Non-linear Wave Equations – Week 3

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1. Consider a global C^2-solution ψ to the inhomogeneous wave equation in $n+1$ dimensions

$$\Box\psi = -\partial_t^2\psi + \Delta\psi = F\,,$$

where $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is smooth. Fix $T > 0, R > 0$ and consider the past light cone through $(t = T + R, \vec{0})$, truncated at $t = 0$ and $t = T$ (as in lectures). Let B_ρ denote the ball of radius ρ around the origin in $\mathbb{R}^n$.

   (a) Prove the following estimate on the truncated cone. For any $0 \leq \tau \leq T$ we have

$$E[\psi]\,(\tau) \leq E[\psi]\,(0) + C\int_0^\tau d\tilde{\tau} E[\psi]\,(\tilde{\tau}) + G\,(\tau)\ , \tag{1}$$

   where

$$E[\psi]\,(\tau) = \frac{1}{2}\int_{\{t=\tau\}\times B_{R+T-\tau}} d^n x \left[(\partial_t\psi)^2 + |\nabla_x\psi|^2 + \psi^2\right]$$

   and

$$G\,(\tau) = \int_0^\tau ds \int_{\{s\}\times B_{R+T-s}} d^n x |F|^2\,.$$

   HINT: Establish the estimate first without the ψ^2-term in $E[\psi](\tau)$.

   (b) Deduce from (1) the estimate

$$E[\psi]\,(\tau) \leq (E[\psi]\,(0) + G\,(\tau))\, e^{C\tau} \quad \text{for any } \tau \in [0, T].$$

   HINT: Use (a slight generalisation of) Gronwall's inequality. Note that G is non-decreasing.

   (c) Deduce the domain of dependence property for classical solutions of the non-linear equation $\Box\psi = (\partial_t\psi)^2$: If ϕ and $\partial_t\phi$ vanish identically in B_{R+T} then the solution necessarily vanishes in the entire lightcone. Can you also infer a uniqueness statement?
   HINT: Use the above with $F = (\partial_t\psi)^2$ and estimate $|F|^2 \leq C(\partial_t\psi)^2$ with C depending on the C^1-norm of the solution in the cone. For uniqueness, use $(\partial_t\psi_1)^2 - (\partial_t\psi_2)^2 = (\partial\psi_1 - \partial\psi_2)(\partial_t\psi_1 + \partial_t\psi_2)$.
   DISCUSSION: Generalisation to other non-linearities and globalising the result.

   (d) Having established the domain of dependence property we revisit Problem 5 of Sheet 1. Construct a blow up solution of $\Box\psi = (\partial_t\psi)^2$ with compactly supported initial data. Next, given $\epsilon > 0$, construct a blow up solution with compactly supported initial data (f, g) such that the initial energy satisfies $\int_{t=0} d^n x \left(|\nabla f|^2 + g^2\right) \leq \epsilon$. Hint: Use the scaling properties of the equation.

2. In this problem we will prove the domain of dependence property for a class of non-constant coefficient wave equations. The class is more restrictive than the one considered in class but will give you the main idea. We consider the equation

$$-\partial_t^2 \phi + \partial_i \left((h^{-1})^{ij} \partial_j \phi \right) = 0 \tag{2}$$

on $\mathbb{R}^{1+n}$ where h_{ij} is a positive definite matrix on $\mathbb{R}^n$ satisfying

$$\sum_{i,j=1}^{n} |h_{ij} - \delta_{ij}| \leq \frac{1}{10} \, .$$

Fix $x_0 \in \mathbb{R}^n$. Suppose there exists a smooth solution to

$$\sum_{i,j=1}^{n} (h^{-1})^{ij} \partial_i q \partial_j q = 1 \quad , \quad q(x_0) = 0 \tag{3}$$

in $B_h(x_0, R)$ such that $q > 0$ in $B_h(x_0, R) \setminus \{x_0\}$.[1] For every $r < R$ we define the set

$$S := \{(t, x) \mid q(x) < r - t \quad , \quad 0 \leq t \leq r\} \, .$$

Show that if the initial data (f, g) vanish in $\{x \mid q(x) \leq r\}$, then the solution ϕ vanishes identically in S.[2]

Discussion: Can you combine problems 1 and 2?

Analysis Review Problems

1. Show that Schwartz functions are dense in $H^s(\mathbb{R}^n)$.
2. Show that $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$ are dual spaces to one another.
3. Prove the following Sobolev inequality for compactly supported functions in $\mathbb{R}^3$:

$$\sup_{\mathbb{R}^3} |u| \leq C \left(\|u\|_{\dot{H}^2(\mathbb{R}^3)} + \|u\|_{\dot{H}^1(\mathbb{R}^3)} \right) \, . \tag{4}$$

DISCUSSION: The norm on the right hand side is conserved for solutions to the standard wave equation leading to a simple proof of uniform boundedness

[1] Some background for people knowing some Riemannian geometry: The level sets of q consist of points of constant distance (measured with respect to the metric h) from x_0 (check the case $h_{ij} = \delta_{ij}$!). The geodesics of the metric h_{ij} are in fact the characteristics of the first order non-linear PDE (3), which is known as the eikonal equation. Note also that with this construction the hypersurfaces of constant $t + q(x)$ (which constitute part of the boundary of S) are "null" with respect to the Lorentzian metric $g = -dt^2 + h_{ij}dx^i dx^j$. Finally, note that at top order (3) agrees with the covariant wave equation $\Box_g \psi = 0$ associated to g.

[2] Geometrically, S is the past light cone of the point $(t = r, x_0)$ with respect to the Lorentzian metric $g = -dt^2 + h_{ij}dx^i dx^j$. Draw a picture!