Non-linear Wave Equations – Week 12

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1. This problem involves some basic estimates involving the Littlewood-Paley projections, some of which having been used explicitly in the lectures.

   (a) Prove that

   $$\|f\|^2_{\dot{H}^s} \approx \sum_{k\in\mathbb{Z}} 2^{2ks}\|P_k f\|^2_{L^2} \quad \text{and} \quad \|f\|^2_{H^s} \approx \sum_{k\in\mathbb{Z}} \left(1+2^k\right)^{2s} \|P_k f\|^2_{L^2} .$$

   (b) Prove that for any $k \in \mathbb{Z}$ we have

   $$\|P_k f\|_{L^\infty} \leq C\|f\|_{\dot{H}^{\frac{n}{2}}}$$

   with C independent of k.

   (c) Define the Besov space $B^s_{2,1}$ as the closure of $C^\infty_0(\mathbb{R}^n)$ relative to the Besov norm

   $$\|f\|_{B^s_{2,1}} := \|P_{\leq 0} f\|_{L^2} + \sum_{k=1}^{\infty} 2^{ks}\|P_k f\|_{L^2} \ .$$

   Define also the homogeneous Besov norm as

   $$\|f\|_{\dot{B}^s_{2,1}} := \sum_{k\in\mathbb{Z}} 2^{ks}\|P_k f\|_{L^2} \ .$$

   Prove that $H^{s+\epsilon} \subset B^s_{2,1} \subset H^s$ for any $\epsilon > 0$ and the embedding inequality $\|f\|_{L^\infty} \leq C\|f\|_{B^{\frac{n}{2}}_{2,1}}$. Recall (and show also) that the estimate $\|f\|_{L^\infty} \leq C\|f\|_{H^{\frac{n}{2}}}$ fails.

2. Prove the calculus estimate

   $$\Big| \int_{-1}^{1} dz\, e^{ikz}(1-z^2)^{\frac{n-3}{2}} \Big| \leq \frac{C}{|k|^{\frac{n-1}{2}}}$$

   used in lectures.

3. Provide the details for the proof of the *inhomogeneous* Strichartz estimate, in particular:

   (a) Repeat the scaling argument to show it suffices to prove the estimate for F supported in a bounded frequency range.

   (b) Derive the Duhamel formula in frequency space and conclude the relevant estimates for the kernel.

   You can of course use freely all the results from the proof of the homogeneous estimate!