Non-linear Wave Equations – Week 7

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June 3, 2021

1. Deduce the local in time Strichartz estimate from the global in time one stated in lectures, i.e. show that one can replace $L^q_t L^r_x$ by $L^q\left([0,T], L^r(\mathbb{R}^n)\right)$ on the left, and $L^{q'}_t L^{r'}_x$ by $L^{q'}\left([0,T], L^{r'}(\mathbb{R}^n)\right)$ on the right of the estimate stated in class.
   HINT: Cut off F in time and use domain of dependence.
2. Show that the conditions $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$ and $\frac{n}{r} + \frac{1}{q} = \frac{n}{r'} + \frac{1}{q'} - 2$ in the Strichartz estimate are necessary for the estimate to hold.
   HINT: Use scaling.

The following problems concern the well-posedness theory of $\Box\phi = |\phi|^{p-1}\phi$ with $3 < p < 5$ in $3+1$ dimensions as discussed in lectures.

3. (Persistence of regularity.) In lectures, we proved local well-posedness for $\Box\phi = |\phi|^{p-1}\phi$ with $3 < p < 5$ and initial data $f \in \dot{H}^1(\mathbb{R}^3)$, $g \in L^2(\mathbb{R}^3)$ in the space $C^0([0,T], \dot{H}^1(\mathbb{R}^3)) \cap C^1\left([0,T], L^2(\mathbb{R}^3)\right) \cap L^{\frac{2p}{p-3}}\left([0,T], L^{2p}(\mathbb{R}^3)\right)$.
   (a) Show that if the data satisfies $f \in H^2(\mathbb{R}^3)$, $g \in H^1(\mathbb{R}^3)$, then the solution is in addition in $C^0([0,T], H^2(\mathbb{R}^3)) \cap C^1\left([0,T], H^1(\mathbb{R}^3)\right)$ for as long as the solution exists. Can you show the solution is classical if the data are sufficiently smooth?
   (b) Derive the conservation of energy for solutions in the space $C^0([0,T], H^2(\mathbb{R}^3)) \cap C^1\left([0,T], H^1(\mathbb{R}^3)\right)$.
   (c) Use an approximation argument to conclude that energy conservation holds even for data in $f \in \dot{H}^1(\mathbb{R}^3)$, $g \in L^2(\mathbb{R}^3)$.
4. (Continuous dependence on data.) The well-posedness result from lectures allows us to define a solution map $S : \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \to X$ where $X = C^0([0,T], \dot{H}^1(\mathbb{R}^3)) \cap C^1\left([0,T], L^2(\mathbb{R}^3)\right) \cap L^{\frac{2p}{p-3}}\left([0,T], L^{2p}(\mathbb{R}^3)\right)$ mapping the data to the unique solution. Show that this map is continuous (in fact Lipschitz).
   HINT: Consider the difference of two solutions arising from data which are close in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.
5. (Global well-posedness in the critical case $p = 5$.) Prove that there exists $\epsilon_0 > 0$ such that for all smooth data satisfying $\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} < \epsilon_0$ the problem $\Box\phi = \phi^5$ with data $\phi(0,x) = f(x)$, $\partial_t\phi(0,x) = g(x)$ has a unique global in time smooth solution.

Analysis Review Problems

1. Show that the mixed spaces $L^q_t L^r_x$ defined in class are indeed Banach spaces.
2. Show that smooth functions of compact support are dense in $L^q_t L^r_x$.
3. Show that the dual of $L^q_t L^r_x$ can be identified with $L^{q'}_t L^{r'}_x$ where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.