Non-linear Wave Equations – Week 5

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May 13, 2021

1. (Poor man's local existence theorem for semi-linear equations using the explicit representation formula.) Consider the following *semi-linear* wave equation in $1+3$ dimensions:

$$\begin{cases} \Box_{3+1}\phi = F(\phi) \\ \phi(0,x) = f(x) \\ \partial_t\phi(0,x) = g(x) \end{cases} \tag{1}$$

with $F : \mathbb{R} \to \mathbb{R}$ smooth, $F(0) = 0$ and $f \in C_0^{k+1}(\mathbb{R}^3)$, $g \in C_0^k(\mathbb{R}^3)$ for $k \geq 2$.

Prove that there exists a $T > 0$ such that there exists unique $C^k([0,T] \times \mathbb{R}^3)$ solution ϕ of the above Cauchy problem. You can follow the outline below.

(a) Derive from the Duhamel formula in the lecture notes the representation formula for solutions of $\Box\phi = F$ with trivial data at $t = 0$:

$$\phi(t,x) = \frac{1}{4\pi}\int_{B(x,t)} \frac{F(t-|y-x|,y)}{|y-x|}dy \quad \text{for } t \geq 0.$$

(b) Define now an iteration scheme (as seen in lectures for the quasi-linear problem) based on a sequence (ϕ_m), with each ϕ_m solving a linear inhomogenous problem, and establish convergence.

(c) Can you prove the same result assuming only that $F \in C^k(\mathbb{R})$?
HINT: Show that for T sufficiently small (ϕ_m) converges in $C^{k-1}\left([0,T] \times \mathbb{R}^3\right)$ and that derivatives of order k remain uniformly bounded along the sequence. Then show that for T sufficiently small the derivatives of order k are equicontinuous and apply the Arzela-Ascoli theorem.

DISCUSSION: Is there an analogue of the persistence of regularity statement proven in lectures in this setting? What about a breakdown criterion? (Last question can be discussed a week later.)

2. Let $\eta > 0$ be a constant. Consider the following quasi-linear equation in dimension $1+3$:

$$\frac{1}{\eta^2}\partial_t^2\phi - \frac{2}{\eta^2}\sum_{i=1}^{3}\partial_i\phi\partial_t\partial_i\phi + \frac{1}{\eta^2}\sum_{i,j=1}^{3}\partial_i\phi\partial_j\phi\partial_i\partial_j\phi - \sum_{i=1}^{3}\partial_i^2\phi = 0\,. \tag{2}$$

This equation can be derived from the Euler equations for an incompressible irrotational fluid.[1] Given smooth and compactly supported initial data (f,g) satisfying $\|f\|_{L^\infty(\mathbb{R}^n)} + \|Df\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\mathbb{R}^n)} < \epsilon$ for sufficiently small ϵ prove the existence of a unique local in time solution.

HINT: Note that the above equation is more non-linear than what we discussed in lectures. The key observation to make is that the equation becomes less quasi-linear after differentiating it.

[1] See for instance Example 1.3 in the Stanford Lecture Notes of J. Luk.

Analysis Review Problems

1. Recall the statement of the Arzela-Ascoli theorem.

2. Prove the following interpolation estimate for $\phi \in L^\infty(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$:

$$\|D\phi\|_{L^4(\mathbb{R}^n)} \leq \|\phi\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^n)} \|\phi\|^{\frac{1}{2}}_{\dot{H}^2(\mathbb{R}^n)} . \tag{3}$$

   Conclude that for $n = 3$, $H^2(\mathbb{R}^n)$ is an algebra.

   HINTS: Prove the estimate first for functions of compact support, then argue (carefully) by density. You might want to start by integrating $\partial_j \left(|D\phi|^2 \phi \partial_j \phi \right)$ over $\mathbb{R}^n$. For the last part recall also the Sobolev estimates from previous sheets.

3. For $1 \leq r \leq s$ prove the (more general) interpolation estimate[2]

$$\|D\phi\|_{L^{\frac{2s}{r}}(\mathbb{R}^n)} \leq C \|\phi\|^{\frac{1}{2}}_{L^{\frac{2s}{r-1}}(\mathbb{R}^n)} \|D^2\phi\|^{\frac{1}{2}}_{L^{\frac{2s}{r+1}}(\mathbb{R}^n)} . \tag{4}$$

   Note that the case $s = 2$, $r = 1$ corresponds to the previous problem.

4. Let $\phi \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Use Problem 3 and induction to establish for $0 < |\alpha| < s$ the estimate

$$\|D^\alpha \phi\|_{L^{\frac{2s}{|\alpha|}}(\mathbb{R}^n)} \leq C_{|\alpha|,s,n} \left(\|\phi\|_{L^\infty(\mathbb{R}^n)} \right)^{1-\frac{|\alpha|}{s}} \left(\|\phi\|_{\dot{H}^s(\mathbb{R}^n)} \right)^{\frac{|\alpha|}{s}} .$$

   This estimate will be crucial for us to establish improved breakdown criteria.

[2] The estimates in Problems 2-4 are examples of Gargliardo–Nirenberg interpolation estimates.