Non-linear Wave Equations – Week 11

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1. (Integrated local energy decay estimate: Part II.) This is a continuation of Problem 2 from Sheet 10. Recall ϕ is a smooth solution of $\Box\phi = 0$ in dimension $1+3$ arising from data (f, g) of compact support at $t = 0$ and that we proved an integrated decay estimate for *angular* derivatives.

   (a) Prove that

   $$\Big| \int_0^T \int_{\mathbb{R}^3} \left[h'(r)(\partial_r\phi)^2 + \frac{h(r)}{r}|\nabla\!\!\!/\phi|^2 - \frac{1}{4}\Delta\left(h'(r) + \frac{2h(r)}{r}\right)\phi^2\right] dxdt \Big| \leq C\left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)}\right)$$

   holds for a constant C independent of T.
   HINT: Integrate by parts the expression $\int_0^T \int_{\mathbb{R}^3} \Box\phi \cdot h(r)(\partial_r\phi) dxdt$ where $h(r)$ is a bounded function satisfying $h'(r) \leq \frac{\tilde{C}}{1+r^2}$. The identity $\Box(\phi^2) = 2\left(-(\partial_t\phi)^2 + |\nabla\phi|^2\right)$ may be useful.

   (b) Choose $h(r) = \frac{1}{1+r}$ and use (c) from the previous sheet to deduce

   $$\int_0^T \int_{\mathbb{R}^3} \left[\frac{1}{(1+r)^2}(\partial_r\phi)^2 + \frac{1}{r}|\nabla\!\!\!/\phi|^2 + \frac{1}{(1+r)^4}|\phi|^2\right] dxdt \leq C\left(\|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)}\right).$$

   (c) Discussion: Can you also control the $(\partial_t\phi)^2$-derivative on the left? Can you improve the r-weights in the last estimate?

2. This question uses the notation used in our proof of shock formation in dimension $1+3$. The goal is to make the argument of Step 2 precise and give a complete proof of the Lemma.

   (a) Extend the bootstrap argument in Step 1 to prov that also

   $$|r^3 L L\psi| \leq C\epsilon$$

   holds in $\mathcal{M}_{T_{max},u_0}$ for all $\epsilon < \epsilon_0$.

   (b) Infer that $|L(rL\mu)(t,u)| \leq C\epsilon\frac{\ln(e+t)}{(1+t)^2}$ holds in $\mathcal{M}_{T_{max},u_0}$. Use the bound to infer that for all $0 \leq s \leq t$ one has

   $$L\mu(s,u) = \frac{1}{r(s,u)}\left((rL\mu)(t,u)\right) + O\left(\epsilon\frac{\ln(e+s)}{(1+s)^2}\right).$$

   and hence for all $0 \leq s \leq t$

   $$\mu(s,u) = 1 + \ln\left(\frac{1-u+s}{1-u}\right) rL\mu(t,u) + \mathcal{O}(\epsilon).$$

   (c) Use the equation for $L\mu$ and previous bounds to deduce

   $$\mu(s,u) = 1 + \ln\left(\frac{1-u+s}{1-u}\right)\left(-\frac{1}{4}\frac{\mu\underline{L}(r\psi)}{1+\psi}(t,u)\right) + \mathcal{O}(\epsilon)$$

   for all $0 \leq s \leq t$. Complete the proof of the Lemma.

3. Consider in dimension $1+3$ the quasi-linear equation

$$\begin{cases} -\partial_t^2\phi + (1+\phi)\Delta\phi = 0 \\ \phi(t=0,x) = \epsilon f(x) \\ \partial_t\phi(t=0,x) = \epsilon g(x). \end{cases} \tag{1}$$

We shall assume that the initial data f and g are smooth and compactly supported in a ball of radius 1. We shall restrict ourselves to **spherically symmetric** solutions.

(a) Show that spherically symmetric initial data yield spherically-symmetric solutions.
(You can assume $\phi > -1$ for as long as the solution exists as otherwise we lose hyperbolicity.)

(b) Determine the characteristic directions L and $\underline{L}$ for (1) and define the null-coordinate $u(t,r)$ (depending on the solution) analogous to what we did in lectures, i.e. solving $Lu = 0$ with initial condition $u(t=0,r) = 1-r$.

(c) Prove that there exists an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, the solution to (1) exists globally in the region $\{t \geq 0\} \cap \{\frac{1}{4} \leq u \leq 1\}$.
HINT: Adapt the bootstrap argument in the proof of shock formation as follows: Do not μ-renormalise the $L\left(\frac{\underline{L}(r\phi)}{1+\phi}\right)$ equation but instead exploit that the right hand side is now integrable. Bootstrap the estimates $|r^2 L\phi| \leq C\epsilon(t+1)^{\frac{1}{4}}$ and $|\mu| + \frac{1}{|\mu|} \leq 3(t+1)^{\frac{1}{4}}$.

REMARK: Small data global existence for (1) holds globally without symmetry. See H. Lindblad, “Global Solutions to Non-Linear wave equations”, American Journal of Mathematics, Vol. 130, No. 1 (Feb., 2008), pp. 115-157 (43 pages).