General Relativity and Black Holes – Week 9

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1 Exercises

1. Let ψ be the smooth solution of $\Box\psi = 0$ in dimension $1+3$ arising from data (ψ_0, ψ_1) of compact support at $t = 0$. Recall that in lectures, we proved an integrated decay estimate for *angular* derivatives:

$$\int_{t=0}^{T} dt \int_{\Sigma_t} \frac{1}{r} |\nabla\!\!\!/ \, \psi|^2 \leq C \left(\|\psi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\psi_1\|^2_{L^2(\mathbb{R}^3)} \right) \tag{1.1}$$

holds for a constant C independent of T.

(a) Prove that

$$\Big| \int_0^T \int_{\mathbb{R}^3} \left[h'(r)(\partial_r \psi)^2 + \frac{h(r)}{r} |\nabla\!\!\!/ \, \psi|^2 - \frac{1}{4} \Delta \left(h'(r) + \frac{2h(r)}{r} \right) \psi^2 \right] dx dt \Big| \leq C \left(\|\psi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\psi_1\|^2_{L^2(\mathbb{R}^3)} \right)$$

holds for a constant C independent of T.
HINTS: Use the vectorfield $X = h(r)\partial_r$ with h a bounded radial function satisfying $h'(r) \leq \frac{\tilde{C}}{1+r^2}$. The identity $\Box(\psi^2) = 2\left(-(\partial_t \psi)^2 + |\nabla \psi|^2 \right)$ may also be useful.

(b) Choose $h(r) = \frac{1}{1+r}$ and use (1.1) to deduce

$$\int_0^T \int_{\mathbb{R}^3} \left[\frac{1}{(1+r)^2} (\partial_r \psi)^2 + \frac{1}{r} |\nabla\!\!\!/ \, \psi|^2 + \frac{1}{(1+r)^4} |\psi|^2 \right] dx dt \leq C \left(\|\psi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\psi_1\|^2_{L^2(\mathbb{R}^3)} \right) .$$

(c) Does the estimate also hold replacing $(\partial_r \psi)^2$ by $(\partial_t \psi)^2$ on the left? Why or why not?

2. Prove Proposition 5.2 of the notes (the $\partial_{t^\star}$-energy identity for the covariant wave equation on Schwarzschild).

2 Problems and Discussion

1. This question again illustrates the *robustness* of the vectorfield method. You should compare with Problem 3 from Sheet 8.

   Consider on Minkowski space $\left(\mathbb{R}^{1+3}, \eta\right)$ the linear wave equation

   $$\Box_\eta \psi + \epsilon b^\mu \partial_\mu \psi = 0 \tag{2.1}$$

   where b is a vector on $\mathbb{R}^{1+3}$ satisfying the pointwise bound $|b^t| + |b^x| + |b^y| + |b^z| \leq \frac{1}{1+r^2}$ on the components and $\epsilon > 0$ is a constant. Assume also that smooth initial data (ψ_0, ψ_1) of compact support are prescribed at $t = 0$.

   Prove that there exists an $\epsilon_0 > 0$ (independent of b and (ψ_0, ψ_1)) such that for all $\epsilon \leq \epsilon_0$ the estimate

   $$\int_0^T \int_{\mathbb{R}^3} \frac{1}{1+r^2} \sum_{i=0}^{3} \left((\partial_i \psi)^2 + \frac{\psi^2}{1+r^2} \right) \leq C \left(\|\psi_0\|^2_{\dot{H}^1(\mathbb{R}^3)} + \|\psi_1\|^2_{L^2(\mathbb{R}^3)} \right)$$

   holds for solutions of (2.1) with data (ψ_0, ψ_1).

   HINTS: Observe first that for $\epsilon = 0$ the estimate has been proven in Exercise 1 above. Now understand how the additional first order term influences the proof and show that the additional terms can be absorbed for sufficiently small ϵ.

2. Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold. Show that for all smooth scalar function f on $\mathcal{M}$, we have

   $$\Delta_g \nabla f = \nabla \Delta_g f + \mathrm{Ric}(g) \cdot \nabla f$$

   and complete the proof of the Bochner identity seen in lectures.