General Relativity and Black Holes – Week 4

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1 Exercises

1. Let (M, g) be a Lorentzian manifold and $p, q \in M$ with $q \in I^+(p)$. Denote by $\Omega(p, q)$ the set of smooth timelike curves γ from p to q. Recall the length functional for timelike curves $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$ given by

$$L[\gamma] = \int_a^b \sqrt{-g(\dot{\gamma}(\tau), \dot{\gamma}(\tau))} d\tau \, .$$

Define also the energy functional

$$E[\gamma] = \int_a^b -g(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) d\tau \, .$$

   (a) Show that the critical points of $E[\gamma]$ as a functional over $\Omega(p, q)$ are precisely the timelike geodesics between p and q . HINT: Compute the Euler-Lagrange equations of the functional. You may use the results of the discussion later in this sheet.

   (b) Show that γ is a critical point of $E[\gamma]$ if and only if γ is a critical point of $L[\gamma]$ and is parametrised by a parameter proportional to proper time.

Discussion: What happens for null geodesics? What happens for spacelike geodesics?
Background: Computing the Euler-Lagrange equations for the functional $E[\gamma]$ is often an efficient way to compute geodesics. It can even be an efficient way to compute Christoffel symbols by reading them off from the resulting equations.

2. (a) Let $\gamma : I \to M$ be a curve whose tangent vector X satisfies the condition

$$\nabla_X X = \alpha(t) X$$

   for some function $\alpha : I \to \mathbb{R}$. Show that there exists some reparametrization $t \to f(t)$ such that $\gamma \circ f$ is a geodesic in a suitable domain.

   (b) Consider a geodesic curve $\gamma : I \to M$. Prove that, given the affine (re)parametrization $t' = at + b$, then $\gamma(t')$, taken with the appropriate domain, is also a geodesic.

   (c) Prove directly the following claim made in lectures for $(1 + n)$-dimensional Minkowski space: If $q \in I^+(p)$, then there exists a unique maximising geodesic, up to affine reparametrization, from p to q.

3. Consider the manifold $\mathbb{R}_t \times \mathbb{R}_r$ equipped with the Lorentzian metric

$$g = -(1 + r^2) dt^2 + \frac{1}{1 + r^2} dr^2 \, .$$

   (a) Show that the manifold is timelike geodesically complete, i.e. all inextendible[1] timelike geodesics can be defined on all of $\mathbb{R}$.

[1] A curve $\gamma : I \to M$ is called *inextendible* if it is not the restriction of a curve $\Gamma : J \to M$ with $I \subset J$ properly.

(b) Show that given $p \in M$, there exist $q \in I^+(p)$ such that there is no timelike geodesic connecting p and q.

Background 1: A corollary of the Hopf-Rinow theorem in *Riemannian* geometry implies that if (M, g) is geodesically complete, then any two points can be connected by a minimizing geodesic.

Background 2: The above metric is the two-dimensional anti-de Sitter metric.

4. We seek to characterize the relationship between curvature and the tendency of geodesics to accelerate toward and away from each other. Specifically, we define a smooth one-parameter family of geodesics $\gamma_s : I \to M$ such that the map $(t, s) \to \gamma_s(t)$ is a diffeomorphism onto its image). Define the vectors $T = \left[\frac{\partial}{\partial t}\right], X = \left[\frac{\partial}{\partial s}\right]$ (where, for each p, we define $\left[\frac{\partial}{\partial t}\right] = \left[\frac{\partial}{\partial t}\right]_p$, $\left[\frac{\partial}{\partial s}\right] = \left[\frac{\partial}{\partial s}\right]_p$).

   (a) Show that there exists a reparametrization $t \to t'(t, s)$ (with corresponding T') such that $g(T', T')$ is constant for all (t', s).

   (b) Show that there exists a new reparametrization $t' \to t''$ such that, in these coordinates, $g(T'', X) = 0$ for all (t'', s) (hint: use the fact that $[T'', X] = 0$).

   (c) We define the acceleration a with respect to the displacement X to be

   $$a = \nabla_T \nabla_T X.$$

   Prove that the *geodesic deviation equation*

   $$a^e = -R^e_{cbd} X^b T^c T^d$$

   holds.

Heuristically, X represents the displacement to an infinitesimally close geodesic. Then, if the right hand side is nonzero, members of γ_s are accelerating towards or away from each other.

2 Problems and Discussion

1. **Discussion: A brief review of variational methods:** We consider a curve $\gamma : [a, b] \to M$ with the constraint that, for some points $p, q \in M$, $\gamma(a) = p$, $\gamma(b) = q$. Consider a *Lagrangian*, defined to be a smooth function

   $$F : [a, b] \times TM \to \mathbb{R}.$$

   Here TM is the tangent bundle on M,

   $$TM = \{(p, S) \mid p \in M, S \in T_pM\}.$$

   We define the *action functional*

   $$S[\gamma] = \int_a^b F(t, \gamma(t), \dot{\gamma}(t))\, dt$$

   acting on curves. Then, a given curve γ is a critical point of S if, for a given coordinate chart, γ satisfies the *Euler-Lagrange equations*,

   $$\frac{\partial F}{\partial \gamma^i}(t, \gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\frac{\partial F}{\partial \dot{\gamma}^i}(t, \gamma(t), \dot{\gamma}(t)) = 0,$$

   for all i.

2. **Problem: Existence of compact Lorentzian manifolds which are not geodesically complete:** We consider the manifold $M = \mathbb{R}^2 - (0, 0)$, endowed with the metric

   $$ds^2 = \frac{2}{u^2 + v^2} du\, dv.$$

(a) Verify that this is a Lorentzian manifold, and that the transformation $(u, v) \to (cu, cv)$ is an isometry for all $c \neq 0$.

(b) Consider the group of isometries $\Gamma = \{(u, v) \to (2^k u, 2^k v)\}$, for $k \in \mathbb{Z}$. Show that $T = M/\Gamma$ is a compact Lorentzian manifold (i.e. take the manifold M subject to the identification $(u, v) \sim (2u, 2v)$).

(c) Write down the geodesic equation and show that there exists a inextendible geodesic $\gamma : I \to T$ with $I \subsetneq \mathbb{R}$. (Hint: consider geodesics on M with $v = 0$). Consequently, T is not geodesically complete.