General Relativity and Black Holes – Week 5

Gustav Holzegel

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1 Exercises

1. Show that the Schwarzschild metric
$$g = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$
on $\mathcal{M} = (-\infty,\infty)_t \times (2M,\infty)_r \times S^2$ is indeed a solution of the vacuum Einstein equations.

2. Show that $(\mathcal{M}, g)$ of Exercise 1 embeds isometrically into $\widetilde{\mathcal{M}} = (-\infty,\infty)_{t^\star} \times (0,\infty)_r \times S^2$ equipped with metric
$$\tilde{g} = -\left(1-\frac{2M}{r}\right)(dt^\star)^2 + \frac{4M}{r}dt^\star dr + \left(1+\frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{1.1}$$
HINT: Use the coordinate transformation $t^\star = t + f(r)$ for appropriate f.
Which region of the Penrose diagram is covered? Show that hypersurfaces of constant $t^\star$ are spacelike and sketch them in the Penrose diagram.

3. Let (M, g) an n-dimensional Lorentzian (or Riemannian) manifold. Suppose one can find coordinates x^i, $i = 1, 2, \ldots, n$, in which the metric is diagonal, i.e. g_{ij} vanishes if $i \neq j$.

   (a) Show that the geodesic differential equations can be expressed as
$$\frac{d}{d\tau}\left(g_{kk}\frac{dx^k}{d\tau}\right) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{dx^i}{d\tau}\right)^2 \quad \text{for } 1 \leq k \leq n,$$
   where we emphasise that there is no summation over k.

   (b) Either from the Euler Lagrange equation (using Sheet 4) or from part (a) obtain the geodesic equations for the Schwarzschild manifold $(\mathcal{M}, g)$ of Exercise 1.
$$\frac{d}{d\tau}\left(\left(1-\frac{2M}{r}\right)\dot{t}\right) = 0$$
$$\frac{d}{d\tau}\left(\left(1-\frac{2M}{r}\right)^{-1}\dot{r}\right) = \frac{1}{2}\left(-\frac{2M}{r^2}\dot{t}^2 - \frac{1}{(1-\frac{2M}{r})^2}\frac{2M}{r^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right) \tag{1.2}$$
$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) = \frac{1}{2}r^2\sin\theta\cos\theta\dot{\phi}^2 \tag{1.3}$$
$$\frac{d}{d\tau}\left(r^2\sin^2\theta\dot{\phi}\right) = 0 \tag{1.4}$$
   Deduce the claim made in lectures that wlog one may consider geodesics lying in the equatorial plane $\theta = \frac{\pi}{2}$.

4. This exercise studies trapped null geodesics in the Schwarzschild geometry $(\mathcal{M}, g)$ of Exercise 1. You can assume exercise 3(b) above.

   (a) Show that through every point of the timelike hypersurface $r = 3M$ of $(\mathcal{M}, g)$ there exists a null geodesic which remains in the timelike hypersurface $r = 3M$ and is future complete.

   (b) Show that through every point of the exterior of the Schwarzschild manifold, $(\mathcal{M}, g)$, there exists a null geodesics which is future complete and asymptotes to the timelike hypersurface $r = 3M$. How many of those are there? Are they stable?

2 Problems and Discussion

1. Consider the metric $h = -x^2 dt^2 + dx^2$ on $N = (-\infty, \infty)_t \times (0, \infty)_x$. Is $x = 0$ a curvature singularity? Can you isometrically embed (N, h) into a larger manifold?

2. Prove the exercise from class that any future directed causal geodesic (in fact, any future directed causal curve) starting in region II of the Penrose diagram reaches $r = 0$ in finite affine parameter time.