General Relativity and Black Holes – Week 11

Gustav Holzegel

January 12, 2022

1 Exercises

1. (Aretakis instability. Blow-up of higher order transversal derivatives for extremal black holes.) Prove that if ψ solves the wave equation on an extremal Reissner-Nordström black hole

$$\Box_{g_{RN}} \psi = 0, \tag{1}$$

(cf. Section 5.7), then ψ satisfies the identity

$$\int_{S^2} \sin\theta \, d\theta \, d\phi \big(\partial_v (\partial_r \partial_r \psi + \tfrac{1}{M} \partial_r \psi - \tfrac{1}{M^2} \psi) + \tfrac{1}{M^2} \partial_r \psi\big) = 0 \tag{2}$$

on the event horizon $r = M$.

Conclude that, if there exists a v for which

$$\int_{S^2} \big(\partial_r \psi + \tfrac{1}{M} \psi\big) \sin\theta \, d\theta \, d\phi \neq 0 \tag{3}$$

at the event horizon and

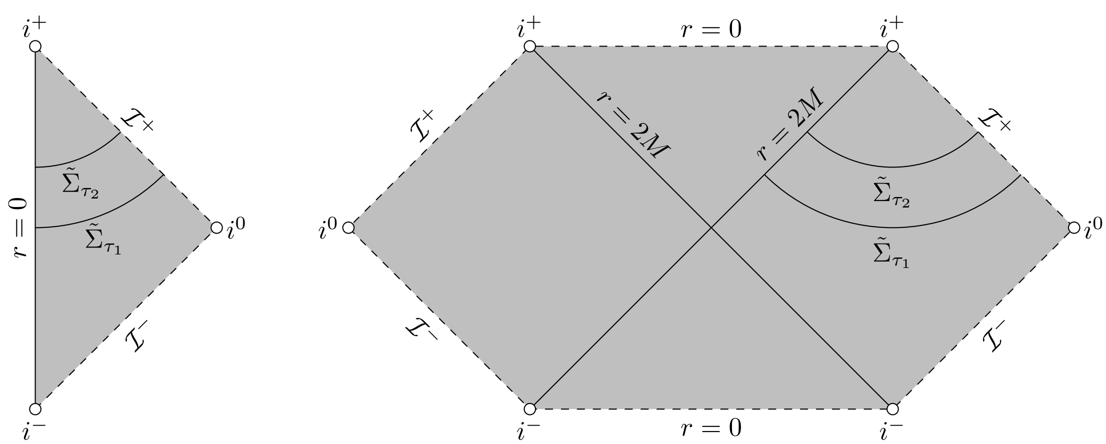
$$\int_{S^2} \psi \, \sin\theta \, d\theta \, d\phi \to 0 \tag{4}$$

along $\mathcal{H}^+$, then

$$\Big| \int_{S^2} \partial_r \partial_r \psi \sin\theta \, d\theta \, d\phi \Big| \to \infty \tag{5}$$

Remark 1.1. *The limit* (4) *indeed holds, but showing it is more involved.*

2. For the Minkowski and the Schwarzschild spacetime, find an explicit parametrisation of the slices $\tilde{\Sigma}_\tau$, which are smooth, spacelike and asymptotically approach a null hypersurface, as in the Penrose diagrams below:



3. The proof of Theorem 5.20 (inverse polynomial decay of the energy through the slices Σ_τ) of the notes left two gaps, which we are filling in here.

   (a) For $\tau_2 \geq \tau_1 \geq 0$ prove the estimate:

$$\int_{\tau_1}^{\tau_2} r^p \left(\frac{1}{4}|\nabla\!\!\!/ \phi|^2 - (\partial_v \phi)^2\right) \sin\theta d\theta d\varphi d\tau\Big|_{r=R} \leq C \int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma \tag{6}$$

   HINT: Multiply the identity from lectures,

$$\partial_u \left(r^p (\partial_v \phi)^2\right) + \frac{1}{2} p r^{p-1} (\partial_v \phi)^2 + \nabla\!\!\!/^A \left(-\frac{1}{2} r^{p-2} \partial_v \phi \nabla\!\!\!/_A \phi\right)$$
$$+ \frac{1}{4} \partial_v \left(r^{p-2} \nabla\!\!\!/^A \phi \nabla\!\!\!/_A \phi\right) + \frac{1}{8}(2-p) r^{p-3} \nabla\!\!\!/^A \phi \nabla\!\!\!/_A \phi = 0\,, \tag{7}$$

   by a radial cut-off function χ equal to 1 in $[R-1, R]$ and equal to zero for $r \leq R-2$ and pull the cut-off inside the boundary terms (producing additional terms). Integrate the resulting identity over the region $\bigcup_{\tau_1 \leq \tau \leq \tau_2} (\Sigma_\tau \cap \{R-2 \leq r \leq R\})$. Control the resulting terms other than the one of the left of (6) by the energy using Proposition 5.19.

   (b) Deduce the step from (194) to (195) in the notes. Show first that this boils down to showing

$$\int_{\tau_1}^{\tau_2} d\tau \int_{S^2_{\tau,R}} \sin\theta d\theta d\varphi r \psi^2 \leq C \int_{\Sigma_{\tau_1}} J^T_\mu[\psi] n^\mu_\Sigma\,. \tag{8}$$

2 Problems and Discussion

1. For the ψ of Theorem 5.20 (and similarly Theorem 5.22) prove the pointwise estimates

$$|\sqrt{r}\psi| \leq C E_2 \tau^{-1} \quad \text{and} \quad |r\psi| \leq C E_2 \tau^{-\frac{1}{2}}\,, \tag{9}$$

   for an appropriate higher order weighted energy E_2 that you should determine.

   HINT: The statement is simple for spacetime points lying on some $\Sigma_\tau \cap \{r \leq R\}$ using standard Sobolev embedding. For points lying on the outgoing null piece of Σ_τ use for instance weighted Hardy inequalities and Sobolev embedding on spheres.