Non-linear Wave Equations – Week 8

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1. (Computational quickies.) Verify the following claims made in lectures:
   (a) The collection of vectorfields from $\mathcal{X}$ defined in lectures indeed form a Lie-Algebra.
   (b) If $\Box\phi = 0$, then $\Box\Gamma\phi = 0$ holds for $\Gamma \in \mathcal{X}$ (Lemma 6.1.1).
   (c) Let χ be a smooth cut-off satisfying $\chi(x) = 1$ for $x \leq 1/2$ and $\chi(x) = 0$ for $x > \frac{3}{4}$. Then $|\partial^\alpha \chi(\frac{r}{t})| \leq \frac{C}{t^{|\alpha|}}$.
   (d) Let χ be a smooth cut-off satisfying $\chi(x) = 1$ for $\frac{1}{2} \leq x \leq 2$ and $\chi(x) = 0$ in $\left[\frac{1}{4}, 4\right]^c$. Then for any $\Gamma \in \mathcal{X}$ and $t + r \geq 1$ we have $|\Gamma\chi(\frac{r}{t})| \leq C$.

2. (The conformal energy) Let ϕ be a solution of the wave equation in $3+1$ dimensions arising from data of compact support.
   (a) Show that
   $$E_K[\phi](t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}(t^2+r^2)\left((\partial_t\phi)^2 + |\nabla\phi|^2\right) + 2tr\partial_r\phi\partial_t\phi + 2t\phi\partial_t\phi - \phi^2\right)(t,x)dx$$
   is independent of time.
   (b) Show that the integral from (a) is in fact positive.
   HINT: Add the integrand to $\frac{1}{2}\partial_i\left(\frac{x_i\phi^2(t^2+r^2)}{r^2}\right)$ and complete the square.
   (c) Conclude the estimate
   $$|\phi(t,x)| \leq \frac{C}{1+t}\sum_{|\alpha|\leq 2} E_K[\Omega^\alpha_{ij}\phi](0) \tag{1}$$
   for $t \geq 0$ and C a uniform constant (not depending on the size of the support).
   HINT: Combine (a) and (b) with Sobolev embedding on spheres (Analysis Review Problem 1 below) with the fundamental theorem of calculus on constant t-slices.

   Discussion: The conformal energy is generated by the vectorfield $u^2\partial_u + v^2\partial_v$, which is a *conformal* isometry of Minkowski space.

3. (The wave map equation) Recall from lectures that the wave map equation is given by
   $$\begin{cases} \Box\phi = \phi\left(\partial_t\phi^T\partial_t\phi - \sum_{i=1}^n \partial_i\phi^T\partial_i\phi\right) \\ \phi(0,x) = \phi_0 \\ \partial_t\phi(0,x) = \phi_1 \end{cases}, \tag{2}$$
   where $\Box$ is the standard wave operator in dimension $1+n$ and ϕ takes values in $\mathbb{R}^{m+1}$.
   (a) Show that if the data also satisfies $\phi_0^T\phi_0 = 1$ and $\phi_1^T\phi_0 = 0$ then, if a solution $\phi : I \times \mathbb{R}^n \to \mathbb{R}^{m+1}$ exists, it satisfies $\phi^T(t)\phi(t) = 1$ for all $t \in I$, i.e. ϕ is indeed a map to the sphere.
   (b) Show that inverse stereographic projection $\Pi : \mathbb{R}^2 \to \mathbb{S}^2$ is a time-independent wave map.

Analysis Review Problems

1. Let ϕ be a smooth function on the unit sphere. Prove the Sobolev inequality

$$\sup_{\mathbb{S}^{n-1}} |\phi| \leq C \left(\sum_{|\alpha| \leq \lfloor \frac{n+1}{2} \rfloor} \int_{\mathbb{S}^{n-1}} |\Omega_{ij}^{\alpha} \phi|^2(\theta) d\sigma_\theta \right)^{\frac{1}{2}} . \tag{3}$$

   HINT: There are (at least) two approaches. One is to use charts (e.g. stereographic projection) and the standard Sobolev embedding on $\mathbb{R}^n$. A second is to use spherical harmonics on the sphere which can be thought of as the analogue of the Fourier transform.