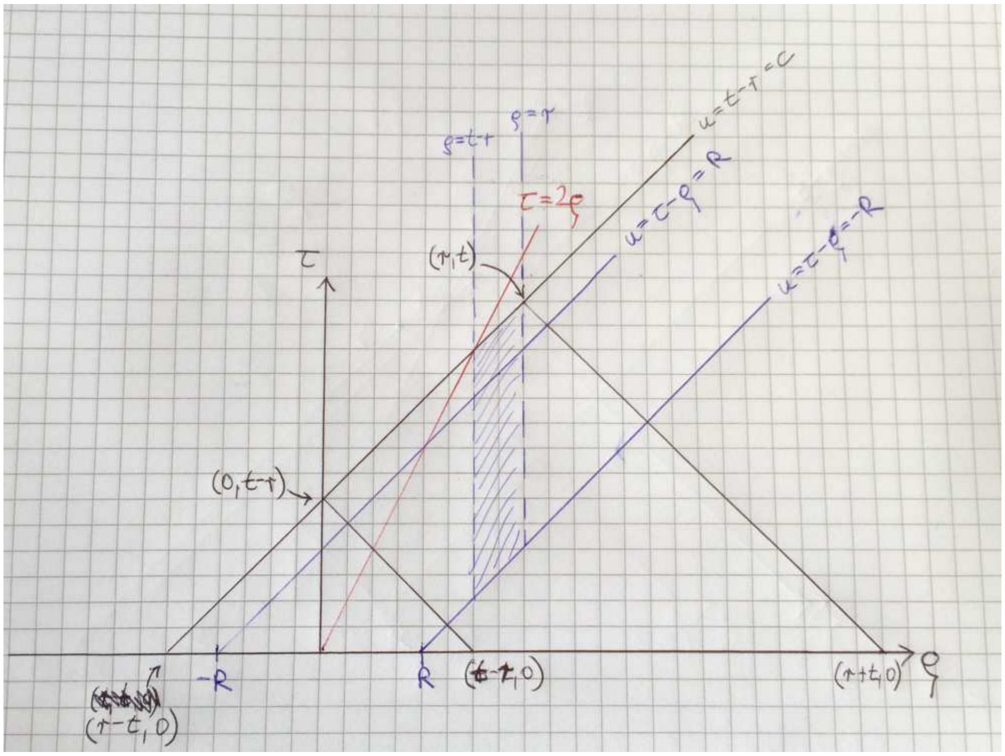
Step 2. We use the Duhamel formula for the inhomogeneous wave equation derived in (2.28) to obtain

$$\overline{\psi}(r,t) = \overline{\psi}_0(r,t) + \frac{1}{2r}\int_{T(r,t)} \rho\overline{(\psi_t)^2}d\rho d\tau \tag{8.2}$$

with (the homogeneous solution realising the data being)

$$\overline{\psi}_0(r,t) = \frac{(r+t)\overline{\psi}(r+t,0) + (r-t)\overline{\psi}(r-t,0)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \rho\overline{\psi}_t(\rho,0)d\rho\,. \tag{8.3}$$

and $T(r,t)$ the characteristic triangle with vertex at (t,r) and basis on the ρ-axis.



Step 3. We now restrict attention to the region

$$\Sigma := \{(\rho,\tau) \mid \rho + R < \tau < 2\rho\}\,.$$

In the picture it is the region to the east of the red line intersected with the region to the north of $u = R$. If $(r,t) \in \Sigma$ we easily see that $\psi_0(r,t) = 0$ because $r + t \geq R$ and $r - t < -R$ and because the integral in (8.3) can be written as

$$\int_{-R}^{R} d\rho\rho\overline{\psi}_t(\rho,0) = 0$$

since one integrates an odd function over an interval symmetric about the origin. We conclude

$$\overline{\psi}\,(r,t) = \frac{1}{2r}\int_{T(r,t)} \rho\overline{(\psi_t)^2} d\rho d\tau\,. \tag{8.4}$$

We now define the trapezoid

$$T^\star\,(r,t) = \text{trapezoid with vertices at } (r,t),\ (0,t-r),\ (t-r,0),\ (t+r,0)$$

Writing the integral in (8.4) as a sum of the integral over the trapezoid and an integral over the characteristic triangle with vertex at $(0,t-r)$ we conclude that the second part vanishes because we are integrating an odd function over an interval that is symmetric around the origin. We conclude

$$\overline{\psi}(r,t) = \frac{1}{2r}\int_{T^\star(r,t)} \rho\overline{(\psi_t)^2} d\rho d\tau \geq \frac{1}{2r}\int_{T^\star(r,t)} \rho(\overline{\psi}_t)^2 d\rho d\tau\,, \tag{8.5}$$

where (for the inequality) we have used that $\overline{(\psi_t)^2} \geq (\overline{\psi}_t)^2$ by Cauchy-Schwarz and that $\rho \geq 0$ holds in $T^\star(r,t)$. Since the integrand is non-negative we can restrict the integration region further to

$$S\,(r,t) = \{(\rho,t) \mid t - r < \rho < r \quad \text{and} \quad -R < u = \tau - \rho < t - r\}\,,$$

(the shaded region in the above picture) to obtain the estimate

$$\overline{\psi}(r,t) \geq \frac{1}{2r}\int_{S(r,t)} \rho(\overline{\psi}_t)^2 d\rho d\tau = \frac{1}{2r}\int_{t-r}^{r} \rho d\rho \int_{\rho-R}^{\rho+t-r} d\tau(\overline{\psi}_t)^2 \tag{8.6}$$

valid for any $(r,t) \in \Sigma$.

Step 4. For $(r,t) \in \Sigma$ we fix the characteristic $u = t - r =: c$ and consider the points along that characteristic, i.e. points parametrised by $(\rho, \rho + c)$ with $\rho \geq c$. Our goal will be to derive an ordinary differential inequality along that characteristic.

Since $\bar{\psi} = 0$ on $u = -R$ we can write by the fundamental theorem of calculus

$$\overline{\psi}(\rho, \rho + c) = \int_{\rho-R}^{\rho+c} \partial_t \overline{\psi}\,(\rho, \tau)\ d\tau\,.$$

Cauchy-Schwarz tells us that

$$|\overline{\psi}(\rho, \rho + c)|^2 \leq (R + c) \int_{\rho-R}^{\rho+c} |\partial_t \overline{\psi}\,(\rho, \tau)\,|^2\, d\tau\,.$$

With this (8.6) becomes

$$\overline{\psi}(r, r + c) \geq \frac{1}{2(R + c)r} \int_c^r \rho|\overline{\psi}(\rho, \rho + c)|^2 d\rho \tag{8.7}$$

for points $(r,t) \in \Sigma$ lying on the fixed characteristic $t - r = c$. Defining

$$\beta\,(r) = \int_c^r \rho|\overline{\psi}(\rho, \rho + c)|^2 d\rho$$

we can write (8.7) as

$$\beta' \geq \frac{1}{4(R + c)^2 r}\beta^2 \tag{8.8}$$

which is the desired ordinary differential inequality along the characteristic $t - r = c$. Assume $\beta \neq 0$ for some $r = r_0$ along the characteristic. Then $\beta(r) \geq \beta(r_0)$ for all $r \geq r_0$. Integrating (8.8) from $r = r_0$ to $r > r_0$ we find

$$\frac{1}{\beta(r_0)} \geq \frac{1}{\beta(r_0)} - \frac{1}{\beta(r)} \geq \frac{1}{4}\frac{1}{(R + c)^2} \log\frac{r}{r_0}$$

which leads to a contradiction for sufficiently large r. We conclude $\beta = 0$ and (by definition of β) $\overline{\psi} = 0$ along the entire characteristic $t - r = c$. Since this works for any $(t,r) \in \Sigma$ we conclude $\overline{\psi} = 0$ in Σ. Going back to (8.5), which holds for any $(r,t) \in \Sigma$, we conclude $\overline{(\psi_t)^2} = 0$ in $T^\star\,(r,t)$ for any $(r,t) \in \Sigma$. Choosing (t,r) with $t - r$ arbitrarily close to R and letting $r \to \infty$ we deduce that $\overline{(\psi_t)^2} = 0$ is identically zero for $t > R$ (why?). Since the spherical average of a non negative function is zero, the function itself has to be zero hence $\psi_t = 0$ identically for $t > R$. In particular also $\psi_{tt} = 0$ for $t > R$ and this means on a constant t slice with $t > R$, the function ψ satisfies $\Delta\psi = 0$ and is of compact support. This immediately implies $\psi = 0$ for $t > R$ (why?). Now by our uniqueness proof (which works backwards and forwards) $\psi = 0$ globally. □

Remark 8.1.2. *It is easy to see that the proof of Theorem 8.1.1 will go through verbatim for* $\Box\psi = F$ *with* $F \geq a\,(\partial_t\psi)^2$ *for* $a > 0$ *a constant. In fact, one can prove the result for more complicated non-linearities including quasi-linear ones. More details in the original paper of F. John (CPAM 34 (1981), 29-51) or the PDE notes of P. Constantin (available from his Princeton webpage) which I have been following closely here.*

8.2 Small data shock formation in $n = 3$

We now want to look at a few quasi-linear problems. Motivated by the introduction, equation (1.5), one would like to study systems of the form

$$\left[\left(g^{-1}\right)^{\alpha\beta}(\Psi)\right]\partial_\alpha\partial_\beta\Psi = \mathcal{N}(\Psi, \partial\Psi) \tag{8.9}$$

where g (with inverse g^{-1}) is a $3+1$-dimensional Lorentzian metric depending (say smoothly) on Ψ. Such system encompass the equations appearing in general relativity and the (irrotational) compressible Euler equations, which are fundamental equations of mathematical physics.

Below, we will allow ourselves to make several simplifications when studying (8.9) with the goal to give you an idea about the phenomena that appear:

- We will look at the case of a **scalar** equation.
- We will look at the **small data** case, i.e. Ψ and $\partial_t\Psi$ are initially small and of compact support. By an affine change of coordinates we can arrange that $\left(g^{-1}\right)^{\alpha\beta}(0) = \eta^{\alpha\beta}$ which we shall henceforth assume. We also assume that the non-linearity $\mathcal{N}(\Psi, \partial\Psi)$ is quadratic or higher order in $\partial\Psi$, i.e. $\mathcal{N} = \mathcal{O}(|\partial\Psi|^2)$ for small Ψ, $\partial\Psi$.
- we will study **spherically symmetric** solutions of (8.9) reducing the problem to a $1+1$ dimensional problem to which the method of characteristics applies.
- we will only look at solutions in a subset of the domain of dependence of the data, the **wave zone**. (This is the potentially "dangerous" region from the intuition we have built so far about the decay of solutions.)

More concretely now, we will look at the following quasi-linear model problem

$$\begin{cases} -\partial_t^2\psi + (1+\psi)\Delta\psi = -\frac{(\partial_t\psi)^2}{1+\psi} \\ \psi(0,x) = \epsilon f(x) \\ \partial_t\psi(0,x) = \epsilon g(x) \end{cases} \tag{8.10}$$

where f and g are radially symmetric smooth functions of compact support and we want to understand the evolution for $\epsilon > 0$ small. We will show

Theorem 8.2.1. *Given f, g smooth, radially symmetric and of compact support and non-trivial ($f^2+g^2 \neq 0$), there exists an ϵ_0 such that for any $\epsilon < \epsilon_0$ the solution of (8.10) must blow up in finite time.*

In fact, in the course of the proof we will obtain detailed information about a blow-up mechanism that necessarily occurs for (8.10) in the wave zone: Shock formation. Note that since (8.10) is quasilinear, it is now the solution itself that determines the null ("characteristic") cones of the solution! We easily read off

$$g(\psi) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{1+\psi} & 0 & 0 \\ 0 & 0 & \frac{1}{1+\psi} & 0 \\ 0 & 0 & 0 & \frac{1}{1+\psi} \end{pmatrix}.$$

Shock formation will correspond to the intersection of characteristic cones. In particular, ψ will remain small and the Euclidean derivatives will blow up at the point of intersection similar to what we have seen in Burger's equation in Section 4.5.2.

Note that in the spherically symmetric case (8.10) reduces to studying the problem

$$\begin{cases} -\partial_t^2(r\psi) + (1+\psi)\partial_r^2(r\psi) = -r\frac{(\partial_t\psi)^2}{1+\psi} \\ r\psi(0,r) = \epsilon\tilde{f}(r) \\ \partial_t(r\psi)(0,r) = \epsilon\tilde{g}(r) \end{cases} \tag{8.11}$$

with $\tilde{f}$, $\tilde{g}$ odd functions of compact support.

