General Relativity and Black Holes – Week 12

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1 Exercises

1. Apply the coordinate transformation

$$t^\star = t + T(r), \qquad \text{with } \frac{dT}{dr} = \frac{r^2+a^2}{\Delta}\ , \tag{1}$$

$$\phi^\star = \phi + A(r) \pmod{2\pi}, \qquad \text{with } \frac{dA}{dr} = \frac{a}{\Delta}. \tag{2}$$

given in lectures to the Kerr metric in Boyer-Lindquist coordinates (Section 5.9). Show that the resulting metric in $(t^\star, r, \theta, \phi^\star)$ coordinates extends to a larger manifold into which the Boyer-Lindquist manifold embeds isometrically and such that $r = r_+$ is a null hypersurface.

2. Show that the spheres of symmetry in region II of the maximally extended Schwarzschild spacetime are trapped spheres. Show that there are trapped spheres in the analogous region of Kerr (cf. Exercise 1).

3. Verify that the Killing tensor for Kerr (given in the lectures notes, see Section 5.9) is indeed a Killing tensor.

2 Problems and Discussion

1. (a) Let (M, g) be a Lorentzian manifold and let $\mathcal{H} \subset \mathcal{M}$ be a Killing horizon of a Killing vector field V, i.e. $\mathcal{H}$ is a null hypersurface with null generators V being Killing. Show that the surface gravity κ defined by $V^\mu \nabla_\mu V_\nu|_{\mathcal{H}} = \kappa V_\nu|_{\mathcal{H}}$ satisfies

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu V^\nu)(\nabla_\mu V_\nu)|_{\mathcal{H}}. \tag{3}$$

   Hint: Use that V is hypersurface orthogonal, i.e. that $V_{[\mu}\nabla_\nu V_{\sigma]}|_{\mathcal{H}} = 0$. To prove the hint you might want to revisit Sheet 7, Problem 2.

   (b) Use the first part to compute the surface gravity of the future event horizon in Kerr.

2. Derive the ODE system of equations for null geodesic flow derived in class. Flesh out the argument (sketched in lectures) establishing the existence of trapped geodesics in the region $[r_1, r_2]$.