General Relativity and Black Holes – Week 3

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1 Exercises

1. Let M be a smooth manifold and X, Y, Z smooth vectorfields. Show that
$$\mathcal{L}_{[X,Y]}Z = \mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z\,.$$
Generalise the above formula to arbitrary (k,l)-tensor fields T:
$$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X\mathcal{L}_Y T - \mathcal{L}_Y\mathcal{L}_X T\,.$$
Conclude that the Killing vectorfields form a Lie algebra as claimed in lectures.

2. Let (M, g) be an $(n+1)$-dimensional connected Lorentzian (Riemannian) manifold. Use the formula
$$\nabla_a\nabla_b K_c = R^d{}_{abc}K_d \tag{1}$$
from lectures to show that the maximum number of linearly independent Killing vectorfields on M is equal to $\frac{(n+1)(n+2)}{2}$. HINT: Derive an ODE system for a Killing vector along a curve.

   Check explicitly that what we defined in lectures as the Killing vectorfields of Minkowski spacetime are indeed Killing vectorfields. Check also that they indeed form a Lie-algebra. Conclude that Minkowski space is maximally symmetric. Discuss also briefly the case of the 2-sphere equipped with the standard metric.

3. Let $(\bar{M}, \bar{g})$ be a Riemannian manifold and $M = \mathbb{R}\times\bar{M}$ equipped with the Lorentzian metric $g = -dt^2 + \bar{g}$. Show that $s \mapsto (\gamma^0(s), \overline{\gamma}(s))$ is a geodesic in M if and only if $s \mapsto \overline{\gamma}(s)$ is a geodesics in $\overline{M}$ and $\gamma^0(s) = \lambda s$ with $\lambda \in \mathbb{R}$.

4. Show that the (second) Bianchi identity for the Riemann tensor implies that the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is divergence free, i.e. $\nabla^i G_{ij} = 0$.

2 Problems and Discussion

1. Let U be a simply connected region on a smooth manifold M equipped with connection ∇. Show that the torsion tensor and the Riemann curvature tensor both vanish if and only if the exists a coordinate system on U with respect to which the connection coefficients all vanish.

2. Let (M, g) be an n-dimensional Lorentzian (or Riemannian manifold) equipped with the Levi-Civita connection.
   (a) Use the symmetries of the Riemann tensor introduced in lectures
      (i) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$,
      (ii) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\nu\mu}$,
      (iii) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$,

(iv) $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$,

to prove that the number of independent components of the Riemann tensor is $\frac{n^2(n^2-1)}{12}$. Conclude that in dimension $n = 3$ the Riemann tensor has the same number of independent components as the Ricci tensor. What happens in dimension $n = 2$?

(b) Define the Weyl tensor

$$C_{\rho\sigma\mu\nu} := R_{\rho\sigma\mu\nu} + \frac{1}{n-2}\left(R_{\rho\nu}g_{\sigma\mu} - R_{\rho\mu}g_{\sigma\nu} + R_{\sigma\mu}g_{\rho\nu} - R_{\sigma\nu}g_{\rho\mu}\right) + \frac{1}{(n-1)(n-2)}R\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right).$$

Show that the Weyl tensor has the same symmetries as the Riemann tensor and is in addition traceless over all indices. Conclude that $C_{\rho\sigma\mu\nu} = 0$ for $n = 3$.

(c) Consider a conformally related metric $\tilde{g} = \phi^2 g$ on M, where $\phi : \mathcal{M} \to \mathbb{R}^+$ is a non-vanishing spacetime function (the conformal factor). Show that the Weyl tensor is conformally invariant in that

$$\tilde{C}^{\mu}{}_{\rho\sigma\tau} = C^{\mu}{}_{\rho\sigma\tau}\,,$$

where the left hand side is the Weyl tensor associated with the metric $\tilde{g}$. Interpret the result. Conclude also that for $n \geq 4$ a necessary condition for the metric to be conformally flat is that its Weyl tensor vanishes. (It is also sufficient.)