General Relativity and Black Holes – Week 6

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November 17, 2021

1 Exercises

1. Consider the semi-linear wave equation $\Box_\eta \psi = (\partial_t \psi)^2$ on Minkowski space in dimension $3+1$ and assume there exists a C^2 solution of this equation existing in a spacetime slab $(-T,T) \times \mathbb{R}^3$.

   (a) Deduce that if $\psi(t=0,x) = 0$ and $\partial_t \psi(t=0,x) = 0$ holds for all $x \in \mathbb{R}^3$, then the solution necessarily vanishes in $(-T,T) \times \mathbb{R}^3$.
   HINT: Repeat the computation on the truncated cone from lectures together with Gronwall's inequality.

   (b) Deduce that two solutions for which both $\psi_1(t=0,x) = \psi_2(t=0,x)$ and $\partial_t \psi_1(t=0,x) = \partial_t \psi_2(t=0,x)$ hold for all $x \in \mathbb{R}^3$ necessarily agree on $(-T,T) \times \mathbb{R}^3$.
   HINT: Use $(\partial_t \psi_1)^2 - (\partial_t \psi_2)^2 = (\partial_t \psi_1 - \partial_t \psi_2)(\partial_t \psi_1 + \partial_t \psi_2)$.

   DISCUSSION: Can you generalise this to other non-linearities?

2. Prove or disprove that the following spacetimes $(\mathcal{M}, g)$ are globally hyperbolic:

   (a) $\mathcal{M} = (-\infty,\infty)_v \times (0,\infty)_r \times S^2$ with

   $$\tilde{g} = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{1}$$

   and time orientation such that $-\partial_r$ is future directed. (HINT: This is region $I + II$ of the maximally extended Schwarzschild spacetime.)

   (b) $\mathcal{M} = \mathbb{R}_t \times \mathbb{R}_r$ with the Lorentzian metric

   $$g = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2$$

   and time orientation such that ∂_t is future directed. (HINT: This is the two-dimensional anti-de Sitter spacetime from Sheet 4.)

2 Problems and Discussion

1. This exercise aims at generalising the formula of the divergence theorem derived in lectures to allow the boundary to contain null pieces.

   Let (M, g) an oriented, time-oriented connected Lorentzian manifold with volume form ϵ and let $N = \partial_{null}M$ be a smooth null hypersurface in M. Assume N is represented as the zero set of some smooth function $u : M \to \mathbb{R}$ (this can always be done locally).

   (a) Show that given u, there exists a unique $n-1$ form η on N such that $\epsilon = du \wedge \eta$. Is η uniquely associated with N? Hint: The function fu for a non-vanishing function f represents the same N.

   (b) Suppose that $\mathcal{R}$ is a compact spacetime region with boundary being the union of two spacelike hypersurfaces Σ_1, Σ_2 (with (timelike) unit outward normals n_1 and n_2) and a null piece N as above. Conclude that

   $$\int_M div X \epsilon = \int_{N=\partial_{null}M} X(u)\eta - \int_{\Sigma_1} g(X, n_1)\epsilon_{\Sigma_1} - \int_{\Sigma_2} g(X, n_2)\epsilon_{\Sigma_2},$$

   for X a smooth vectorfield on M.

   (c) As an illustration, repeat the basic energy conservation computation on the truncated cones applying the divergence theorem.

2. Let $(\mathbb{R}^3, h)$ be a Riemannian manifold. Consider on $\mathbb{R}^4$ the (static) Lorentzian metric

   $$g = -dt^2 + h_{ij}(x)dx^i dx^j.$$

   Fix $x_0 \in \mathbb{R}^3$ and denote by $B_h(x_0, R)$ the ball of radius R (i.e. the set of points with Riemannian distance from x_0 smaller than R). Suppose given $R > 0$ sufficiently small there exists a smooth solution $q : B_h(x_0, R) \to \mathbb{R}^+_0$ of the problem

   $$\sum_{i,j=1}^{n} h^{ij}\partial_i q \partial_j q = 1, \quad q(x_0) = 0$$

   such that $q > 0$ in $B_h(x_0, R) \setminus \{x_0\}$.[1] For every $r < R$ define the set

   $$S := \{(t, x) \mid q(x) = r - t \quad , \quad 0 \leq t \leq r\}. \tag{2}$$

   What does S describe geometrically? Consider now a smooth solution to the covariant wave equation $\Box_g \psi = 0$. Prove that if the initial data $\psi(t = 0, x)$ and $\partial_t \psi(t = 0, x)$ vanish for $\{q \mid q(x) \leq r\}$, then the solution necessarily vanishes in S.

3. For this problem you can assume that the set $I^+(p)$ in a Lorentzian manifold is always open. (This is intuitively clear and we will show it later by studying convex neighbourhoods.) You can also assume, as discussed in lectures, that in a globally hyperbolic spacetime the sets $J^+(p) \cap J^-(q)$ are compact. Show that in a globally hyperbolic spacetime (M, g) the sets $J^{\pm}(p)$ for $p \in M$ are always closed. Show that the assumption of global hyperbolicity is necessary.

[1] For Riemannian geometers: The existence of q can be inferred from the existence of gedeodesic normal coordinates at p.