Non-linear Wave Equations – Week 9

Gustav Holzegel

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1. Consider the wave map equation in dimension $n = 1$ as in lectures. Show that sufficiently regular solutions satisfy
$$\Box\left(\partial_t\phi^T\partial_t\phi + \partial_x\phi^T\partial_x\phi\right) = 0\,.$$
Use this to give an alternative proof of global existence.

2. Show that the estimate of Proposition 7.4.2 (the integrated decay one) in the lecture notes fails if $\overline{\partial}$ is replaced by ∂.

3. (From Tao, Nonlinear Dispersive Equations, Exercise 6.11) Let $n \in \{2, 3\}$ and $\phi : \mathbb{R}^{1+n} \to S^n$ be a smooth *equivariant* function, i.e. ϕ is of the form
$$\phi(t,x) = \begin{pmatrix}\sin(f(t,r))\frac{x}{r}\\ \cos(f(t,r))\end{pmatrix} \tag{1}$$
where we denote $r = |x|$, for some smooth function $f : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$. Show that ϕ satisfies the wave maps equation if and only if f solves
$$-\partial_{tt}f + \partial_{rr}f + \frac{n-1}{r}\partial_r f = \frac{(n-1)\sin(f)\cos(f)}{r^2} \tag{2}$$
Conclude that in $n = 2$ there is the explicit solution $f(t,r) = 2\arctan(r)$. Convince yourself that ϕ is thus in fact a harmonic map.

   Conclude further that in $n = 3$ there is the explicit self-similar solution $f(t,r) = 2\arctan\left(\frac{r}{t}\right)$.

   (a) Verify by inserting the solution back into the ansatz (1) that ϕ in this case is given by $\phi(t,x) = \Psi\left(\frac{x}{t}\right)$ with $\Psi : \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$ the *stereographic projection* given by $\Psi(y) = \left(\frac{2y}{1+|y|^2}, \frac{|y|^2-1}{|y|^2+1}\right)$.

   (b) Convince yourself that this solution develops a singularity at the spacetime origin $(t,x) = (0,0)$. Use the symmetries of the wave maps equation to construct a solution that blows up at any point $(T,X) \in \mathbb{R}^{1+3}$.

   (c) Finally use a finite speed of propagation argument to conclude the existence of solutions with compactly supported smooth data which blow up in finite time.

4. Formulate and prove a small data global existence result in dimension $n = 3$ for the problem
$$\begin{cases}\Box\phi = \phi\left(\partial_t\phi\right)^2\\ \phi(0,x) = \phi_0\\ \partial_t\phi(0,x) = \phi_1\,.\end{cases} \tag{3}$$

Analysis Review Problems

1. (This will be useful in the proof of the Strichartz estimate that we will do in 2-3 weeks from now.) Recall the statement and the proof of the Riesz-Thorin interpolation theorem. If this is entirely foreign to you, I might do this explicitly in class.