General Relativity and Black Holes – Week 2

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1 Exercises

1. Prove that a smooth manifold always admits a Riemannian metric. *Bonus question*: Can you construct a *complete* Riemannian metric?

2. Let M be a smooth manifold. Show that a map
$$\tau : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{l \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \to C^\infty(M)$$
is induced by a (k,l)-tensor field iff it is multilinear over $C^\infty(M)$. Similarly, show that a map
$$\tau : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{l \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k \text{ times}} \to \mathcal{X}(M)$$
is induced by a $(k+1,l)$-tensor field iff it is multilinear over $C^\infty(M)$.

3. Let X, Y, Z be smooth vectorfields on a smooth manifold M. Let ∇ be a torsion free connection on M (*i.e.* such that $\nabla_X Y - \nabla_Y X = [X,Y]$ for all smooth vectorfields X, Y). Show that
$$(\nabla\nabla Z)(X,Y) - (\nabla\nabla Z)(Y,X) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

4. Let V be an $(n+1)$-dimensional Lorentzian inner product space. Prove that orthogonal null vectors are collinear. Prove that orthogonal non-spacelike vectors are null hence collinear.

5. (a) Let M, N be two smooth manifolds and $\Phi : M \to N$ be a smooth map. For all smooth 1-forms α on N we define the *pull-back* $\Phi^*\alpha$ of α by
$$(\Phi^*\alpha)|_p := \alpha|_{\Phi(p)} \circ \mathrm{d}\Phi_p.$$
Show that $\Phi^*\alpha$ is a smooth 1-form on M.

   (b) Assume additionally that Φ is a local diffeomorphism. We define the *push-forward* of a 1-form α on M to be the 1-form $\Phi_*\alpha := (\Phi^{-1})^*\alpha$ on N. Propose a definition of the pull-back and push-forward operations for vectorfields consistent with contractions with 1-forms. Generalise this definition to (k,l)-tensor fields.

   (c) Let M be a smooth manifold and X a smooth vectorfield on M. We define the *Lie derivative* of a (k,l)-tensor field F on X to be
$$\mathcal{L}_X F := \lim_{t\to 0} \frac{(\Phi_t)^*F - F}{t},$$
where Φ_t is the flow associated to the vectorfield X (see the lecture notes for definition). Show that $(\Phi_t)^* = (\Phi_{-t})_*$ and hence that this definition is consistent with the definition given in lectures.

   (d) Show the Leibniz rule for the Lie derivative, *i.e.*
$$\mathcal{L}_X(F \otimes G) = \mathcal{L}_X F \otimes G + F \otimes \mathcal{L}_X G,$$
for all smooth tensor fields F, G on M. Show that the Lie derivative commutes with tensorial contractions.

   (e) Write the expression in coordinates for the Lie derivative of a general (k,l)-tensor field.

2 Problems

1. Prove that if (M, g) is a Lorentzian manifold which is not time-orientable then there exists a time-orientable Lorentzian manifold $(\tilde{M}, \tilde{g})$ which is a double cover of (M, g).

2. Let (M, g) be a Lorentzian (or Riemannian) manifold, consider a point $p \in M$ and let x^μ be a local coordinate system centred at p (*i.e.* $x^\mu(p) = 0$). Let $X, Y, Z \in T_pM$ be three tangent vectors. Let $0 < \varepsilon, \delta \ll 1$ be very small. We first parallely propagate Z along the curve γ, *i.e.*, first from 0 along the straight coordinate line to εX^μ and then along the straight coordinate line to $\varepsilon X^\mu + \delta Y^\mu$ to obtain the vector $Z_\gamma(\varepsilon X^\mu + \delta Y^\mu)$. Now, we parallely propagate Z along the curve γ', *i.e.*, first from 0 to δY^μ and then to $\varepsilon X^\mu + \delta Y^\mu$, both along straight coordinate lines. Denote the resulting vector by $Z_{\gamma'}(\varepsilon X^\mu + \delta Y^\mu)$. Show that to leading order in ε and δ we have

$$Z_\gamma(\varepsilon X^\mu + \delta Y^\mu) - Z_{\gamma'}(\varepsilon X^\mu + \delta Y^\mu) = -\varepsilon\delta R^\rho_{\kappa\alpha\beta}(0) Z^\kappa X^\alpha Y^\beta,$$

thus giving another interpretation of curvature. (*Hint*: Can you justify that the parallel transport of Z from 0 to εX^μ is to leading order $Z^\mu - \varepsilon \Gamma^\mu_{\kappa\sigma}(0) X^\kappa Z^\sigma$?)

3. Let V be an n-dimensional vector space with $n \geq 2$. Let g be a Lorentzian inner product on V and assume that $q : V \times V \to \mathbb{R}$ is a bilinear form which is positive definite in the null directions of g. Then we can choose a basis on V such that $g(e_a, e_b) = \eta_{ab}$ and $q(e_a, e_b) = \lambda_a \delta_{ab}$.
(*Hint*: You can consider the minimum of the function

$$X \mapsto \arctan\left(\frac{q(X, X)}{g(X, X)}\right)$$

where X is a vector ranging in (a compact subdomain of) the future time cone of V and argue similarly as in the variational proof of the spectral theorem for quadratic forms in Euclidean vector space.)