8.2.1 Warm-up: Semi-linear problem with null condition

To get into the spirit of integrating along characteristics and the bootstrap argument, let us first consider the following problem (recall the null coordinates $u = t - r$, $v = t + r$)

$$\begin{cases} -\partial_t^2(r\psi) + \partial_r^2(r\psi) = -4\partial_u\partial_v(r\psi) = -4r(\partial_u\psi)(\partial_v\psi) \\ r\psi(0,r) = \epsilon\tilde{f}(r) \\ \partial_t(r\psi)(0,r) = \epsilon\tilde{g}(r) \end{cases} \tag{8.12}$$

Note this is the spherically symmetric version of the problem $\Box\psi = Q_0(\psi,\psi)$ for which we have already shown small data global existence! Suppose we want to understand the solution in the strip

$$\mathcal{R} = \{t \geq 0\} \cap \left\{ -1 \leq u \leq u_0 = -\frac{1}{4} \right\}.$$

Say initially

$$\sup_r |r\psi(t=0,r)| + \sup_r |\partial_u(r\psi)(t=0,r)| + \sup_r |r\partial_u\psi(t=0,r)| + \sup_r |r^2\partial_v\psi(t=0,r)| \leq C_{init}\epsilon\,.$$

Proposition 8.2.2. *Fix $\tilde{f}$, $\tilde{g}$. For sufficiently small ϵ (depending only on C_{init}), the solution of (8.12) exists in all of $\mathcal{R}$.*

Remark 8.2.3. *Note that we have obviously proven a much stronger statement already (no symmetry assumptions and no restrictions on the domain). The point here is to prove the weaker statement in the most elementary fashion.*

Proof. We define the bootstrap region $\mathcal{B} \subset [0,\infty)$ by

$$\mathcal{B} = \{t' \in [0,\infty) \;\big|\; \text{the estimates } \; |r\psi| \leq C\epsilon \;\;,\;\; |r\partial_u\psi| \leq C\epsilon \;\;,\;\; |r^2\partial_v\psi| \leq C\epsilon \;\text{ hold in } \mathcal{R}\cap\{t<t'\}\},$$

where $C = 8C_{init}$. The region $\mathcal{B}$ is clearly closed and non-empty (why?).[1]

We will now show that the region is also open for sufficiently small ϵ (depending only on C_{init}) by successively "improving" the three bounds.

$$\partial_u(r\psi)(u,v) = \partial_u(r\psi)(u, v_{data} = -u) + \int_{v_{data}}^{v} \partial_u\psi r\partial_v\psi r^2 \frac{2\partial_v r}{r^2} dv \tag{8.13}$$

which after inserting the bootstrap assumptions leads to

$$|\partial_u(r\psi)(u,v)| \leq C_{init}\epsilon + 8C^2\epsilon^2 \leq 2C_{init}\epsilon \tag{8.14}$$

for sufficiently small ϵ and all (u,v) in $\mathcal{R}\cap\{t<t'\}$. Integrating in the short direction, from $u = u_0$ (or the data) we have

$$|r\psi(u,v)| \leq 3C_{init}\epsilon\,.$$

and combining the bounds

$$|r\partial_u\psi(u,v)| \leq 4C_{init}\epsilon\,.$$

Finally, from

$$\partial_u(r\partial_v\psi) = -\partial_u\psi + r\partial_u\psi\partial_v\psi \tag{8.15}$$

we deduce

$$|r\partial_v\psi(u,v)| \leq \frac{C_{init}}{r(u,v)} + C_{init}\int_{-1}^{u} d\bar{u}\frac{1}{r(\bar{u},v)} + \int_{-1}^{u} d\bar{u}\frac{C^2\epsilon^2}{r^2(\bar{u},v)}$$

and by the fact that r decreases as u increases (and the u-length is at most 1)

$$|r\partial_v\psi(u,v)| \leq \frac{4C_{init}}{r(u,v)} \quad \text{for all } (u,v) \text{ in } \mathcal{R}\cap\{t<t'\}.$$

This finishes the proof as all bounds have been improved. By continuity $\mathcal{B}$ is also open and hence $\mathcal{B} = [0,\infty)$ which was the claim. □

Note again the crucial role played by the null condition in this proof!

[1] Recall also from our breakdown criteria that the solution can be continued as long as we have C^1 bounds on the solution.

8.2.2 The proof of Theorem 8.10

To prove Theorem 8.10 we study the system (8.12). We will assume that the data $\tilde{f}$, $\tilde{g}$ are supported in $[-1, 1]$. Note this is without loss of generality, as we can rescale the coordinates t and r to achieve this.

Step 0: Preliminary considerations and definition of the domain of analyis
Let us suppose we are given a local in time solution, so in particular ψ is "known". We then define the characteristic directions

$$L = \partial_t + \sqrt{1+\psi}\partial_r \quad \text{ and } \quad \underline{L} = \partial_t - \sqrt{1+\psi}\partial_r \,. \tag{8.16}$$

Note that these vectors are indeed null vectors for the metric g in (8.2) and reduce to the familiar ingoing and outgoing null-directions $2\partial_u = \partial_t + \partial_r$ and $2\partial_v = \partial_t + \partial_r$ if $\psi = 0$. Corresponding to L there is an optical function $u(t, r)$ defined by the first order PDE

$$Lu(t, r) = 0 \quad u(t = 0, r) = 1 - r \,. \tag{8.17}$$

In other words, the integral curves of L are precisely the $u = const$ curves, where you should note that the definition of these curves depends on the solution itself.[2] We denote the leaves of the foliation (i.e. the constant u-curves or null-hypersurfaces in the physical space picture) by C_u. This gives rise to a new coordinate system $(\tilde{t}, u)$ defined by

$$\tilde{t} = t \quad , \quad u = u(t, r) \,. \tag{8.18}$$

We would like to understand the solution in the region

$$\mathcal{M}_{t,u_0} := \{(t', r) \;\; | \;\; 0 \leq t' < t \;\text{ and }\; 0 \leq u(t', r) \leq u_0\} \tag{8.19}$$

with $u_0 = \frac{3}{4}$ (say) so that we are away from the center of symmetry.

We next define the (inverse) foliation density

$$\mu^{-1} = \partial_t u(t, r) \,, \tag{8.20}$$

which measures the density of the leaves C_u with respect to the Minkowskian time-coordinate. Now, the change of variables (8.18) remains (C^1)-regular as long as $\mu > 0$ (why?). Note that initially $\mu^{-1} = \partial_t u(0, r) = -\sqrt{1+\psi}\partial_r u(0, r) = \sqrt{1+\psi} \approx 1$.

The idea now is to write the wave equation as a transport equation along the characteristics. We compute

$$L\left(\frac{\underline{L}(\psi r)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left((\underline{L}\psi)^2 - L\psi\underline{L}\psi\right) + \frac{1}{2}\frac{1}{(1+\psi)^{\frac{3}{2}}}L\psi\cdot\psi \,, \tag{8.21}$$

$$\underline{L}\left(\frac{L(\psi r)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left((L\psi)^2 - L\psi\underline{L}\psi\right) - \frac{1}{2}\frac{1}{(1+\psi)^{\frac{3}{2}}}\underline{L}\psi\cdot\psi \,. \tag{8.22}$$

We can modify this system further as follows. We first compute a propagation equation for the foliation density μ as follows:

$$L\mu^{-1} = L\partial_t u = \partial_t Lu(t, r) - [\partial_t, L]u(t, r) = -\frac{1}{2\sqrt{1+\psi}}\partial_t\psi\partial_r u = \frac{1}{4}\frac{1}{1+\psi}\mu^{-1}\left(L\psi + \underline{L}\psi\right) \,,$$

where we have used $Lu(t, r) = \partial_t u(t, r) + \sqrt{1+\psi}\partial_r u(t, r) = 0$ by definition in the last two steps. In other words,

$$L\mu = -\frac{1}{4}\frac{1}{1+\psi}\mu\left(L\psi + \underline{L}\psi\right) \tag{8.23}$$

We can now rewrite (8.21)–(8.22) as

$$L\left(\mu\frac{\underline{L}(\psi r)}{1+\psi}\right) = \frac{1}{2}\frac{r}{(1+\psi)^2}\left((-L\psi\mu\underline{L}\psi\right) + \frac{3}{4}\frac{1}{(1+\psi)^{\frac{3}{2}}}\mu L\psi\cdot\psi + \frac{1}{4}\frac{1}{(1+\psi)^{\frac{3}{2}}}\mu\underline{L}\psi\cdot\psi \,, \tag{8.24}$$

$$\mu\underline{L}\left(\frac{L(\psi r)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left(\mu(L\psi)^2 - L\psi\mu\underline{L}\psi\right) - \frac{1}{2}\frac{1}{(1+\psi)^{\frac{3}{2}}}\mu\underline{L}\psi\cdot\psi \,. \tag{8.25}$$

[2] In a semilinear problem, the above is of course solved by $u = t - r + 1$.

Note that the term $r|\underline{L}\psi|^2$ (which is the slowest decaying term and which we believe to be driving the blow-up) has disappeared from the first equation by renormalising with μ! This suggests that we should be able to prove that the renormalised weighted derivatives $r\mu\underline{L}\psi$ and $r^2 L\psi$ and $r\psi$ will remain small for as long as the solution exists!

Step 1: Show smallness of renormalised quantities for as long as solution exists. Let us define

$$T_{max,u_0} = \sup_{t>0}\{\ \psi \text{ is a } C^2\text{-solution in the strip } \mathcal{M}_{t,u_0}\}.$$

Proposition 8.2.4. *There exists an $\epsilon_0 > 0$ such that the following estimates hold in $\mathcal{M}_{T_{max},u_0}$ for all $\epsilon < \epsilon_0$.*

(1) $|r^2 L\psi| \leq C\epsilon$

(2) $|r\mu\underline{L}\psi| \leq C\epsilon$

(3) $|r\psi| \leq C\epsilon$

(4) $|\mu - 1| \leq C\epsilon \ln(e+t)$

(5) $|1 - r + t - u| \leq C\epsilon \ln(e+t)$.

Here C is an explicitly computable constant depending only on C_{init}.

Proof. The proof is a bootstrap argument. We define $\mathcal{B} \subset [0, T_{max,u_0})$ to be the set of all t' such that the estimates (1)–(5) hold in $\mathcal{M}_{T_{max},u_0} \cap \{t \leq t'\}$. The set is clearly non-empty and closed and we will show that it is also open by improving the estimates (1)–(5).

To achieve this we will integrate the equations (8.24)–(8.25) in $\mathcal{M}_{T_{max},u_0} \cap \{t \leq t'\}$ along the characteristics, i.e. the integral curves of L and $\underline{L}$, which in the new coordinates are given by

$$L = \partial_{t^\star} \quad \text{and} \quad \mu\underline{L} = 2\partial_u + 2\mu\partial_{t^\star}\,. \tag{8.26}$$

Note in this context that we can parametrise a past directed integral curve of $\mu\underline{L}$ emanating from (u_f, t_f) by $\gamma(u) = \big(u, \hat{t}(u)\big)$, where $\hat{t}(u)$ is defined by solving $\hat{t}'(u) = \frac{1}{2}\mu\left(\hat{t}(u), u\right)$ with initial condition $\hat{t}(u_f) = t_f$. This curve intersects the characteristic $u = 0$ in $(u = 0, t)$ and we have $t_f - t = \int_{u=0}^{u_f} \hat{t}'(u)du$ and hence $|t_f - t| \leq \frac{1}{2} + C\epsilon \ln(e + t_f)$, which means in particular $\frac{1}{t} \leq \frac{1}{t_f - |t_f - t|} \leq \frac{2}{t_f}$ for $t_f \geq 2$. In other words, we have $t \sim t_f$ along the integral curves of $\underline{L}$. (Hence we can take out $\frac{1}{t_f+1}$ weights of the integral along the integral curves of $\mu\underline{L}$.) As far as the integration itself is concerned, we have

$$\int_\gamma f(\gamma)d\gamma := \int_0^{u_f} f(\gamma(u))|\gamma'(u)|du \leq \int_0^{u_f} f(\gamma(u))\sqrt{1 + \frac{1}{4}\mu^2(\gamma(u))}du\,,$$

which we will typically estimate by the u-lengths of the integral (≤ 1) times the maximum of the integrand (where we again use the fact that $t \sim t_f$ in the region of integration and that $|\mu| \leq 2\log(e + t)$.

