Non-linear Wave Equations – Week 6

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On this sheet, $\Box = -\partial_t^2 + \sum_{i=1}^3 \partial_i^2$ denotes the standard wave operator in dimension $3+1$.

1. (Global existence with infinite energy.) Consider the equation
$$\Box\phi = \phi|\phi|^2$$
in $\mathbb{R} \times \mathbb{R}^3$. We have shown in class that finite energy smooth initial data give rise to global-in-time finite energy smooth solutions. Show that smooth initial data, $(f,g) \in C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$, i.e. data without the assumption on the finiteness of the initial energy, give rise to global smooth solutions.

2. (Nirenberg example revisited.) Consider the Cauchy problem for the equation
$$\Box\phi = (\partial_t \phi)^2 - \sum_{i=1}^3 (\partial_i \phi)^2$$
with $\phi : I \times \mathbb{R}^3 \to \mathbb{R}$.
   (a) Show that there exist smooth and compactly supported initial data such that the solution blows up in finite time. HINT: Use the transformation $\psi = e^\phi - 1$ from Example Sheet 1.
   (b) Consider the following statement:

   For initial data $f \in H^{s+1}(\mathbb{R}^n)$ and $g \in H^s(\mathbb{R}^n)$ there exists a unique solution ϕ in the space $C\left([0,T], H^{s+1}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^s(\mathbb{R}^n)\right)$ such that the time of existence T depends only on the $H^{s+1}(\mathbb{R}^n)$ norm of f and the $H^s(\mathbb{R}^n)$-norm of g.

   Clearly, the above statement is true for $s = 6$ by our wellposedness theorem from lectures. Show using (a) that it is false for $s = 0$. Determine the smallest $s \leq 6$ such that the statement is true.

3. (Keller's blow-up theorem) Consider the non-linear wave equation
$$\Box\phi = -\phi^2 \quad , \quad u(t=0,x) = f(x) \quad , \quad \partial_t u(t=0,x) = g(x) \tag{1}$$
with $f, g \in C^\infty(\mathbb{R}^3)$. For any given constant $p, q > 0$ define
$$E := \frac{q^2}{2} - \frac{p^3}{3} \quad \text{and} \quad T := \int_p^\infty \left| \frac{u^3}{3} + E \right|^{-\frac{1}{2}} du\,.$$
   (a) Prove that if $f(x) = p$ and $g(x) = q$ then the solution to (1) blows up at time T.
   HINT: Use the energy conservation law for the corresponding ODE.
   (b) Prove that if $f(x) = p$ and $g(x) \geq q$ holds for all $|x| \leq T$ in (1), then the corresponding solution ϕ blows up on or before time T.
   HINT: Consider the difference with the solution from (a) and use the representation formula.

DISCUSSION: Discuss briefly the case of other dimensions and possible generalisations of the comparison principle underlying the theorem. See also *J. B. Keller, On solutions of nonlinear wave equations, Comm. Pure Appl. Math. 10, 1957, pp. 523-530*

Analysis Review Problems

1. Let H be a Hilbert space and (x_n) a sequence in H. Show that if (x_n) converges weakly in H and $\|x_n\| \to \|x\|$, then (x_n) in fact converges strongly. [This is relevant if you attempt the additional problem on page 3 below!]

2. Show that there exists a smooth function $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which is such that $\psi(t) : \mathbb{R}^n \to \mathbb{R}$ is compactly supported for all t but still $\psi \notin C^0\left(\mathbb{R}, L^2(\mathbb{R}^n)\right)$.
HINT: Let $\psi(t,x) = \chi\left(x_1 - \frac{1}{t}, x_2, \ldots, x_n\right)$ for $\chi \in C_0^\infty(\mathbb{R}^n)$ if $t > 0$ and $\psi(t,x) = 0$ for $t \leq 0$.

3. Prove the following Sobolev embedding estimate. For $0 \leq s < \frac{n}{2}$ there exists a constant C depending only on s and n such that
$$\|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \leq C \|\phi\|_{\dot{H}^s(\mathbb{R}^n)} .$$
In particular, $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$.

   Consult a standard PDE reference (e.g. Evans) if you have never seen this before!

An additional problem in case you are bored during Pentecost...

(Regularity properties of the limit in the iteration scheme.) Recall the iteration scheme for the well-posedness theorem proven in lectures. We constructed a sequence of (smooth) functions $\left(\phi^{(i)}\right)$ with the property that

- $\left(\phi^{(i)}(t), \partial_t \phi^{(i)}(t)\right)$ uniformly bounded for all $t \in [0,T]$ in $H^{n+3}(\mathbb{R}^n) \times H^{n+2}(\mathbb{R}^n)$ and
- $\phi^{(i)}$ converges in $C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$ to a limit ϕ in that space.

