Non-linear Wave Equations – Week 4

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1. (Uniqueness of distributional solutions to non-constant coefficient linear wave equations in the space $C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$.)

   Recall that to prove uniqueness of our class of linear wave equations we need to show that the only distributional solution $\Phi \in C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$ with zero initial data is $\Phi \equiv 0$.

   Prove that if $\Phi \in C^0\left([0,T], H^1(\mathbb{R}^n)\right) \cap C^1\left([0,T], L^2(\mathbb{R}^n)\right)$ satisfies

   $$\int_0^T \int_{\mathbb{R}^n} \Phi \partial_\alpha \left(a^{\alpha\beta} \partial_\beta \psi\right) dxdt = 0 \quad \text{for all } \psi \in C_0^\infty\left([0,T), \mathbb{R}^n\right) \tag{1}$$

   then $\Phi \equiv 0$ in $[0,T] \times \mathbb{R}^n$. (You can follow the outline below.)

   (a) Show that the above assumption and the regularity of Φ implies that

   $$\int_0^T \int_{\mathbb{R}^n} \partial_\alpha \Phi \left(a^{\alpha\beta} \partial_\beta \psi\right) dxdt = 0 \tag{2}$$

   holds for all $\psi \in H^1\left([0,T) \times \mathbb{R}^n\right)$.

   (b) Fix now $s \in (0,T)$ and define the function $v : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ by

   $$v(t,x) = \int_t^s \Phi(\tau, x) d\tau \quad \text{for } 0 \leq t \leq s \quad \text{and } v \equiv 0 \text{ on } (s,T) \times \mathbb{R}^n.$$

   Observe that $v \in H^1\left([0,T) \times \mathbb{R}^n\right)$ and insert it in (2) as a test function obtaining an estimate

   $$\|\Phi\|^2_{L^2(\mathbb{R}^n)}(s) + \|a^{ij} \partial_i v(0,\cdot) \partial_j v(0,\cdot)\|_{L^1(\mathbb{R}^n)} \leq \dots$$

   Set now $w(t) = \int_0^t \Phi(\tau) d\tau$ to deduce an estimate to which Gronwall's inequality can be applied.

2. Verify the estimates for the expressions denoted by β and γ in the lecture notes!

3. (Ill-posedness of the Cauchy problem for elliptic equations.)

   Consider the Cauchy problem for Laplace's equation, i.e. $\Delta \phi = \frac{\partial^2}{\partial t^2}\phi + \frac{\partial^2}{\partial x_1^2}\phi + \dots \frac{\partial^2}{\partial x_n^2}\phi = 0$ on $\mathbb{R}^{1+n}$ with prescribed Cauchy data $\phi(t=0,x) = f(x)$ and $\partial_t \phi(t=0,x) = g(x)$. Show that:

   (a) If f and g are analytic near the origin in $\mathbb{R}^n$, then an analytic solution exists in small neighbourhood of the origin in $\mathbb{R}^{1+n}$.

   (b) Any classical solution of $\Delta u = 0$ in an open set $\mathcal{U}$ is actually analytic in $\mathcal{U}$. (Feel free to consult a standard PDE reference (e.g. Evans) if you have not seen this before.) Conclude that the Cauchy problem cannot be solved outside the analytic class.

   (c) Set $n = 1$ and check that with $k \in \mathbb{N}$

   $$\phi_k(t,x) = \frac{1}{k^2} \sin(kx) \sinh(kt)$$

   is a family of solutions to the Cauchy problem with data $\phi(t=0,x) = 0$, $\partial_t \phi(t=0,x) = \frac{1}{k}\sin(kx)$. Prove that for any $T > 0$, the following is true: Given any $C > 0$ (large) and $\delta > 0$ (small), one can find initial data with $\|f\|_{C^1(\mathbb{R})} + \|g\|_{C^0(\mathbb{R})} < \delta$ such that $\sup_{[0,T]\times\mathbb{R}} |\phi| \geq C$. Compare and contrast with the wave equation!

Analysis Review Problems

1. Prove the following interpolation inequality

$$\|f\|_{H^s(\mathbb{R}^n)} \le \|f\|^{\theta_1}_{H^{s_1}(\mathbb{R}^n)} \|f\|^{\theta_2}_{H^{s_2}(\mathbb{R}^n)} \tag{3}$$

   for some $C = C(s_1, s_2, s, n)$ where $0 \le s_1 \le s \le s_2$, $\theta_1 + \theta_2 = 1$ and $\theta_1 s_1 + \theta_2 s_2 = s$.
   HINT: Use Hölder's inequality.

2. Prove the following *basic version of the Banach Alaoglu theorem*: Let (u_k) be a bounded sequence in a Hilbert space H, i.e. $\|u_k\|_H \le C$. Then there exists a subsequence which converges weakly in H.
   HINT: Use the following outline

   (a) Pick an ONB (e_k) and use a diagonal argument to show that for a subsequence of the (u_k), denoted $(u_n^{(n)})$ (arising from a Cantor diagonal argument) we have that

   $$\langle u_n^{(n)}, e_k \rangle \to v_k \in \mathbb{R} \quad \text{holds for all } e_k.$$

   (b) Show that $\sum_{k=1}^{\infty} |v_k|^2 < \infty$ and hence $v = \sum_k v_k e_k \in H$.

   (c) Show that $u_n^{(n)} \rightharpoonup v$.