Integrating (8.24) we infer

$$\left|\frac{\mu\underline{L}(r\psi)}{1+\psi}(t,u)\right| \leq C_{init}\epsilon + \int_0^t \frac{C^2\epsilon^2}{(1+\bar{t})^2}d\bar{t} + \int_0^t \frac{C^2\epsilon^2\left(\ln(e+\bar{t})\right)}{(1+\bar{t})^3}d\bar{t} \leq 2C_{init}\epsilon \tag{8.27}$$

for sufficiently small ϵ. Integrating (8.25) and using the remarks above, we infer (note that the data is trivial for large $t \geq 3$).

$$\left|\frac{L(\psi r)}{1+\psi}(t,u)\right| \leq \frac{C_{init}\epsilon}{(1+t)^2} + \frac{C^2\epsilon^2 \ln(e+t)}{(1+t)^2} \leq 2\frac{C_{init}\epsilon}{(1+t)^{3/2}}\,. \tag{8.28}$$

Integrating forwards from data $\psi r(t,u) = \psi r(0,u) + \int_0^t L(\psi r)d\tilde{t}$ and using the bound above we deduce

$$|\psi r| \leq 4C_{init} \tag{8.29}$$

which improves (3). Using this improved bound with (8.27) and (8.28) we easily improve (1) and (2). To improve (4) we integrate (8.23) and use that the right hand side has been improved to deduce

$$|\mu(t,u)-1| \leq |\mu(t,u)-\mu(0,u)| + |\mu(0,u)-1| \leq \int_0^t \frac{10C_{init}\epsilon}{1+\bar{t}} d\bar{t} + C_{init}\epsilon$$

which after integration improves (4). Finally, to improve (5) integrate $L(1-r+t-u)$ from 0 to t observing that in view of (8.29)

$$|L\,(1-r+t-u)\,| = |-\sqrt{1+\psi}+1| \leq \frac{2C_{init}\epsilon}{1+t}$$

□

Remark 8.2.5. *One can similarly bootstrap that the estimate*

$$|r^3 L^2 \Psi| \leq C\epsilon \tag{8.30}$$

must hold in $\mathcal{M}_{T_{max},u_0}$. See Sheet 11.

We now know that for as long as the solution exists, the (renormalised) quantities appearing in (1)–(5) remain small in the strip. The second step is to use these estimates to prove that $\mu \to 0$ in finite time.

Step 2. Show that $\mu \to 0$ in finite time along one of the characteristics.

From Step 1 (in particular the estimates $(1)-(5)$) and our breakdown criteria, we know that the solution can only blow up if $|\underline{L}\psi| \to \infty$, which is only consistent with (2) if $\mu \to 0$ at the same time. We now show that this breakdown mechanism is necessarily in operation for $\epsilon < \epsilon_0$ sufficiently small: Assuming the solution exists for a sufficiently long time, we prove that (a) $\mu \to 0$ in finite time along at least one of the characteristics and (b) that the Minkowskian derivative $\underline{L}\psi$ (i.e. the one transversal to the outgoing cones) blows up along this characteristic.The following Lemma proves this with an additional assumption on the data which we shall remove in Step 3.

Lemma 8.2.6. *Assume that*

$$\max_{u\in[0,\frac{3}{4}]} \frac{\mu\underline{L}(\psi r)}{1+\psi}(t=0,u) \geq d\epsilon \quad \textit{for some } d>0 \textit{ and all } \epsilon \textit{ sufficiently small.}$$

Then there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ one has $\mu \to 0^+$ and $|\underline{L}\psi| \to \infty$ in finite time along at least one characteristic.

Proof. We give a sketch of the argument and provide the rigorous proof on Example Sheet 11. We expect that for large times (8.23) becomes[3]

$$L\mu = -\frac{1}{4}\frac{1}{1+t}\frac{\mu\underline{L}(\psi r)}{1+\psi} + \text{small integrable error}\,, \tag{8.31}$$

as the $\mu L\psi$ term decays much faster along the strip than $\mu\underline{L}\psi$. The idea now is to prove a lower bound for the right hand side along the characteristic $u = u_\star$ where the maximum in the assumption of the lemma is achieved. Indeed, using the assumption of the Lemma and the propagation equation (8.24) we deduce that

$$\frac{\mu\underline{L}(\psi r)}{1+\psi} \geq \frac{d}{2}\epsilon \tag{8.32}$$

holds along $u = u_\star$. From (8.31) we infer

$$L\mu \leq -\frac{1}{4}\frac{d}{2}\frac{\epsilon}{1+t}$$

and

$$\mu \leq \mu_{data} - \frac{d}{8}\epsilon \cdot \log(1+t)\,,$$

which implies that $\mu \to 0$ in finite time as $\mu_{data} \approx 1$. In view of the bound (8.32) we deduce also that the Minkowskian derivative $\underline{L}\psi$ has to blow up along this characteristic. □

[3]Of course, for small times $L\psi$ and $\underline{L}\psi$ are comparable so one needs to be more careful here. See Sheet 11.

Step 3: Deducing the assumption in Lemma 8.2.6 from compact support of the data.
We first prove two simple lemmata:

Lemma 8.2.7. *Suppose the odd functions $\tilde{f}$ and $\tilde{g}$ in (8.12) are compactly supported in $[-R,R]$ for some $R>0$. Then*

$$\max_{[-R,R]}\left[\tilde{g}(r)-\partial_r\tilde{f}(r)\right]=\frac{3}{2}d>0 \quad \textit{for some } d>0.$$

Proof. Define $h=\tilde{g}-\partial_r\tilde{f}$ and note $\int_{-R}^{R}h(r)dr=0$ since $\tilde{g}$ is odd and $\tilde{f}$ vanishes at $r=\pm R$. It follows that either $h>0$ somewhere in $[-R,R]$ or h vanishes identically. In the former case, we are done, while in the latter we have $\tilde{g}=\partial_r\tilde{f}$, which implies that $\tilde{f}$ is even. Since $\tilde{f}$ is also odd it follows that $\tilde{f}=0$ identically and hence also $\tilde{g}=0$ identically, which contradicts the fact that $\tilde{g}$ and $\tilde{g}$ are assumed to be non-trivial. □

Lemma 8.2.8. *For the Cauchy problem (8.12) with $\tilde{f}$, $\tilde{g}$ compactly supported in $[-R,R]$, there exists $\epsilon_0>0$ such that for all $\epsilon<\epsilon_0$ we have*

$$\max_{[-R,R]}\left.\frac{\mu\underline{L}(\psi r)}{1+\psi}\right|_{t=0}\geq d\cdot\epsilon \tag{8.33}$$

Proof. Note that the left hand side can be written as

$$\max_{[-R,R]}\left[\epsilon\left(\tilde{g}-\partial_r\tilde{f}\right)+\mathcal{O}(\epsilon^2)\right]$$

and use Lemma 8.2.7. □

With the help of the lemmata above, we can show that the assumption in Lemma 8.2.6 follows from the assumption of compact support of the data. To see this, assume without loss of generality that compactly supported data for (8.12) are prescribed at $t=-\frac{3}{4}$ and that the data is supported in $[-\frac{1}{4},\frac{1}{4}]$. Then, by the previous lemmata there exists an $\epsilon_0>0$ such that for all $\epsilon<\epsilon_0$ we have

$$\max_{[-\frac{1}{4},\frac{1}{4}]}\left.\frac{\mu\underline{L}(\psi r)}{1+\psi}\right|_{t=-\frac{3}{4}}\geq d\epsilon \quad \text{for some } d>0. \tag{8.34}$$

Now use domain of dependence and integration along characteristics to deduce that at $t=0$ the solution is supported in $[-1,1]$ and there exists a point on the interval $\{0\}\times[\frac{1}{4},1]$ where

$$\max_{[-1,1]}\left.\frac{\mu\underline{L}(\psi r)}{1+\psi}\right|_{t=0}\geq\frac{3d}{4}\epsilon\,.$$

This is the setting for which we have proven shock formation in Steps 1 and 2.

8.2.3 Outlook: The non-symmetric case

John, Alinhac, Christodoulou, some latest developments....

Main difficulties: (1) Characteristic directions are not know explicitly in terms of the solution but have to be determined by solving the eikonal equation $\left(g^{-1}\right)^{\mu\nu}(\psi)\partial_\mu u\partial_\nu u=0$. (2) Integration along characteristics loses derivatives so μ-weighted L^2-estimates need to be employed.

Chapter 9

Proof of the Strichartz estimate

9.1 Littlewood Paley theory on $\mathbb{R}^n$

9.1.1 The definitions

Let $\phi(\xi)$ be a radial bump function with $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$.

Let $\psi(\xi) = \phi(\xi) - \phi(2\xi)$. So ψ is a bump function supported in the annulus $\{1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{k\in\mathbb{Z}} \psi\left(\frac{\xi}{2^k}\right) = 1 \quad \text{for all } \xi \neq 0 \qquad \text{(partition of unity)}.$$

To see this note that for any ξ there is only a finite number of non-zero summands and that

$$\sum_{-M\leq k\leq N} \psi\left(\frac{\xi}{2^k}\right) = \left[\phi\left(\frac{\xi}{2^N}\right) - \phi\left(\frac{\xi}{2^{-M-1}}\right)\right]. \tag{9.1}$$

Now for any $\xi \neq 0$ there exist a $M', N' \in \mathbb{N}$ such that $2^{-M} \leq |\xi| \leq 2^N$ holds for all $M \geq M'$ and $N \geq N'$. For all such such M, N the right hand side equals 1 and the result follows.

The Littlewood-Paley (LP) projections $P_k f$ and $P_{\leq k} f$ are defined by

$$\widehat{P_k f}\,(\xi) = \psi\left(\frac{\xi}{2^k}\right)\hat{f}(\xi) \quad \text{and} \quad \widehat{P_{\leq k} f}\,(\xi) = \phi\left(\frac{\xi}{2^k}\right)\hat{f}(\xi) \tag{9.2}$$

the latter projecting the function to frequencies of size $\leq 2^k$, the former projecting to frequencies in a dyadic annulus of size 2^k. In physical space we have

$$P_k f = m_k \star f \quad \text{and} \quad P_{\leq k} f = \overline{m}_k \star f$$

with $m_k(x) = 2^{nk} m(2^k x)$ and $m(x)$ the inverse Fourier transform of $\psi(\xi)$ (and similarly for $\overline{m}$ with ϕ replacing ψ. This can be checked from the behaviour of the Fourier transform under scaling and the fact that the Fourier transform of a product in Fourier space is the convolution in physical space. Note also that

$$\|m_k\|_{L^1} = \|m\|_{L^1} \quad \text{and} \quad \|\overline{m}_k\|_{L^1} = \|\overline{m}\|_{L^1} \tag{9.3}$$

as follows from a simple change of variables.