In this problem we prove that it follows that $\phi \in C^0\left([0,T], H^{n+3}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2}(\mathbb{R}^n)\right)$.

1. Establish that $\phi \in C^0\left([0,T], H^{n+3-\epsilon}(\mathbb{R}^n)\right) \cap C^1\left([0,T], H^{n+2-\epsilon}(\mathbb{R}^n)\right)$ holds for any $\epsilon > 0$. Conclude that for sufficiently large n the limit ϕ is a classical (i.e. C^2) solution of the non-linear wave equation. HINT: Use the interpolation estimate from Sheet 4 for the first part. Use Sobolev embedding and the equation for the second.

2. Prove that the limit ϕ is weakly continuous in the sense that for every bounded linear functional $\mathcal{F}$ on $H^{n+3}(\mathbb{R}^n)$ we have that $\mathcal{F}[\phi(t,\cdot)]$ is a continuous function of t. Show similarly that $\partial_t\phi$ is weakly continuous as a function with values in $H^{n+2}(\mathbb{R}^n)$.
HINT: Represent the functional $\mathcal{F}$ by $\mathcal{F}[\psi] = \int_{\mathbb{R}^n} \hat{v}\hat{\psi}$ for a $v \in H^{-n-3}(\mathbb{R}^n)$. Then show that $\sup_{t\in[0,T]} |\mathcal{F}[\phi(t,\cdot)] - \mathcal{F}[\phi^{(i)}(t,\cdot)]| \to 0$ as $i \to \infty$.

3. Define for any $\psi(t), \varphi(t) \in H^{k+1}(\mathbb{R}^n)$ and $\partial_t\psi(t), \partial_t\varphi(t) \in H^k(\mathbb{R}^n)$ the energy
$$E_k[\psi,\varphi](t) = \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} -a^{tt}(\psi)|\partial_t\partial^\alpha\varphi|^2 + a^{ij}(\psi)\partial_i\partial^\alpha\varphi\partial_j\partial^\alpha\varphi + |\partial^\alpha\varphi|^2. \tag{2}$$
Prove that the limit ϕ satisfies
$$\limsup_{t\to 0^+} E_{n+2}[\phi,\phi](t) \le E_{n+2}[\phi,\phi](0)\,.$$
HINT: Start by deriving the energy estimate
$$E_{n+2}\left[\phi^{(i)}, \phi^{(i+1)}\right](t) \le E_{n+2}\left[\phi^{(i)}, \phi^{(i+1)}\right](0) + \int_0^t d\bar{t}\ldots$$
as in lectures. Deduce that
$$\limsup_{i\to 0} E_{n+2}\left[\phi, \phi^{(i+1)}\right](t) \le E_{n+2}\left[\phi,\phi\right](0) + \int_0^t d\bar{t}\left(C_1 + C_2 \limsup_{i\to 0} E_{n+2}\left[\phi,\phi^{(i+1)}\right](\bar{t})\right)$$
and apply Gronwall. Finally, use the weak continuity and Cauchy-Schwarz to prove $E_{n+2}[\phi,\phi](t) \le \limsup_{i\to 0} E_{n+2}\left[\phi,\phi^{(i+1)}\right](t)$.

4. Establish strong continuity at $t = 0$, i.e. the estimate
$$\lim_{t\to 0+}\left(\|\phi(t,\cdot) - \phi(0,\cdot)\|_{H^{n+3}(\mathbb{R}^n)} + \|\partial_t\phi(t,\cdot) - \partial_t\phi(0,\cdot)\|_{H^{n+2}(\mathbb{R}^n)}\right) = 0 \tag{3}$$
HINT: Define an inner-product on $H^{n+3}(\mathbb{R}^n) \times H^{n+3}(\mathbb{R}^n)$ by
$$\langle(\psi_1,\psi_2),(\varphi_1,\varphi_2)\rangle := \sum_{|\alpha|\le n+2}\int_{\mathbb{R}^n} \partial^\alpha\psi_2\partial^\alpha\varphi_2 + a^{ij}[\phi(0,x)]\partial_i\partial^\alpha\psi_1\partial_j\partial^\alpha\varphi_1 + \partial^\alpha\psi_1\partial^\alpha\varphi_1\,.$$
Now adapt the proof of Analysis Review Problem 1 below in conjunction with (b) and (c) to conclude.

5. Explain briefly how the above argument can be repeated to establish continuity at any $t \in [0,T]$.

Reference: H. Ringström, *The Cauchy problem in General Relativity*, ESI Lectures in Mathematical Physics, EMS; Chapter 9.3