Non-linear Wave Equations – Week 2

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1. **The Klein-Gordon equation (wave equation for massive fields), from notes of J. Luk.** Suppose ϕ satisfies the Klein-Gordon equation, i.e.
$$\Box\phi - \phi = 0$$
on $\mathbb{R}\times\mathbb{R}^n$ with initial data $\phi(0,x) = f(x)$ and $\partial_t\phi(0,x) = g(x)$ where $f, g \in C_0^\infty(\mathbb{R}^n)$.
   (a) Show using the Fourier transform that if the solution is sufficiently regular, it is given by
$$\hat{\phi}(t,\xi) = \hat{f}(\xi)\cos(\sqrt{4\pi^2|\xi|^2+1}\cdot t) + \frac{\hat{g}}{\sqrt{4\pi^2|\xi|^2+1}}\sin(\sqrt{4\pi^2|\xi|^2+1}\cdot t).$$
   (b) For $n = 1$ prove (following the steps below) that there exists a constant $C(f,g) > 0$ such that
$$(\sup_{x\in\mathbb{R}}|\phi|)(t) \leq \frac{C}{\sqrt{1+t}}\,.$$
      i. Establish first that it suffices to show that
$$\Big|\int_{\mathbb{R}} e^{it\left(\frac{2\pi x\xi}{t}+\sqrt{4\pi^2|\xi|^2+1}\right)}\hat{f}(\xi)d\xi\Big| \leq \frac{C}{\sqrt{t}} \quad \text{holds for large } t.$$
      ii. Defining the phase $\varphi = \frac{2\pi x\xi}{t} + \sqrt{4\pi^2|\xi|^2+1}$ deduce
$$\frac{\partial}{\partial\xi}\varphi = 0 \qquad \Longrightarrow \qquad \xi^2 = \frac{x^2}{4\pi^2(t^2-x^2)}$$
      and
$$\left(\frac{\partial}{\partial\xi}\right)^2\varphi \geq \frac{4\pi^2}{(4\pi^2|\xi|^2+1)^{\frac{3}{2}}}\,.$$
      iii. Finally, split the integral into $\int_{|\xi^2-\frac{x^2}{4\pi^2(t^2-x^2)}|\leq\delta} + \int_{|\xi^2-\frac{x^2}{4\pi^2(t^2-x^2)}|\geq\delta}$, estimate each part with the above observation and optimise in δ to obtain the desired decay result.
   (c) Prove now for general n that
$$(\sup_{x\in\mathbb{R}^n}|\phi|)(t) \leq \frac{C}{(1+t)^{\frac{n}{2}}}.$$
2. Recall the H^s estimates that we obtained for solutions to the wave equation from the Fourier theory. In analogy to what was done in lectures, one can also construct generalised solutions in $C^0(\mathbb{R}, H^s)\cap C^1(\mathbb{R}, H^{s-1})$ for any $s\in\mathbb{R}$. As an example, solve the initial value problem for the wave equation in $1+1$ dimensions with data $u(0,x) = 0$, $\partial_t u(0,x) = \delta(x)$, where $\delta(x)$ is the Dirac delta distribution. (This is an example of propagation of singularities along characteristics.)
3. **Lorentz invariance of the wave equation**; Fritz John, Problem 1, Chapter 5.1(c) Let S denote a spacelike hyperplane with equation $t = \gamma x_1$ in xt-space. Show that the Cauchy problem for $\Box\phi = 0$ with data on S can be reduced to the initial value problem (i.e. posed at $t' = 0$) for the same equation by introducing new independent variables x', t' by the *Lorentz transformation*
$$x_1' = \frac{x_1-\gamma t}{\sqrt{1-\gamma^2}} \quad , \quad x_2' = x_2 \quad , \quad x_3' = x_3 \quad , \quad t' = \frac{t-\gamma x_1}{\sqrt{1-\gamma^2}}\,.$$
Discusssion: Discuss the Lorentz invariance of the wave operator.

Analysis Review Problems

1. Find (non)-examples of Schwartz functions. Construct a $u \in \mathcal{S}(\mathbb{R})$ that is not exponentially small at infinity, i.e. for all $k > 0$ one has $e^{k|x|^k} u \notin L^\infty(\mathbb{R})$.

2. Prove that $\mathcal{S}(\mathbb{R}^n)$ equipped with the metric ρ defined in the notes is a complete metric space. Is the metric induced by a norm?

3. Prove that the Fourier transform is a continuous bijective transformation from $\mathcal{S}(\mathbb{R}^n)$ to itself. Prove the Plancherel identity for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \,.$$

   Conclude that the Fourier transform extends to an isometric isomorphism $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.
   Discussion: Extension of the Fourier transform to L^p.