9.1.2 Basic Properties

Lemma 9.1.1. *If $f \in L^2$ we have $f = \sum_{k\in\mathbb{Z}} P_k f$ with the convergence being in L^2.*

Proof. Use Parseval and (9.1) to write $\|f - \sum_{-M\le k\le N} P_k\|_{L^2} =$

$$\Big\|\hat f - \sum_{-M\le k\le N} \widehat{P_k f}(\xi)\Big\|_{L^2} = \Big\|\hat f - \sum_{-M\le k\le N} \psi\left(\frac{\xi}{2^k}\right)\hat f(\xi)\Big\|_{L^2} = \Big\|\hat f(\xi)\left[1-\phi\left(\frac{\xi}{2^N}\right)+\phi\left(\frac{\xi}{2^{-M-1}}\right)\right]\Big\|_{L^2}$$

$$\le \int_{|\xi|\ge 2^N} |\hat f(\xi)|^2 d\xi + \int_{|\xi|\le 2^{-M}} |\hat f(\xi)|^2 d\xi\,. \tag{9.4}$$

The right hand side goes to zero as $M, N \to \infty$. □

Lemma 9.1.2. *The P_k are selfadjoint and $P_{k_1}P_{k_2} = 0$ for $|k_1 - k_2| \ge 2$ and moreover for $f \in L^2$*

$$\|f\|_{L^2}^2 \approx \sum_{k\in\mathbb{Z}} \|P_k f\|_{L^2}^2\,.$$

Proof. Self-adjointness follows from Parseval and the fact that in Fourier space the operator P_k is just multiplication by a cut-off:

$$\langle P_k f, g\rangle = \langle \widehat{P_k f}, \hat g\rangle = \langle \psi\left(\frac{\cdot}{2^k}\right)\hat f, \hat g\rangle = \langle \hat f, \psi\left(\frac{\cdot}{2^k}\right)\hat g\rangle = \langle f, P_k g\rangle\,.$$

Showing $P_{k_1}P_{k_2} = 0$ for $|k_1 - k_2| \ge 2$ is an easy exercise (use that the supports of the associated cut-offs in frequency space are disjoint). Finally, we have

$$\sum_{k\in\mathbb{Z}} \|P_k f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}} \|\widehat{P_k f}\|_{L^2}^2 \le \sum_{k\in\mathbb{Z}} \|\psi\left(\frac{\xi}{2^k}\right)\hat f(\xi)\|_{L^2}^2 = \sum_{k\in\mathbb{Z}} \int_{\{2^{k-1}\le|\xi|\le 2^{k+1}\}} |\hat f|^2 d\xi \le 3\|\hat f\|_{L^2}^2 = 3\|f\|_{L^2}^2. \tag{9.5}$$

We have

$$\|f\|_{L^2}^2 = \sum_k\sum_{k'} \langle P_k f, P_{k'} f\rangle = \sum_{|k-k'|\le 1} \langle f, P_k P_{k'} f\rangle = \sum_{|k-k'|\le 1} \langle P_k f, P_{k'} f\rangle \le 3\|P_k\|_{L^2}^2\,. \tag{9.6}$$

□

Lemma 9.1.3. *We have for $1 \le p \le \infty$ and any k*

$$\|P_{\le k} f\|_{L^p} \le C\|f\|_{L^p}\,.$$

We have for $1 \le p \le q \le \infty$ the Bernstein inequality

$$\|P_k f\|_{L^q} \le C 2^{kn(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}\,.$$

Proof. For the first estimate we note that by the definition of $P_{\le k}$ (9.2) and Young's inequality (Theorem 11.1.5) we have

$$\|P_{\le k} f\|_{L^p} = \|\overline{m}_k \star f\|_{L^p} \le \|\overline{m}_k\|_{L^1}\|f\|_{L^p} = \|\overline{m}\|_{L^1}\|f\|_{L^p}$$

with the last step following from (9.3). Now $\overline{m}$ being the (inverse) Fourier transform of a compactly supported smooth function we easily see $\|\overline{m}\|_{L^1} \le C$ and the claim follows.

For the second estimate we repeat the previous argument using Young's inequality

$$\|P_k f\|_{L^q} = \|m_k \star f\|_{L^q} \le \|m_k\|_{L^r}\|f\|_{L^p} \quad \text{for } \frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r}.$$

We now observe

$$\|m_k\|_{L^r} = 2^{nk}\left(\int_{\mathbb{R}^n} |m(2^k x)|^r dx\right)^{\frac{1}{r}} = 2^{nk\left(1-\frac{1}{r}\right)}\left(\int_{\mathbb{R}^n} |m(x)|^r dx\right)^{\frac{1}{r}} = 2^{nk\left(1-\frac{1}{r}\right)}\|m\|_{L^r}$$

and again $\|m\|_{L^r} \le C$ independently of m. Combining the inequalities yields the second estimate. □

Corollary 9.1.4. *We have for $1 \le p \le q \le \infty$ the inequality $\|P_0 f\|_{L^q} \le C\|f\|_{L^p}$.*

Lemma 9.1.5. *We have the inequalities*

$$\|\partial P_k f\|_{L^p} \le C \cdot 2^k \|f\|_{L^p} \quad \textit{and} \quad 2^k \|P_k f\|_{L^p} \le \|\partial f\|_{L^p} . \tag{9.7}$$

Proof. For the first estimate note that

$$\partial_{x^j} P_k f = \partial_{x^j} \int 2^{nk} m\left(2^k(x-y)\right) f(y)dy = 2^k \int 2^{nk} (\partial_{x^j} m)\left(2^k(x-y)\right) f(y)dy$$

and hence defining $(\partial_{x^j} m)_k = 2^{nk}(\partial_{x^j} m)(2^k x)$ we can estimate

$$\|\partial_{x^j} P_k\|_{L^p} = 2^k \|(\partial_{x^j} m)_k \star f\|_{L^p} \le C \cdot 2^k \|(\partial_{x^j} m)_k\|_{L^1} \|f\|_{L^p} = C \cdot 2^k \|\partial_{x^j} m\|_{L^1} \|f\|_{L^p} .$$

The result follows from $\|\partial_{x^j} m\|_{L^1} \le C$.

For the second estimate we first note that for $\xi \neq 0$ we can write $\hat{f}(\xi) = \sum_{j=1}^n \frac{\xi_j}{i|\xi|^2} \widehat{\partial_{x^j} f}$. Hence

$$2^k \widehat{P_k f} = \sum_{j=1}^n 2^k \psi\left(\frac{\xi}{2^k}\right) \frac{\xi_j}{i|\xi|^2} \widehat{\partial_{x^j} f} = \sum_{j=1}^n \chi_j\left(\frac{\xi}{2^k}\right) \cdot \widehat{\partial_{x^j} f}$$

where we have defined $\chi_j(\xi) = \psi(\xi) \frac{\xi_j}{i|\xi|^2}$. Let h_j denote the inverse Fourier transform of χ_j. Then

$$\widehat{\chi_j\left(\frac{\cdot}{2^k}\right)}(x) = 2^{nk} h_j\left(2^k \cdot x\right) =: (h_j)_k$$

and hence

$$2^k P_k f = \sum_{j=1}^n (h_j)_k \star (\partial_{x^j} f) .$$

Applying as usual Young's inequality (Theorem 11.1.5) and using $\|(h_j)_k\|_{L^1} = \|h_j\|_{L^1} \le C$ for all j we obtain the result since $\|\partial_{x^j} f\|_{L^p} \le C\|\partial f\|_{L^p}$. □

9.1.3 The Littlewood Paley inequality

We consider the operator

$$Sf(x) := \left(\sum_{k \in \mathbb{Z}} |P_k f(x)|^2\right)^{\frac{1}{2}} .$$

Theorem 9.1.6. *For $1 < p < \infty$ and $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\|f\|_{L^p} \le C\|Sf\|_{L^p} \le C\|f\|_{L^p} .$$

Remark 9.1.7. *The theorem fails for $p = 1$ and $p = \infty$. For $p = 2$ it follows directly from Lemma 9.1.2. For later applications in the Strichartz estimate we will only need the estimate $\|Sf\|_{L^p} \le C\|f\|_{L^p}$ for $2 \le p < \infty$, which in turn will follow by duality from $\|f\|_{L^p} \le C\|Sf\|_{L^p}$ for $1 < p \le 2$. We will therefore focus on proving these two estimates in detail. However, the proof is postponed to Section 11.2*

9.2 The homogeneous Strichartz estimate

We consider the solution of the Cauchy-Problem in $\mathbb{R}^{1+n}$ for $n \ge 2$:

$$\Box u = 0 \quad , \quad u(0,x) = f(x) \;\; , \;\; \partial_t u(0,x) = g(x) . \tag{9.8}$$

We can assume that f, g are Schwartz and hence ψ is smooth. As usual, once certain estimates are proven on the solution we can argue by density to obtain statements in lower regularity.

Theorem 9.2.1. *For any $f, g \in \mathcal{S}$, the solution of (9.8) satisfies the estimate*

$$\|u\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}\,, \tag{9.9}$$

where $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$ provided (q, r) is wave admissible, i.e.

$$2 \leq q \leq \infty \quad , \quad 2 \leq r < \infty \quad , \quad \frac{2}{q} \leq \frac{n-1}{2}\left(1 - \frac{2}{r}\right). \tag{9.10}$$

Proof. The proof is contained in the following subsections. We will prove the estimate excluding the so-called endpoint cases $1 = \frac{2}{q} = \frac{n-1}{2}\left(1 - \frac{2}{r}\right)$. These cases are proven in Keel–Tao (1998). ☐

Remark 9.2.2. *One of the main points of the estimate is its low regularity on the data. One may prove relatively easily that the left hand side of (9.9) can be estimated by higher norms on the data (e.g. using vectorfields).*

9.2.1 Step 1: Reduction of the problem to bounded frequency

Note that $\Box$ and P_k commute and hence that each piece $P_k u$ of the solution u of (9.8) satisfies

$$\Box P_k u = 0 \quad , \quad P_k u(0, x) = P_k f(x) \;\; , \;\; \partial_t P_k u(0, x) = P_k g(x)\,. \tag{9.11}$$

The rough idea now is to estimate each piece separately and then add them up. In that regard the next proposition establishes that (using the Littlewood Paley inequality of Theorem 9.1.6 and elementary scaling arguments) to prove Theorem 9.2.1 it actually suffices to estimate solutions with frequency supported in a band $\frac{1}{2} \leq |\xi| \leq 2$ (in particular away from zero and bounded):

Proposition 9.2.3. *Suppose that for any $f, g \in \mathcal{S}$ the solution u of (9.8) satisfies the estimate*

$$\|P_0 u\|_{L^q_t L^r_x} \lesssim \|P_0 f\|_{L^2_x} + \|P_0 g\|_{L^2_x}\,. \tag{9.12}$$

Then the estimate (9.9) holds.

Proof. Given u a solution to (9.8) we define for fixed $k \in \mathbb{Z}$:

$$u_k(t, x) := u(2^{-k}t, 2^{-k}x) \quad , \quad f_k(x) := f(2^{-k}x) \quad , \quad g_k(x) := 2^{-k}g(2^{-k}x)\,. \tag{9.13}$$

By the scale invariance of $\Box$, we see that u_k satisfies $\Box u_k = 0$ with data $u_k(0, 2^{-k}x) = f_k$, $\partial_t u_k(0, 2^{-k}x) = g_k$. In particular u_k is a solution of (9.8) and applying (9.12) we deduce

$$\|P_0 u_k\|_{L^q_t L^r_x} \lesssim \|P_0 f_k\|_{L^2_x} + \|P_0 g_k\|_{L^2_x}\,. \tag{9.14}$$

Now a simple calculation (write down the definition of $\|P_k u\|_{L^q_t L^r_x}$ and apply the change of variables $t = 2^{-k}t'$, $x = 2^{-k}x'$ (and $y = 2^{-k}y'$ in the convolution integral)) yields

$$\|P_0 u_k\|_{L^q_t L^r_x} = 2^{(\frac{n}{r}+\frac{1}{q})k}\|P_k u\|_{L^q_t L^r_x} \quad , \quad \|P_0 f_k\|_{L^2_x} = 2^{\frac{nk}{2}}\|P_k f\|_{L^2_x} \quad , \quad \|P_0 g_k\|_{L^2_x} = 2^{(\frac{n}{2}-1)k}\|P_k f\|_{L^2_x}\,. \tag{9.15}$$

Combining (9.14) and (9.15) yields that

$$\|P_k u\|_{L^q_t L^r_x} \lesssim 2^{sk}\|P_k f\|_{L^2_x} + 2^{(s-1)k}\|P_k g\|_{L^2_x} \quad \text{holds for all } k \in \mathbb{Z} \text{ and } s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}. \tag{9.16}$$

Now we have

$$\begin{aligned} \|u\|_{L^q_t L^r_x} \lesssim \left\| \left(\sum_{k\in\mathbb{Z}} |P_k u|^2\right)^{\frac{1}{2}} \right\|_{L^q_t L^r_x} \lesssim \left(\sum_{k\in\mathbb{Z}} \|P_k u\|^2_{L^q_t L^r_x}\right)^{\frac{1}{2}} &\lesssim \left(\sum_{k\in\mathbb{Z}} \left(2^{2sk}\|P_k f\|^2_{L^2_x} + 2^{2(s-1)k}\|P_k g\|^2_{L^2_x}\right)\right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}\,, \end{aligned} \tag{9.17}$$

where the first estimate follows from Theorem 9.1.6 (note that since $r \geq 2$ we are – as promised in Remark 9.1.7 – only using $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$ for $2 \leq p < \infty$), the second estimate is Minkowski inequality,[1] the third estimate is inserting (9.16) and the last estimate follows from the definition of the homogeneous Sobolev space. □

9.2.2 Step 2: Reduction to operator boundedness

To prove (9.12) for solutions u of (9.8), we consider the Cauchy problem (9.8) with data $u(0,x) = P_0 f(x)$ and $\partial_t u(0,x) = P_0 g(x)$. Let $\hat{u}(t,\xi)$ denote the Fourier transform of $x \to u(t,x)$. We have $\partial_t^2 \hat{u} + |\xi|^2 \hat{u} = 0$ with data $\hat{u}(0,\xi) = \hat{f}\psi$ and $\partial_t \hat{u}(0,\xi) = \hat{g}\psi$ which yields the solution

$$\hat{u}(t,\xi) = \psi(\xi)\hat{f}(\xi)\frac{e^{it|\xi|} + e^{-it|\xi|}}{2} + \frac{\psi(\xi)\hat{g}(\xi)}{|\xi|}\frac{e^{it|\xi|} - e^{-it|\xi|}}{2i}\,. \tag{9.18}$$

Note that ψ (and hence the solution) is supported in $\frac{1}{2} \leq |\xi| \leq 2$. We next define the propagator $e^{\pm it\sqrt{-\Delta}}$ by its action in Fourier space through

$$\widehat{e^{\pm it\sqrt{-\Delta}} f}(\xi) = e^{\pm it|\xi|}\hat{f}(\xi)\,.$$

Since both $\psi\hat{f}$ and $\psi\frac{\hat{g}}{\xi}$ are in L^2 we conclude

Proposition 9.2.4. *Suppose we can establish for any $f \in L^2$ (or by density $f \in \mathcal{S}$) the estimate*

$$\|P_0 e^{it\sqrt{-\Delta}} f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x}\,. \tag{9.19}$$

Then the estimate (9.12) holds.

