General Relativity and Black Holes – Week 7

Gustav Holzegel

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1 Exercises

The following problems investigate the relation between the intrinsic and extrinsic geometry of (spacelike) hypersurfaces in a spacetime. (The concepts introduced below are in fact much more general and work for submanifolds of arbitrary co-dimension in a pseudo-Riemannian manifold.)

Let $(\overline{M}, g)$ be a spacetime and M be a spacelike hypersurface with (locally defined) future directed unit normal n. A smooth map $X : M \to T\overline{M}$ such that $\pi \circ X(p) = p$ (where $\pi : T\overline{M} \to \overline{M}$ is the standard projection) will be called an $\overline{M}$ vectorfield on M. Note that

- the restriction of a spacetime vectorfield $X : \overline{M} \to T\overline{M}$ to M is an $\overline{M}$ vectorfield on M.
- we can decompose an $\overline{M}$ vectorfield on M into its normal and tangential (to TM) components, which we denote by **tan** and **nor**.
- vectorfields on M, $X : M \to TM$, can be considered as $\overline{M}$ vectorfields (with vanishing normal component) using the push forward of the inclusion map.

1. (The induced connection.) For X, Y vectorfields on M define a map $D : X(M) \times X(M) \to X(M)$ by
$$D_X Y := \mathbf{tan}[\nabla_X Y]\,. \tag{1}$$
Here $[\nabla_X Y]$ on the right is defined by first extending X and Y to vectorfields on $\overline{M}$, then taking the covariant derivative and finally restricting the result to M.
   (a) Show that $[\nabla_X Y]$ and hence (1) are well defined, i.e. in particular that the definitions do not depend on how one extends X and Y from M to $\overline{M}$.
   (b) Show that the map D defines a connection on M.
   HINT: Recall the properties of a connection from the notes.
   (c) Show that D is in fact the Levi-Civita connection of the Riemannian manifold (M, h).
   HINT: Write out the formula for $2g(\nabla_X Y, Z)$ in the proof of Proposition 2.54 of the notes for X, Y, Z vectorfields on M extended to $\overline{M}$ and take the restriction to M.

2. (The second fundamental form.) We define the second fundamental form of M in $\overline{M}$ as the $(0, 2)$-tensor field on M given by
$$K(X, Y) := g(\nabla_X n, Y),$$
for all $X, Y \in X(M)$.
   (a) Check that this is indeed a tensorfield and that $K(X, Y) = K(Y, X)$, for all $X, Y \in X(M)$.
   (b) Show that $[\nabla_X Y] = D_X Y + K(X, Y)n$, for all $X, Y \in X(M)$, where $[\nabla_X Y]$ is defined as in Exercise 1.

3. (The Gauss equation.) Prove that for X, Y, Z, W vectorfields all tangent to M we have
$$h\left(Riem(X, Y)Z, W\right) = g\left(\overline{Riem}(X, Y)Z, W\right) + K(X, Z)K(Y, W) - K(X, W)K(Y, Z)\,. \tag{2}$$
Here K is defined as in Exercise 2, h denotes the induced Riemannian metric on M, $\overline{Riem}$ the Riemann tensor of $\overline{M}$ and $Riem$ the Riemann tensor of M.

2 Problems and Discussion

1. (The Codazzi equation.) This problem is a direct follow-up of Exercise 3. Prove that for X, Y, Z vectorfields all tangent to M we have

$$g\left(\overline{Riem}(X,Y)Z, n\right) = (\nabla_Y K)(X,Z) - (\nabla_X K)(Y,Z) \tag{3}$$

Here K is defined as in Exercise 2 and $\overline{Riem}$ denotes the Riemann tensor of $\overline{M}$ and $Riem$ the Riemann tensor of M. (You should give meaning to $\nabla_Y K$.)

2. (Generators of null hypersurfaces.) Let (M, g) be a spacetime and C a smooth null hypersurface in M. Let N be a normal vectorfield for C. Show that the integral curves of N are pre-geodesics, i.e. they become geodesics after a change of parametrisation.
Remark: The integral curves are called the *generators* of the null hypersurface (why?). What are the generators in the case of the event horizon of the Schwarzschild spacetime?