General Relativity and Black Holes – Week 8

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1 Exercises

1. Let (Σ, h) be an n-dimensional Riemannian manifold and $\phi \in C^\infty(\Sigma)$. Consider the conformally related metric $\overline{h} = \phi^{\frac{4}{n-2}} h$ on Σ. Prove the following transformation formula for the scalar curvature for $n > 2$:

$$R(\overline{h}) = \phi^{\frac{-(n+2)}{(n-2)}} \left(\phi R(h) - \frac{4(n-1)}{n-2} \Delta_h \phi \right) \tag{1.1}$$

where $\Delta_h = h^{ij} \nabla_i \nabla_j$ and ∇ denotes the Levi-Civita connection of (Σ, h). What happens for $n = 2$?

2. (The conformal method for the constraint equations.) In this exercise, we present a simple method to generate solutions of the constraint equations (going back to Lichnerowicz (1944)). It will generate solutions to the constraint equations with $tr_h K = 0$ corresponding to slices in the spacetime whose mean curvature vanishes.[1]

We discuss the physical case $n = 3$. The starting point is to choose an arbitrary Riemannian metric $\tilde{h}$ on Σ and construct[2] a symmetric $\tilde{h}$-*traceless* covariant 2-tensor $\tilde{K}$ satisfying the *linear* equation

$$\tilde{\nabla}^a_{\tilde{h}} \tilde{K}_{ab} = 0\,. \tag{1.2}$$

We now set $h = \phi^4 \tilde{h}$ for an unknown function $\phi \in C^\infty(\Sigma)$ and $K = \phi^{-2} \tilde{K}$.

(a) Show that K satisfies

$$\nabla^a_h K_{ab} = 0\,.$$

(b) Show using the result of Exercise 1 that h satisfies

$$R(h) = |K|^2_h$$

provided ϕ satisfies the non-linear elliptic equation

$$\Delta_{\tilde{h}} \phi - \frac{1}{8} R(\tilde{h}) \phi + \frac{1}{8} \phi^{-7} \tilde{K}^{ab} \tilde{K}_{ab} = 0\,. \tag{1.3}$$

Conclude that we have reduced solving the constraint equations to solving the decoupled (1.2) and (1.3).

(c) *Bonus question:* Set $\Sigma = \mathbb{R}^3$ and $\tilde{h}_{ij} = \delta_{ij}$. How would you construct solutions to the constraints close to trivial data $(\Sigma, \delta_{ij}, 0)$ for flat space? (Which well-known functional theorem could you use?)

(d) Of course setting $\tilde{K} = 0$ is a solution of (1.3).[3] Show that

$$h = \left(1 + \frac{M}{2r}\right)^4 \delta$$

is a scalar flat metric on $\Sigma = \mathbb{R}^3 \setminus \{0\} \approx \mathbb{R} \times S^2$. Can you relate (Σ, h) to a slice in the Schwarzschild spacetime? HINT: Apply the coordinate transformation $r = \rho \left(1 + \frac{M}{2\rho}\right)^2$ to the Schwarzschild metric in standard (t, r, θ, ϕ) coordinates.

[1] Such slices are called *maximal* (why?). Not every spacetime admits maximal slices so this is an a priori restriction.
[2] This is a relatively easy task but we will not follow this up here.
[3] Such data are called time-symmetric and this is a much more severe restriction than the vanishing of the trace.

3. Consider $\mathbb{R}^{1+3}$ equipped with the standard global (t, x) coordinates and a Lorentzian metric g satisfying $\sum_{\mu,\nu=0}^{3} |g_{\mu\nu} - \eta_{\mu\nu}| \leq \frac{1}{10}$, where $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ and $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$. Assume in addition that $T = \partial_t$ satisfies

$$\sum_{i,j} |{}^{(T)}\pi_{\mu\nu}| \leq \frac{1}{(1+|t|)^{1+\delta}}$$

for some $\delta > 0$. Prove that any C^2 solution of the covariant wave equation $\Box_g \psi = 0$ on $(\mathbb{R}^{1+3}, g)$ satisfies

$$\sup_{t\in\mathbb{R}} \|\psi(t,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)} \leq C_\delta \|\psi(0,\cdot)\|_{\dot{H}^1(\mathbb{R}^3)}$$

for some uniform constant $C_\delta > 0$ depending only on δ.

2 Problems and Discussion

1. Fill in the details of the proof of the local uniqueness statement for the vacuum Einstein equations of Section 4.9 of the notes.

2. Let (Σ, g) be a smooth Riemannian manifold. We say that (Σ, g) is *asymptotically flat* if there exists an open pre-compact subset K of Σ such that $\Sigma \setminus \overline{K}$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and such that for the coordinates (x, y, z) induced by this diffeomorphism on $\Sigma \setminus \overline{K}$, we have the following asymptotic behaviour at infinity

$$\partial^{\leq k}(g_{ij} - \delta_{ij}) = O_{r\to+\infty}(1/r^k),$$

for all $k \in \mathbb{N}$ and where $r := \left(x^2 + y^2 + z^2\right)^{1/2}$. We define the *mass* of an asymptotically flat Riemannian manifold (Σ, g) to be

$$M := \frac{1}{16\pi} \lim_{r\to+\infty} \int_{S_r} \sum_{i,j=1}^{3} \left(\partial_{x^j} g_{ij} - \partial_{x^i} g_{jj}\right) N^i \mathrm{d}A$$

where S_r is the coordinate sphere of radius r, N is the exterior unit normal to it and $\mathrm{d}A$ is its area element. Show that Schwarzschild initial slice is asymptotically flat and compute its mass (HINT: use Question 2d).

Let us quote the following fundamental result which was proved in two different ways by Schoen and Yau (1979) and by Witten (1981).

Theorem 2.1 (Positive mass theorem). *Let (Σ, g) be a smooth asymptotically flat Riemannian manifold. Then, $M \geq 0$ with equality iff (Σ, g) is isometric to Euclidean space.*