9.2.3 Step 3: The $TT^\star$ argument

We now define the operator

$$T : L^{q'}_t L^{r'}_x \supset C^\infty_0(\mathbb{R}_t \times \mathbb{R}^n_x) \to L^2_x \tag{9.20}$$

by

$$TF(x) = \int_{\mathbb{R}} dt \int_{\mathbb{R}^n} d\xi e^{i(x\cdot\xi - t|\xi|)}\bar{\psi}(\xi)\hat{F}(t,\xi) = \int_{\mathbb{R}} dt \mathcal{F}^{-1}\left[e^{-it|\xi|}\bar{\psi}(\xi)\hat{F}(t,\xi)\right] \tag{9.21}$$

for $F \in C^\infty_0(\mathbb{R}_t \times \mathbb{R}^n_x)$. Since $\hat{F}(t,\xi)$ is still compactly supported in t and smooth in ξ one easily sees that TF is indeed in L^2_x (and in fact Schwartz).[2] The reason for defining T in this way is that its adjoint is the operator we are interested in. Indeed, from the definition we have $T^\star : L^2_x \to (C^\infty_0(\mathbb{R}_t \times \mathbb{R}^n_x))^\star$ with

$$\begin{aligned} T^\star f[F] = \langle f, TF\rangle &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}} dt \int_{\mathbb{R}^n} d\xi e^{-i(x\cdot\xi - t|\xi|)}\psi(\xi)\overline{\hat{F}(t,\xi)}f(x) \\ &= \int_{\mathbb{R}} dt \int_{\mathbb{R}^n} d\xi e^{it|\xi|}\psi(\xi)\hat{f}(\xi)\overline{\hat{F}(t,\xi)} \end{aligned} \tag{9.22}$$

[1] Note that since $q, r \geq 2$ we have

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |P_k u|^2\Big)^{\frac{1}{2}} \Big\|_{L^q_t L^r_x} = \Big(\Big\|\sum_{k\in\mathbb{Z}} |P_k u|^2\Big\|_{L^{\frac{q}{2}}_t L^{\frac{r}{2}}_x}\Big)^{\frac{1}{2}} \lesssim \Big(\sum_{k\in\mathbb{Z}} \Big\||P_k u|^2\Big\|_{L^{\frac{q}{2}}_t L^{\frac{r}{2}}_x}\Big)^{\frac{1}{2}} = \Big(\sum_{k\in\mathbb{Z}} \|P_k u\|^2_{L^q_t L^r_x}\Big)^{\frac{1}{2}}\,.$$

[2] Use the identity $\sum_{j=1}^n \frac{-ix_j}{|x|^2}\partial_{\xi^j} e^{ix\xi} = e^{ix\xi}$ and integrate by parts to change the integrand into something compactly supported in t and ξ and decaying like $|x|^{-N}$ for sufficiently large N.

where the last equality is true if $f \in L^1 \cap L^2$ by Fubini's theorem. So we have for $f \in L^1 \cap L^2$ the formula

$$T^\star f(t,x) = \int_{\mathbb{R}^n} d\xi e^{ix\xi} e^{it|\xi|}\psi(\xi)\hat{f}(\xi) = \mathcal{F}^{-1}\left[e^{it|\xi|}\psi(\xi)\hat{f}(\xi)\right] , \tag{9.23}$$

which is precisely the operator (9.19) whoses boundedness we need to establish. We compute

$$\begin{aligned}(T^\star T\, F)(t,x) &= \int_{\mathbb{R}} dt' \mathcal{F}^{-1}\left[e^{i(t-t')|\xi|}|\psi(\xi)|^2 \hat{F}(t,\xi)\right] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \left(K(t-t',y) \star F(t',x-y)\right) dy dt' = \int_{\mathbb{R}} \left(K(t-t',\cdot) \star F(t',\cdot)\right)(x) dt' ,\end{aligned} \tag{9.24}$$

where we have defined the (smooth) kernel

$$K(s,x) := \int_{\mathbb{R}^n} e^{i(x\xi + s|\xi|)}|\psi(\xi)|^2 d\xi\,. \tag{9.25}$$

Lemma 9.2.5. *If we can show* $\|T^\star T F\|_{L^q_t L^r_x} \leq \|F\|_{L^{q'}_x L^{r'}_t}$. *Then* $\|T^\star f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x}$ *holds.*[3]

Proof. We have for $F \in C^\infty_0 (\mathbb{R}_t \times \mathbb{R}^n_x)$

$$\|TF\|^2_{L^2_x} = \langle TF, TF\rangle_{L^2} = |T^\star T F[F]| \leq \|T^\star T F\|_{L^q_t L^r_x} \|F\|_{L^{q'}_x L^{r'}_t} \leq \|F\|^2_{L^{q'}_t L^{r'}_x} , \tag{9.26}$$

hence T is bounded and extends by continuity to a map $\bar{T} : L^{q'}_t L^{r'}_x \to L^2_x$. Its adjoint $\bar{T}^\star$ maps L^2_x to $L^q_t L^r_x$ and we easily see $\bar{T}^\star = T^\star$. We finally conclude $\bar{T}^\star$ is bounded in view of (write $\|F\| = \|F\|_{L^{q'}_t L^{r'}_x}$ in the formula below)

$$\|\bar{T}^\star f\| = \sup_{\|F\|=1} |\bar{T}^\star f[F]| = \sup_{\|F\|=1} \langle f, \bar{T} F\rangle_{L^2} \leq \sup_{\|F\|=1} \|f\|_{L^2_x} \|\bar{T} F\|_{L^2} \leq \|f\|_{L^2_x},$$

with the last step following from (9.26). ∎

9.2.4 Step 4: Proving boundedness of $T^\star T$

We claim

Proposition 9.2.6. *We have* $\|T^\star T F\|_{L^q_t L^r_x} \leq \|F\|_{L^{q'}_x L^{r'}_t}$ *and hence by Lemma 9.2.5,* $\|T^\star g\|_{L^q_t L^r_x} \lesssim \|g\|_{L^2_x}$.

Proof. Recall the form of the operator $T^\star T$ given by (9.24). In the next lemma, we first establish estimates in L^r_x on the kernel (with the help of an additional oscillation lemma). The proof will then conclude by taking the L^q_t norm of these estimates.

Lemma 9.2.7 (Spatial estimates for the kernel)**.** *For* $F \in C^\infty_0 (\mathbb{R}_t \times \mathbb{R}^n_x)$, *the kernel* $K(s,x)$ *satisfies the two estimates*

$$\begin{aligned}\|K(t-t',\cdot) \star F(t',\cdot)\|_{L^2_x} &\leq \|F(t',\cdot)\|_{L^2_x} && L^2 - L^2 \text{ estimate,} \\ \|K(t-t',\cdot) \star F(t',\cdot)\|_{L^\infty_x} &\leq \frac{C\|F(t',\cdot)\|_{L^1_x}}{(1+|t-t'|)^{\frac{n-1}{2}}} && L^\infty - L^1 \text{ estimate.}\end{aligned} \tag{9.27}$$

By interpolation we also have for $r \geq 2$ *and* $\gamma(r) = \frac{n-1}{2}\left(1 - \frac{2}{r}\right)$:

$$\|K(t-t',\cdot) \star F(t',\cdot)\|_{L^r_x} \leq \frac{C\|F(t',\cdot)\|_{L^{r'}_x}}{(1+|t-t'|)^{\gamma(r)}}\,. \tag{9.28}$$

[3] Hence (9.19), hence (9.12) and hence (9.9) holds.

Proof. The interpolation estimate follows directly from Theorem 11.1.9. The first estimate is also simple since

$$\|K(t-t',\cdot)\star F(t',\cdot)\|_{L^2_x} \lesssim \|\widehat{K}(t-t',\cdot)\hat{F}(t',\cdot)\|_{L^2_\xi} = \|e^{is|\xi|}\psi(\xi)^2\hat{F}(t',\xi)\|_{L^2_\xi} \leq \|\hat{F}(t',\xi)\|_{L^2_\xi} = \|F(t',x)\|_{L^2_x}.$$

It remains to prove the decay estimate, which is more complicated. Note that by Young's inequality (Theorem 11.1.5) it suffices to prove for all (t,x) the estimate

$$|K(t,x)| \leq \frac{C}{(1+|t|+|x|)^{\frac{n-1}{2}}}\,. \tag{9.29}$$

and this is what we will prove. In fact, we can assume $|t|+|x|\geq 1$ since K given by (9.25) is clearly uniformly bounded for all (t,x) and hence (9.29) holds for $|t|+|x|\leq 1$. We write the integral defining K in polar coordinates: $\rho=|\xi|$, $\xi=\rho\omega$, where ω is a vector on the unit-sphere:

$$K(t,x)=\int e^{i\rho(t+x\cdot\omega)}|\psi(\rho)|^2\rho^{n-1}d\rho d\sigma(\omega)\,. \tag{9.30}$$

If $|t|\geq 2|x|$ we have $|t+x\omega|\geq\frac{|t|}{2}$ and the usual integration by parts argument using $\frac{1}{i(t+x\cdot\omega)}\frac{d}{d\rho}e^{i\rho(t+x\omega)}=e^{i\rho(t+x\omega)}$ and the fact that the integrand is compactly supported in ρ gives (arbitrary inverse polynomial power) decay in t (and hence in x) establishing (9.29) for $|t|\geq 2|x|$. The argument breaks down if $|x|$ and $|t|$ are of a similar size. However, if $|x|\geq\frac{|t|}{2}$ is large, the decay of (9.29) an be inferred by exploiting the oscillations in the $n-1$ angular variables:

Lemma 9.2.8 (Auxiliary oscillation lemma)**.** *We have for $n\geq 2$*

$$\Big|\int_{S^{n-1}} e^{i\zeta\omega}d\sigma(\omega)\Big| \leq \frac{C}{(1+|\zeta|)^{\frac{n-1}{2}}}\,. \tag{9.31}$$

Applying Lemma 9.2.8 with $\zeta=\rho x$ and noting that $|\zeta|=\rho|x|>\frac{1}{2}|x|$ holds in the support of the integral (9.30) we easily establish $|K(t,x)|\leq(1+|x|)^{-\frac{n-1}{2}}$, which implies (9.29) also in the region $|x|\geq\frac{|t|}{2}$. □

Proof of Lemma 9.2.8. The proof requires the n-dimensional version of the spherical coordinates. Note that by rotational invariance we can assume $\zeta=(0,...,0,k)$. We can also assume $|k|\geq 1$ since the integral (9.31) is clearly uniformly bounded for all ζ. We then need to prove for $|k|\geq 1$

$$\Big|\int_0^\pi d\varphi_1\int_0^\pi d\varphi_2...\int_0^\pi d\varphi_{n-2}\int_0^{2\pi} d\varphi_{n-1}\; e^{ik\cos\varphi_1}\left(\sin\varphi_1\right)^{n-2}\left(\sin\varphi_2\right)^{n-3}...\left(\sin\varphi_{n-2}\right)^1\Big| \leq \frac{C}{|k|^{\frac{n-1}{2}}}\,.$$

After a change of variables $z=\cos\varphi_1$ this boils down to proving the one-dimensional estimate

$$\Big|\int_{-1}^1 dz\, e^{ikz}(1-z^2)^{\frac{n-3}{2}}\Big| \leq \frac{C}{|k|^{\frac{n-1}{2}}}\,.$$

We leave this to the exercises. Note that the $n=3$ case is immediate (and the case of n odd a straightforward induction). For even n one may do a variant of stationary phase (split the integral into $\int_0^{1-\frac{1}{k}}+\int_{1-\frac{1}{k}}^1$). □

We are ready to conclude the proof. We have by (9.24), Minkowski's inequality and the estimate (9.28)

$$\|T^\star TF\|_{L^q_tL^r_x}\leq C\Big\|\int_{\mathbb{R}}\frac{\|F(t',\cdot)\|_{L^{r'}_x}}{(1+|t-t'|)^{\gamma(r)}}dt'\Big\|_{L^q_t}\,. \tag{9.32}$$

Note that the integral is a convolution and we can hence apply Young's inequality.

$$\|T^\star TF\|_{L^q_tL^r_x}\leq C\|F\|_{L^{q'}_tL^{r'}_x}\Big\|\frac{1}{(1+|t|)^{\gamma(r)}}\Big\|_{L^{\tilde{q}}_t} \tag{9.33}$$

with $\frac{1}{q} = \frac{1}{q'} + \frac{1}{\tilde{q}} - 1 = -\frac{1}{q} + \frac{1}{\tilde{q}}$ hence $\tilde{q} = \frac{q}{2}$. The last term is bounded by a constant for $\frac{2}{q} < \gamma(r)$ and the desired estimate follows in that case. For the case of equality $\frac{2}{q} = \gamma(r)$ (and $q \neq 2$) we have to use a different argument based on the Hardy-Littlewood Sobolev inequality. Indeed, by duality we can rewrite (9.32)

$$\|T^\star TF\|_{L^q_t L^r_x} \leq C \sup_{\phi \in L^{q'}_t, \|\phi\|_{L^{q'}_t}=1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|F(t',\cdot)\|_{L^{r'}_x} \phi(t)}{(1+|t-t'|)^{\gamma(r)}} dt' dt \,. \tag{9.34}$$

Applying Corollary 11.1.7 with $n = 1$, $\alpha = \gamma(r) = \frac{2}{q} < 1$ yields the desired result since $\frac{1}{q'} + \frac{1}{q'} + \gamma(r) = 2$. □

9.3 The inhomogeneous Strichartz estimate

We consider the solution of the Cauchy-Problem in $\mathbb{R}^{1+n}$ for $n \geq 2$:

$$\Box u = F \quad , \quad u(0,x) = f(x) \ , \ \partial_t u(0,x) = g(x) \,. \tag{9.35}$$

As before we can assume that $f, g : \mathbb{R}^n \to \mathbb{R}$ are Schwartz and that $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is also Schwartz. By density we can infer statements in lower regularity from the estimates.

Theorem 9.3.1. *The solution of (9.8) satisfies the estimate*

$$\|u\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{L^{q'}_t L^{r'}_x} \,, \tag{9.36}$$

where $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$ provided (q, r) is wave admissible, i.e.

$$2 \leq q \leq \infty \quad , \quad 2 \leq r < \infty \quad , \quad \frac{2}{q} \leq \frac{n-1}{2}\left(1 - \frac{2}{r}\right) \tag{9.37}$$

and the scaling condition

$$\frac{n}{r} + \frac{1}{q} = \frac{n}{r'} + \frac{1}{q'} - 2 \tag{9.38}$$

holds.

Proof. We leave the proof as an exercise but sketch the main steps. By linearity and Theorem 9.2.1 it suffices to consider the case $f = g = 0$. It also suffices, just as in the proof of Theorem 9.2.1, to prove the estimate for F with frequency support in $\frac{1}{2} \leq |\xi| \leq 2$.[4] Assuming now that $F = P_0 F$ we recall the Duhamel representation of the solution in Fourier space

$$\hat{u}(t,\xi) = -\int_0^t \frac{\psi(\xi)\hat{F}(s,\xi)}{|\xi|} \frac{e^{i(t-s)|\xi|} - e^{-i(t-s)|\xi|}}{2i} ds \,. \tag{9.39}$$

Now one can rewrite this in physical space as

$$u(t,x) = \int_0^t ds (K(t-s,\cdot) \star F(s,\cdot))(x) \tag{9.40}$$

for a kernel which has the same structure as the one in the proof of Theorem 9.2.1. The estimate $\|P_0 u\|_{L^q_t L^r_x} \lesssim \|P_0 F\|_{L^{q'}_t L^{r'}_x}$ then follows directly from the estimate of Proposition 9.2.6. □

[4]To repeat the required scaling argument from the proof of Theorem 9.2.1 you will need to prove the estimate

$$\Big\| \|P_k F\|_{L^{q'}_t L^{r'}_x} \Big\|_{\ell^2} \leq \Big\| \|P_k F\|_{\ell^2} \Big\|_{L^{q'}_t L^{r'}_x} \quad \text{valid for } r' \leq 2,\ q' \leq 2.$$

It follows by applying Minkowski's inequality twice (starting by rewriting the left-hand side using the $\ell^2_{\frac{2}{q'}}$ norm).

Chapter 10

Applications to General Relativity

10.1 Wellposedness for the Einstein vacuum equations

10.2 Stability of Minkowski space

Chapter 11

Appendix: Analysis Background Material

11.1 Background on L^p spaces

11.1.1 Some L^p-inequalities

Theorem 11.1.1 (Hölder's inequality). *Let $1 \leq p \leq \infty$. For $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(\mathbb{R}^n)$ and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}\,.$$

Proof. (0) Assume $1 < p < \infty$, the $p = \infty$ and $p = 1$ cases being seen directly. (1) Show $A^\theta B^{1-\theta} \leq \theta A + (1-\theta) B$ for $A, B \geq 0$ and $0 \leq \theta \leq 1$. (2) Can assume $\|f\|_{L^p} \neq 0$ and $\|g\|_{L^q} \neq 0$. Replace f by $\frac{f}{\|f\|_{L^p}}$ and g by $\frac{g}{\|g\|_{L^q}}$ and show $\|fg\|_{L^1} \leq 1$ by setting $A = |f(x)|^p$, $B = |g(x)|^q$ and $\theta = 1/p$ in (1). □

Theorem 11.1.2 (Generalised Hölder). *Let $N \geq 2$ and $f_i \in L^{p_i}(\mathbb{R}^n)$ for $i = 1, 2, ..., N$ and $\sum_{i=1}^N \frac{1}{p_i} = 1$. Then $\Pi_{i=1}^N f_i \in L^1$ and*

$$\|\Pi_{i=1}^N f_i\|_{L^1} \leq \Pi_{i=1}^N \|f_i\|_{L^{p_i}}\,.$$

Proof. Use induction and Theorem 11.1.1. □

Theorem 11.1.3 (Minkowski's inequality for sums). *Let $1 \leq p < \infty$. If $f, g \in L^p$ then $f + g \in L^p$ and*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof. To show that $f + g \in L^p$, note $|f(x) + g(x)|^p \leq 2^p \left(|f(x)|^p + |g(x)|^p\right)$ (consider separately $|f(x)| \leq |g(x)|$ and $|g(x)| \leq |f(x)|$). Estimate obvious for $p = 1$ or if $f + g = 0$ a.e. Otherwise, use $|f + g|^p \leq (|f| + |g|)|f + g|^{p-1}$ and apply Hölder on the right. □

Theorem 11.1.4 (Minkowski's inequality for integrals). *Let $1 \leq p \leq \infty$. We have for $f : \mathbb{R}^n_x \times \mathbb{R}^n_y \to \mathbb{R}$ non-negative the inequality*

$$\Big\| \int_{\mathbb{R}^n} f(x,y) dy \Big\|_{L^p_x} \leq \int_{\mathbb{R}^n} \|f(x,y)\|_{L^p_x} dy\,.$$

Proof. Note $p = 1$ follows from Tonelli's theorem and $p = \infty$ follows from the monotonicity of the integral. For $1 < p < \infty$ and $1/p + 1/q = 1$ we have for $g \in L^q(\mathbb{R}^n)$ by Tonelli and Hölder

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y) dy \right) |g(x)| dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x,y) |g(x)| dx dy \leq \|g\|_{L^q} \int_{\mathbb{R}^n} \|f(x,y)\|_{L^p_x} dy\,. \tag{11.1}$$

It therefore suffices to show

$$\sup_{\|g\|_{L^q}=1} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,y) dy \right) |g(x)| dx = \Big\| \int_{\mathbb{R}^n} f(x,y) dy \Big\|_{L^p_x}\,. \tag{11.2}$$

The $\geq$ follows from choosing $g(x) = \frac{\left(\int_{\mathbb{R}^n} f(x,y)dy\right)^{p-1}}{\| \int_{\mathbb{R}^n} f(x,y)dy\|^{p-1}_{L^p_x}}$ and the $\leq$ by Hölder. ◻

Theorem 11.1.5 (Young's Inequality)**.** *Suppose* $1 \leq p, q, r \leq \infty$ *satisfies* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$*. Then*

$$\|f \star g\|_{L^q} \leq \|f\|_{L^p}\|g\|_{L^r} .$$

Proof. Can assume $1 < p, q, r < \infty$ as $q = \infty$ follows directly from Hölder and $p = \infty$ or $r = \infty$ necessitates $q = \infty$. Can assume $f \geq 0$, $g \geq 0$ (as we can show the inequality for the various combinations arising from $f = f_+ + f_-$, $g = g_+ + g_-$. Use

$$f(x)g(x-y) = f(y)^a g(x-y)^b [f(y)^{1-a} g(x-y)^{1-b}]$$

with $a = 1 - \frac{p}{q}$ and $b = 1 - \frac{r}{q}$. Now apply Theorem 11.1.2 with $p_1 = \frac{p}{1-\frac{p}{q}}$, $p_2 = \frac{r}{1-\frac{r}{q}}$, $p_3 = q$ to deduce

$$\Big| \int f(y)g(x-y)dy \Big| \leq \|f\|_{L^p}^{1-\frac{p}{q}} \|g\|_{L^r}^{1-\frac{r}{q}} \left(\int f(y)^p g(x-y)^r dy \right)^{\frac{1}{q}}$$

Take both sides to the power q, integrate in x and take the q^{th} root. ◻

11.1.2 Maximal function and weak type estimates

For $f \in L^1_{loc}(\mathbb{R}^n)$ we define the maximal function

$$f^\star(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)|dy$$

with the supremum taken over all balls containing x. Now $f^\star$ is not quite in L^1 but one can prove the weak type inequality (see my Measure Theory lecture notes)

$$m\left(x : f^\star(x) > \alpha\right) \leq \frac{A}{\alpha} \|f\|_{L^1(\mathbb{R}^n)} .$$

Once can then proceed to show, for $f \in L^p(\mathbb{R}^n)$ with $p < 1$ the inequality

$$\|f^\star\|_{L^p} \leq A_p \|f\|_{L^p} \tag{11.3}$$

For the (not very complicated) proof see Stein-Sakarchi Volume 4, Section 2.4, page 70.

11.1.3 Proof of Hardy Littlewood Sobolev

Theorem 11.1.6. *Let* $f \in \mathcal{S}(\mathbb{R}^n)$*. Then*

$$\Big\| f \star \frac{1}{|x|^\alpha} \Big\|_{L^r} \leq C\|f\|_{L^p} \tag{11.4}$$

provided $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{n}$ *and* $0 < \alpha < n$.

Proof of Theorem 11.1.6. Estimate is obvious if $f = 0$ a.e. Otherwise we decompose

$$\left(f \star \frac{1}{|x|^\alpha} \right)(x) = \int_{|x-y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy + \int_{|x-y| > R} \frac{f(y)}{|x-y|^\alpha} dy \tag{11.5}$$

We first estimate the first integral. Choose the smallest integer K with $2^{K+1} \geq R \geq 2^K$ and

$$\Big| \int_{|x-y| \leq R} \frac{f(y)}{|x-y|^\alpha} dy \Big| \leq \int_{|x-y| \leq R} \frac{|f(y)|}{|x-y|^\alpha} dy \leq \sum_{k=-\infty}^{N} \int_{2^k \leq |x-y| \leq 2^{k+1}} \frac{|f(y)|}{|x-y|^\alpha} \leq \sum_{k=-\infty}^{K} \frac{1}{2^{k\alpha}} \int_{|x-y| \leq 2^{k+1}} |f(y)|$$

$$\leq C_n \sum_{k=-\infty}^{K} \frac{2^{(k+1)n}}{2^{k\alpha}} \frac{1}{B(x, 2^{k+1})} \int_{|x-y| \leq 2^{k+1}} |f(y)| \leq C_{n,\alpha} 2^{K(n-\alpha)} \frac{\int_{|x-y| \leq 2^{k+1}} |f(y)|}{B(x, 2^{k+1})} \leq C_{n,\alpha} R^{n-\alpha} f^\star(x) ,$$

where $f^\star$ is the maximal function. To estimate the second integral we apply Hölder:

$$\int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha} dy \leq \|f\|_{L^p} R^{-\frac{n}{r}} .$$

We now choose R for each x by $R(x) = \left(\frac{\|f\|_{L^p}}{f^\star(x)}\right)^{\frac{p}{n}}$. Note $f^\star(x) > 0$ for non-trivial f. This yields

$$\left(f \star \frac{1}{|x|^\alpha}\right)(x) \leq (C_{n,\alpha} + 1)\|f\|_{L^p}^{1-\frac{p}{r}} |f^\star|^{\frac{p}{r}} .$$

Taking both sides to the power r and integrating yields the result after using (11.3). □

For later purposes, we note a simple corollary following from an application of Hölder's inequality:

Corollary 11.1.7. *Let $0 < \alpha < 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{\alpha}{n} = 2$. Then for $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{f(x)g(y)}{|x-y|^\alpha} \leq C\|f\|_{L^p}\|g\|_{L^q} . \tag{11.6}$$

11.1.4 Interpolation

We next state and prove the complex interpolation theorem due to M. Riesz. We recall the space $L^p + L^q$ consisting of measurable functions that can be written as $f = f_1 + f_2$ with $f_1 \in L^p$ and $f_2 \in L^q$ equipped with the norm $\|f\|_{L^p+L^q} = \inf_{f=f_1+f_2} (\|f_1\|_{L^p} + \|f_2\|_{L^q})$ with the infimum taken over all decompositions.

Exercise 3. *Let $1 \leq p < q \leq \infty$. Show that $L^p + L^q$ is a Banach space. Show that $f \in L^r$ wit $p \leq r \leq q$ implies $f \in L^p + L^q$.*

Proof. We first check the above is a norm. Only the triangle inequality is non-trivial: Given $f, g \in L^p + L^q$ we find, for any $\epsilon > 0$, $f = f_1 + f_2$ and $g = g_1 + g_2$ with $\|f\|_{L^p+L^q} \geq \|f_1\|_{L^p} + \|f_2\|_{L^q} - \epsilon$, $\|g\|_{L^p+L^q} \geq \|g_1\|_{L^p} + \|g_2\|_{L^q} - \epsilon$. Now $\|f+g\|_{L^p+L^q} \leq \|f_1+g_1\|_{L^p} + \|f_2+g_2\|_{L^q} \leq \|f_1\|_{L^p} + \|f_2\|_{L^q} + \|g_1\|_{L^p} + \|g_2\|_{L^q} \leq \|f\|_{L^p+L^q} + \|g\|_{L^p+L^q} + 2\epsilon$. Since this holds for all $\epsilon > 0$ we are done.

For the completeness we use the equivalence of the statements: (1) X is complete (2) For any sequence (x_n) with $\sum \|x_n\| < \infty$ we have that $\sum x_n$ converges in X. [For (1) $\implies$ (2) show that the partial sums converge. For (2) $\implies$ (1) construct a "telescopic" subsequence.] So let (f_n) be a sequence with $\sum_n \|f_n\|_{L^p+L^q} < \infty$. By the definition of the norm we can find, for each n, $f_n = f_n^1 + f_n^2$ such that $\|f_n\|_{L^p+L^q} \geq \|f_n^1\|_{L^p} + \|f_n^2\|_{L^q} - \frac{1}{2^n}$, in particular $\sum_n \|f_n^1\|_{L^p} + \|f_n^2\|_{L^q} < \infty$. Since L^p and L^q are complete we have $\sum_n f_n^1 \to f_1 \in L^p$ and $\sum_n f_n^2 \to f_2 \in L^q$. Hence $\sum f_n = \sum f_n^1 + \sum f_n^2 \to f_1 + f_2 \in L^p + L^q$.

For the last part, we need to show any $f \in L^r$ can be written as $f = f_1 + f_2$ with $f_1 \in L^p$ and an $f_2 \in L^q$. Given $f \in L^r$ we decompose $f = f\chi_{\{x|f(x)\leq 1\}} + f\chi_{\{x|f(x)\geq 1\}}$. The first part is in L^q, the second in L^p. □

Before stating the Riesz interpolation theorem, we prove the three-lines lemma from complex analysis:

Lemma 11.1.8. *Let Φ be a continuous and bounded function on the strip $0 \leq \Re(z) \leq 1$ that is holomorphic in the interior of the strip. If $|\Phi(z)| \leq M_0$ for $\Re(z) = 0$ and $|\Phi(z)| \leq M_1$ for $\Re(z) = 1$, then $|\Phi(z)| \leq M_0^{1-t} M_1^t$ for $\Re(z) = t$ and $0 < t < 1$.*

Proof. For $\epsilon > 0$ we define $\Phi_\epsilon(z) = \Phi(z) M_0^{z-1} M_1^{-z} \exp(\epsilon z(z-1))$. One checks that Φ_ϵ satisfies the assumptions in the lemma with $M_0 = 1$ and $M_1 = 1$ and that $|\Phi_\epsilon| \to 0$ as $|\Im(z)| \to \infty$. We thus have $|\Phi_\epsilon(z)| \leq 1$ on the boundary of the rectangle $0 \leq \Re(z) \leq 1$, $-A \leq \Im(z) \leq A$ if A is chosen sufficiently large. By the maximum modulus principle $|\Phi_\epsilon(z)| \leq 0$ on the entire rectangle and therefore on the entire strip $0 \leq \Re(z) \leq 1$. Letting $\epsilon \to 0$ produces the desired result. □

We remark that the statement is independent of the actual bound on Φ in the strip. However at least a mild growth assumption is generally necessary for the result to hold.

Theorem 11.1.9. *Suppose T is a linear mapping from $L^{p_0}+L^{p_1}$ to $L^{q_0}+L^{q_1}$. Assume*

$$\|Tf\|_{L^{q_0}} \leq M_0\|f\|_{L^{p_0}} \quad \textit{and} \quad \|Tf\|_{L^{q_1}} \leq M_1\|f\|_{L^{p_1}}. \tag{11.7}$$

Then T is bounded from L^p to L^q

$$\|Tf\|_{L^q} \lesssim M\|f\|_{L^p}$$

provided that (p,q) satisfies

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for some $0 \leq t \leq 1$. Moreover, the bound M satisfies $M \leq M_0^{1-t}M_1^t$.

Proof. To show $\|Tf\|_{L^q} \leq M\|f\|_{L^p}$ is suffices to show the estimate

$$\|Tf\|_{L^q} = \sup_{g \text{ simple}, \|g\|_{L^{q'}}=1} \Big|\int (Tf)g\Big| \leq M\|f\|_{L^p} \quad \text{where } \frac{1}{q}+\frac{1}{q'}=1.$$

Note the equality follows from Hahn-Banach (there exists a functional g on L^q of norm 1 and such that $g(Tf) = \|Tf\|_{L^q}$). By rescaling we can also assume that $\|f\|_{L^p} = 1$.

Let us first assume $p < \infty$ and $q > 1$ and that f is simple, $\|f\|_{L^p} = 1$, g simple and $\|g\|_{L^{q'}} = 1$. We define for $z \in \mathbb{C}$

$$f_z = |f|^{\gamma(z)}\frac{f}{|f|} \quad \text{where } \gamma(z) = p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right),$$

$$g_z = |g|^{\delta(z)}\frac{g}{|g|} \quad \text{where } \delta(z) = q'\left(\frac{1-z}{q_0'}+\frac{z}{q_1'}\right),$$

with $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{q_1}+\frac{1}{q_1'}=1$ and $\frac{1}{q_2}+\frac{1}{q_2'}=1$. With this we have

$$f_t = f \quad , \quad \|f_z\|_{L^{p_0}} = 1 \text{ if } \Re(z)=0 \quad , \quad \|f_z\|_{L^{p_1}} = 1 \text{ if } \Re(z)=1,$$

$$g_t = g \quad , \quad \|g_z\|_{L^{q_0'}} = 1 \text{ if } \Re(z)=0 \quad , \quad \|g_z\|_{L^{q_1'}} = 1 \text{ if } \Re(z)=1.$$

We now consider the map $\Phi : \mathbb{C} \to \mathbb{C}$ given by

$$\Phi(z) = \int (Tf_z)g_z\,. \tag{11.8}$$

This is well defined since $f = \sum a_k\chi_{E_k}$ is a finite sum with E_k disjoint and of finite measure and similarly $g = \sum b_j\chi_{F_j}$. We find

$$\Phi(z) = \sum_{j,k}|a_k|^{\gamma(z)}|b_j|^{\delta(z)}\frac{a_k}{|a_k|}\frac{b_k}{|b_k|}\left(\int T(\chi_{E_k})\chi_{F_j}\right).$$

We see that Φ is holomorphic in the strip $0 < \Re(z) < 1$ and bounded and continuous in its closure. From (11.8) we check that for $\Re(z) = 0$ we have $|\Phi(z)| \leq \|Tf_z\|_{L^{q_0}} \leq M_0\|f\|_{L_0^p} \leq M_0$ with the second inequality following from the assumption in the theorem that T is bounded from L^{p_0} to L^{q_0}. Similarly, we check that for $\Re(z) = 1$ we have $|\Phi(z)| \leq \|Tf_z\|_{L^{q_1}} \leq M_1\|f\|_{L_1^p} \leq M_1$. Applying the three-lines lemma, we conclude

$$|\Phi(t)| \leq M_0^{1-t}M_1^t\,.$$

Taking the sup over all g we have proven the result when f is simple and for $p < \infty$, $q > 1$.

Now let $f \in L^p$ and wlog $p_0 \leq p \leq p_1$ (otherwise switch p_0 and p_1). We approximate f by a sequence (f_n) of simple functions with $|f_n| \leq |f|$ and $f_n \to f$ a.e., hence also $\|f_n - f\|_{L^p} \to 0$ by dominated convergence. We have $\|Tf_n\|_{L^q} \leq M\|f_n\|_{L^p}$, so in particular Tf_n is Cauchy in L^q. Below we will show $T(f_n) \to T(f)$ almost everywhere for a subsequence of (f_n). This is sufficient because by Fatou's Lemma:

$$\|Tf\|_{L^q} = \|\lim_{n\to\infty} Tf_n\|_{L^q} \leq \liminf \|Tf_n\|_{L^q} \leq M\liminf\|f_n\|_{L^p} \leq M\|f\|_{L^p}\,.$$

Given (f_n) we let $f = f^U + f^L$ where $f^U = f\chi_E$ with $E = \{x \mid |f(x)| \geq 1\}$ and $f^L = f\chi_{E^c}$. Note $f \in L^p$ implies $f^U \in L^{p_0}$ and $f^L \in L^{p_1}$. Now with $f_n^U = f_n\chi_E$ we have

$$\int |f_n^U - f^U|^{p_0} \to 0$$

by dominated convergence. Hence $f_n^U \to f^U$ in L^{p_0}. A similar argument yields $f_n^L \to f^L$ in L^{p_1}. Now by the assumption in the theorem $\|T\left(f_n^U - f^U\right)\|_{L^{q_0}} \to 0$ and $\|T\left(f_n^L - f^L\right)\|_{L^{q_1}} \to 0$. There is a subsequence (denoted yet again by (f_n)) such that $T(f_n^U) \to T(f^U)$ pointwise a.e. and a further subsequence such that also $T(f_n^L) \to T(f^L)$ pointwise a.e. We conclude $T(f_n) \to Tf$ almost everywhere for the final subsequence. This concludes the proof for $p < \infty$ and $q > 1$. The case $p = \infty$ (which implies $p_0 = p_1 = \infty$) follows directly from an application of Hölder's inequality. The case $q = 1$ (which implies $q_0 = q_1 = 1$) can be treated by following the steps of the main proof with $g_z = g$. ☐

11.2 Proof of the Littlewood-Paley inequality

The proof requires some Calderon-Zygmund theory, which concerns mapping properties of singular integral operators. While this is a huge subject on its own, the following definition (which concerns a certain type of singular convolution operators appearing in the proof of the Littlewood-Paley inequality) will suffice for our purposes. Let H and H' be separable Hilbert spaces, which for us will be $H = \mathbb{C}$ and $H' = \ell^2(\mathbb{Z})$ (or the other way around, when we consider adjoints).[1] We recall that $L^2(\mathbb{R}^n, H)$ denotes the space of measurable functions $f : \mathbb{R}^n \to H$ such that $\int_{\mathbb{R}^n} \|f\|_H^p dx$ is finite. We denote by $B(H, H')$ the space of bounded linear operators from H to H'.

Definition 11.2.1. *A Calderon-Zygmund operator T from H to H' is a linear operator $T : L^2(\mathbb{R}^n, H) \to L^2(\mathbb{R}^n, H')$, which is bounded*

$$\||Tf|_{H'}\|_{L^2} \lesssim \||f|_H\|_{L^2}, \tag{11.9}$$

and such that there exists a (say smooth) kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \mid x \in \mathbb{R}^n\} \to B(H, H')$ satisfying the bounds

$$\|K(x,y)\|_{B(H,H')} \leq \frac{C}{|x-y|^n} \quad \text{and} \quad \|\partial K(x,y)\|_{B(H,H')} \leq \frac{C}{|x-y|^{n+1}} \quad \text{for } x \neq y \tag{11.10}$$

and such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \tag{11.11}$$

holds for all $f \in L^2\left(\mathbb{R}^d, H\right)$ of compact support and y not in the support of f.

Note that T is in particular well defined for all $f \in \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. We will also see that the arguments below work for much weaker assumptions on the kernel.

The key theorem that will allow us to prove Theorem 9.1.6 is the following:

Theorem 11.2.2. *Let T be a Calderon-Zygmund operator as in Definition 11.2.1 with $H = \mathbb{C}$ and $H' = \ell^2(\mathbb{Z})$. Then the operator extends to a bounded operator from L^p into L^p for any $1 < p < \infty$, i.e. we have the estimate*

$$\|Tf\|_{L^p(\mathbb{R}^n, \ell^2(\mathbb{Z}))} \leq \|f\|_{L^p}. \tag{11.12}$$

Proof. The proof is quite involved and hence postponed to Section 11.2.1. ☐

[1] We recall $\ell^2(\mathbb{Z})$ is the space of square integrable sequences of complex numbers.

Proof of Theorem 9.1.6. We define the operator

$$\mathbf{S}f(x) := (P_k f(x))_{k\in\mathbb{Z}} \ .$$

Step 1. We show that $\mathbf{S}$ is a Calderon-Zygmund operator as in Definition 11.2.1 with $H = \mathbb{C}$ and $H' = \ell^2(\mathbb{Z})$.

We easily see that

$$\|\mathbf{S}f\|^2_{L^2(\mathbb{R}^n,\ell^2(\mathbb{Z}))} = \int_{\mathbb{R}^n} |P_k f|^2_{\ell^2} dx = \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |P_k f|^2 dx = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |P_k f|^2 dx \leq C\|f\|_{L^2} \ ,$$

with the last step following from Lemma 9.1.2 and the dominant convergence theorem having been used in the penultimate step. So $\mathbf{S} : L^2(\mathbb{R}^n) \to L^2\left(\mathbb{R}^n, \ell^2(\mathbb{Z})\right)$ is a bounded linear operator.

We next check the conditions (11.10). By the definition of P_k we can write

$$\mathbf{S}f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

with

$$K(x,y) = \left(2^{nk} m\left(2^k(x-y)\right)\right)_{k\in\mathbb{Z}}$$

and m the Fourier transform of the cut-off ψ. We have for $x \neq y$

$$|K_k(x,y)| = \left| 2^{nk} \int d\xi e^{i2^k(x-y)\xi} \psi(\xi) \right| \leq C 2^{nk} \leq C \frac{1}{|x-y|^n} \left(2^k|x-y|\right)^n \ .$$

On the other hand, we can integrate by parts in ξ ($n+2$ times) to produce the bound

$$|K_k(x,y)| = \left| 2^{nk} \int d\xi e^{i2^k(x-y)\xi} \psi(\xi) \right| \leq C \frac{1}{|x-y|^n} \left(2^k|x-y|\right)^{-2} \ .$$

Now for given x, y choose K_0 such that $2^{K_0-1} \leq |x-y| \leq 2^{K_0}$. Then

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |K_k(x,y)| &= \sum_{k=-\infty}^{-K_0} |K_k(x,y)| + \sum_{k=-K_0+1}^{\infty} |K_k(x,y)| \\ &\leq \frac{C}{|x-y|^n} \left[2^{K_0 n} \sum_{k=-\infty}^{-K_0} 2^{kn} + 4^{-K_0+1} \sum_{k=-K_0+1}^{\infty} 4^{-k} \right] \leq \frac{C}{|x-y|^n} \ . \end{aligned} \tag{11.13}$$

The bound for the derivative is left to the reader.

Step 2. We prove $\|Sf\|_{L^p} \leq C\|f\|_{L^p}$ using Step 1 and Theorem 11.2.2.

It suffices to note that $\|Sf\|_{L^p} = \||\mathbf{Sf}|_{\ell^2}\|_{L^p} = \|\mathbf{S}f\|_{L^p(\mathbb{R}^n,\ell^2(\mathbb{Z}))} \leq C\|f\|_{L^p}$.

Step 3. We prove $\|f\|_{L^p} \leq C\|Sf\|_{L^p}$ using Step 2 and a duality argument. We have for $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 < p < \infty$ the string of inequalities

$$\begin{aligned} \int_{\mathbb{R}^n} fg dx = \int_{\mathbb{R}^n} \sum_{k,k'} P_k f P_{k'} g dx &\leq \int_{\mathbb{R}^n} \sum_k P_k f \left(P_{k-1} g + P_k g + P_{k+1} g\right) dx \\ &\leq 3 \int_{\mathbb{R}^n} |Sf||Sg| dx \leq C\|Sf\|_{L^p} \|Sg\|_{L^{p'}} \leq C\|Sf\|_{L^p} \|g\|_{L^{p'}} \end{aligned} \tag{11.14}$$

where we have used the orthogonality of the P_k, Cauchy-Schwarz, Hölder and the estimate from Step 2. Taking now the sup over all g with $\|g\|_{L^{p'}} = 1$ yields $\|f\|_{L^p}$ on the left and hence the desired estimate. ☐

Remark 11.2.3. *Note that to prove the two estimates mentioned in Remark 9.1.7 it suffices to use Theorem 11.2.2 for* $1 < p \leq 2$.

11.2.1 Proof of Theorem 11.2.2 assuming a weak-type estimate

We first claim that Theorem 11.2.2 would follow if we could establish the following weak-type estimate: There exists a constant $A > 0$ such that for any $\alpha > 0$ we have the estimate

$$m\left(\{x \,\middle|\, |Tf(x)|_{\ell^2} > \alpha\}\right) \leq \frac{A}{\alpha}\|f\|_{L^1} \tag{11.15}$$

for $f \in L^1 \cap L^2$. Assuming (11.15) we deduce the stronger estimate (for a different A)

$$m\left(\{x \,\middle|\, |Tf(x)|_{\ell^2} > \alpha\}\right) \leq A\left(\frac{1}{\alpha}\int_{|f|>\alpha}|f|dx + \frac{1}{\alpha^2}\int_{|f|\leq\alpha}|f|^2 dx\right) \tag{11.16}$$

for $f \in L^1 \cap L^2$. Indeed, given $\alpha > 0$ we can decompose $f = f_1 + f_2$ with $f_1(x) = f(x)$ if $|f(x)| > \alpha$ and zero otherwise, $f_2(x) = f(x)$ if $|f(x)| \leq \alpha$ and zero otherwise. Then by the triangle inequality

$$m\left(\{x \,\middle|\, |Tf(x)|_{\ell^2} > \alpha\}\right) = m\left(\{x \,\middle|\, |Tf_1(x)|_{\ell^2} > \frac{\alpha}{2}\}\right) + m\left(\{x \,\middle|\, |Tf_2(x)|_{\ell^2} > \frac{\alpha}{2}\}\right).$$

Now by (11.15) applied to f_1 we have

$$m\left(\{x \,\middle|\, |Tf_1(x)|_{\ell^2} > \frac{\alpha}{2}\}\right) \leq \frac{2A}{\alpha}\|f\|_{L^1} = \frac{2A}{\alpha}\int_{|f|>\alpha}|f|dx\,.$$

For the other part we use the L^2 boundedness of T and Tchebychev's inequality:

$$m\left(\{x \,\middle|\, |Tf_2(x)|_{\ell^2} > \frac{\alpha}{2}\}\right) \leq \left(\frac{2}{\alpha}\right)^2 \||Tf_2|_{\ell^2}\|_{L^2}^2 \leq \left(\frac{2}{\alpha}\right)^2 C\|f_2\|_{L^2}^2 = C\left(\frac{2}{\alpha}\right)^2\int_{|f|\leq\alpha}|f|^2 dx$$

to conclude (11.16). We now use the identity for the distribution function[2]

$$\int_{\mathbb{R}^n}|Tf(x)|_{\ell^2}^p dx = \int_0^\infty \lambda\left(\alpha^{\frac{1}{p}}\right)d\alpha \quad \text{with } \lambda(\alpha) = m\left(\{x \,\middle|\, |Tf(x)|_{\ell^2} > \alpha\}\right). \tag{11.17}$$

Using (11.16) this yields

$$\int_{\mathbb{R}^n}|Tf(x)|_{\ell^2}^p dx \leq A\left(\int_0^\alpha d\alpha\frac{1}{\alpha^{\frac{1}{p}}}\int_{|f|>\alpha^{\frac{1}{p}}}|f|dx + \int_0^\alpha d\alpha\frac{1}{\alpha^{\frac{2}{p}}}\int_{|f|\leq\alpha^{\frac{1}{p}}}|f|^2 dx\right). \tag{11.18}$$

We can now change the order of integration in both integrals (exercise), which requires $p > 1$ for the first and $p < 2$ for the second to produce (recall that the estimate below holds for $p = 2$ by assumption)

$$\||Tf\|_{\ell^2}\|_{L^p} \leq C\|f\|_{L^p} \quad \text{for } 1 < p < 2. \tag{11.19}$$

A duality argument can be employed to extend the estimate to $2 < p < \infty$. We leave the details to the reader and sketch the main idea. We have

$$\begin{aligned}\|Tf\|_{L^p(\mathbb{R}^n,\ell^2(\mathbb{Z}))} &= \sup_{\|g\|_{L^q(\mathbb{R}^n,\ell^2(\mathbb{Z}))}=1}\int\langle Tf(x), g(x)\rangle_{\ell^2}dx = \sup_{\|g\|_{L^q(\mathbb{R}^n,\ell^2(\mathbb{Z}))}=1}\int\sum_{k\in\mathbb{Z}}T_k f g_k dx \\ &= \sup_{\|g\|_{L^q(\mathbb{R}^n,\ell^2(\mathbb{Z}))}=1}\sum_{k\in\mathbb{Z}}\int fT_k g_k dx \leq \sup_{\|g\|_{L^q(\mathbb{R}^n,\ell^2(\mathbb{Z}))}=1}\Big\|\sum_{k\in\mathbb{Z}}T_k g_k\Big\|_{L^q}\|f\|_{L^p}\,,\end{aligned} \tag{11.20}$$

so if we could establish $\|T^\star g\|_{L^q} := \left\|\sum_{k\in\mathbb{Z}}T_k g_k\right\|_{L^q} \lesssim \|g\|_{L^q(\mathbb{R}^n,\ell^2(\mathbb{Z}))}$ for $1 < q < 2$, we would be done. To achieve this one checks that $T^\star$ is a Calderon-Zygmund operator from $L^2\left(\mathbb{R}^n, \ell^2(\mathbb{Z})\right) \to L^2(\mathbb{R}^n, \mathbb{C})$ as in Definition 11.2.1. One may therefore repeat the proof which lead to (11.19) for $T^\star$ producing the desired estimate.

11.2.2 Proof of the weak-type estimate (11.15)

To be done (see Stein's Functional Analysis book).

[2] This identity holds in the extended sense. To prove it, note that the case $p = 1$ is Tonelli's heorem ($Tf(x)|_{\ell^2}$ is a measurable function since the norm is continuous). Then set $G(x) = |Tf(x)|^p$ and apply the $p = 1$ case for G.